COLLECTED PAPERS OF G. H. HARDY

INCLUDING JOINT PAPERS WITH J. E. LITTLEWOOD AND OTHERS

EDITED BY A COMMITTEE APPOINTED BY THE LONDON MATHEMATICAL SOCIETY

VI

OXFORD

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VOLUME VI

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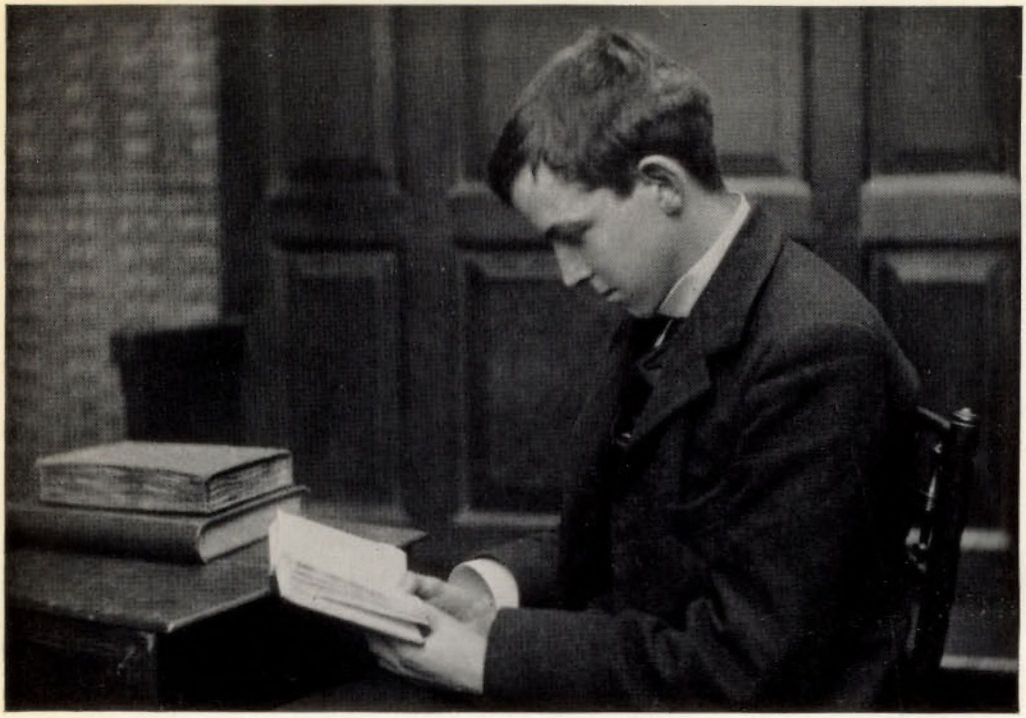
The main object of this publication is to render more accessible the papers of the great mathematician, which in their original form appeared in many journals over a period of about 50 years. The editors have kept in view a second object also; that of rendering the work useful to mathematicians generally by providing introductions to groups of papers, or comments where appropriate. These editorial additions, while not always systematic or exhaustive, will (it is hoped) assist the reader to view Hardy's papers in proper perspective.

The work will be completed in seven volumes.



SCHOLAR OF WINCHESTER, c. 1890

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EDITORIAL NOTE

The work will comprise seven volumes

FOR convenience of reference, papers are numbered according to years, e.g. 1912, 4. A complete list of Hardy's papers will be found at the end of this volume (pp. 837–53) and is reproduced at the end of each volume. This list is based on that compiled by Titchmarsh (*Journal of the London Mathematical Society*, 25 (1950), 89–101).

Where reference is made, in corrections or comments, to the pages of a paper, the numbers used are those of the original pagination and not the consecutive page numbers of the volume.

CONTENTS OF VOLUME VI

THEORY OF SERIES

INTRODUCTION TO THE PAPERS ON SERIES

THE papers in this volume are arranged in order of publication. The Comments contain references† to earlier writers and writers who have filled gaps or answered questions arising out of the papers. References to the many writers who have developed and transformed the subject during the past twenty-five years are, in general, avoided. These may be found in Zeller‡ or *Mathematical Reviews*. Bibliographies up to 1930 are given by Kogbetliantz§ and Moore.‖ See also Hardy's *Divergent series* and Hardy and Riesz's *General theory of Dirichlet's series*; these are referred to as D.S. and H.R. References to D.S. are either to the text or to the Notes at the ends of the chapters.

L. S. B.

† The dates given are those of the published volumes.

‡ K. Zeller, Theorie der Limitierungsverfahren, *Ergebnisse der Mathematik und ihre Grenzgebiete*, 15, 1st edn. 1956, 2nd edn. (with W. Beekmann) 1970. Springer, Berlin.

§ E. Kogbetliantz, Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques, *Mémorial des sci. math.* 51, 1931.

‖ C. N. Moore, Summable series and convergence factors, *American Math. Soc. Colloquium Publications*, 22, 1938.

ABBREVIATED TITLES

THE following works, and also those mentioned in the Introduction, are referred to by abbreviated titles.

A. F. Andersen, Studier over Cesàro's Summabilitetsmetode, *Copenhagen dissertation* (1921).

H. Bohr (1) Bidrag til de Dirichlet'ske Rækkers Theori, *Copenhagen dissertation* (1910). *Collected works*, Vol. I (Danish), Vol. III (English translation); (2) *Collected mathematical works*, Vols. I–III. Danish math. Soc., Copenhagen, 1952.

É. Borel, *Leçons sur les séries divergentes*, 1st edn. 1901, 2nd edn. (revised by G. Bouligand) 1928. Gauthier–Villars, Paris.

T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, 1st edn. 1908, 2nd edn. (with T. M. MacRobert) 1926. Macmillan, London. (Edn. 1 contains a chapter on Borel's and other methods of summation, not included in edn. 2.)

G. H. Hardy, *Divergent series* (1949). Oxford University Press; referred to as D.S.

—— and M. Riesz, The general theory of Dirichlet's series, *Cambridge Tracts in Math. and Math. Phys.* No. 18 (1915, reprinted 1952). Cambridge University Press; referred to as H.R.

ON THE CONVERGENCE OF CERTAIN MULTIPLE SERIES

By G. H. Hardy.

1. The most important of the few known tests for the conditional convergence of simple series are derived from an elementary theorem generally known as Abel's lemma. In this paper I propose to extend this theorem in such a way as to derive similar tests for the conditional convergence of multiple series. So far as I am aware, no one has yet proved the convergence of any general class of multiple series whose terms are not all positive, though the general theory of such series has been worked out in considerable detail by Pringsheim.*

2. Abel's lemma may be written in the form

$$\sum_{i=1}^{p} a_i u_i = \sum_{i=1}^{p-1} (a_i - a_{i+1}) \sum_{k=1}^{i} u_k + a_p \sum_{k=1}^{p} u_k.$$

This is an almost obvious identity. Hence

$$\sum_{i=1}^{p} \sum_{j=1}^{q} a_{i,j} u_{i,j} = \sum_{j=1}^{q} \left[\sum_{i=1}^{p-1} \beta_{i,j} v_{i,j} + a_{p,j} v_{p,j} \right], \tag{1}$$

where

$$\beta_{i,j} = a_{i,j} - a_{i+1,j},$$

$$v_{i,j} = \sum_{k=1}^{i} u_{k,j}.$$

But

$$\sum_{j=1}^{q} \beta_{i,j} v_{i,j} = \sum_{j=1}^{q-1} (\beta_{i,j} - \beta_{i,j+1}) \sum_{l=1}^{j} v_{i,l} + \beta_{i,q} \sum_{l=1}^{q} v_{i,l} \tag{2}$$

and

$$\sum_{j=1}^{q} a_{p,j} v_{p,j} = \sum_{j=1}^{q-1} \gamma_{p,j} \sum_{l=1}^{j} v_{p,l} + a_{p,q} \sum_{l=1}^{q} v_{p,l}, \tag{3}$$

where

$$\gamma_{i,j} = a_{i,j} - a_{i,j+1}.$$

It is convenient to put $\Delta = a_{p,q}$,

$$\Delta_i = \beta_{i,q} = a_{i,q} - a_{i+1,q}, \qquad \Delta_j = \gamma_{p,j} = a_{p,j} - a_{p,j+1},$$

$$\Delta_{i,j} = \beta_{i,j} - \beta_{i,j+1} = \gamma_{i,j} - \gamma_{i+1,j} = a_{i,j} - a_{i+1,j} - a_{i,j+1} + a_{i+1,j+1}.$$

---

* *Sitzungsberichte d. Ak. d. Wiss. zu München*, Vol. XXVII. See also later papers by Pringsheim and London in the *Math. Annalen*.

Then, from (1), (2), and (3),

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{i,j}u_{i,j} = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}\Delta_{i,j}\sum_{k=1}^{i}\sum_{l=1}^{j}u_{k,l} + \sum_{i=1}^{p-1}\Delta_i\sum_{k=1}^{i}\sum_{l=1}^{q}u_{k,l}$$
$$+\sum_{j=1}^{q-1}\Delta_j\sum_{k=1}^{p}\sum_{l=1}^{j}u_{k,l} + \Delta\sum_{k=1}^{p}\sum_{l=1}^{q}u_{k,l}. \quad \text{(A)}$$

3. The corresponding equation for n-ple series may be written in the form

$$\sum_{i_1=1}^{p_1}\sum_{i_2=1}^{p_2}\dots\sum_{i_n=1}^{p_n} a_{i_1,i_2,\dots,i_n}u_{i_1,i_2,\dots,i_n} = \Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1,k_2,\dots,k_n})]. \quad \text{(A}')$$

To form the right-hand side we proceed as follows. We take any selection of the suffixes $i_1, i_2, \dots, i_n$ as a suffix for Δ. If i_1 does not occur in this selection, we put $i_1 = p_1$ in $a_{i_1,\dots,i_n}$. If it does, we substitute

$$a_{i_1,i_2,\dots,i_n} - a_{i_1+1,i_2,\dots,i_n} \quad \text{for} \quad a_{i_1,i_2,\dots,i_n}.$$

We repeat this for each suffix, and the result is the corresponding Δ; thus, *e.g.*,

$$\Delta_{i_{n-1},i_n} = a_{p_1,p_2,\dots,i_{n-1},i_n} - a_{p_1,p_2,\dots,i_{n-1}+1,i} - a_{p_1,p_2,\dots,i_{n-1},i_n+1}$$
$$- a_{p_1,p_2,\dots,i_{n-1}+1,i_n+1}.$$

In the summation in round brackets the limits for k_ν are 1 and i_ν if i_ν is a suffix of Δ; 1 and p_ν otherwise. The summation within the square brackets applies to every i_ν which is a suffix of Δ, and the limits are 1 and $p_\nu - 1$. The outside summation applies to all selections of the suffixes, including that in which no suffix is selected. In this case $\Delta = a_{p_1,p_2,\dots,p_\nu}$.

It is easy to prove (2) by induction. We assume it for n indices, and suppose each a and u affected with a new suffix i_{n+1}. We then sum from $i_{n+1} = 1$ to $i_{n+1} = p_{n+1}$, and apply Abel's lemma to each of the terms on the right. Then it is almost obvious that we obtain

$$\Sigma[\Sigma\Delta(\Sigma\Sigma\dots\Sigma u_{k_1,k_2,\dots,k_n,k_{n+1}})]. \quad \text{(A}'')$$

For the term of (A′) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega$ gives the two terms of (A″) whose characteristic suffixes are $i_\theta, i_\phi, \dots, i_\omega, i_{n+1}$ and $i_\theta, i_\phi, \dots, i_\omega$ respectively.

The theorem expressed by the equation (A′) is therefore true generally.

4. Now suppose that $\Delta'_{i_\theta,i_\phi,\dots,i_\omega}$ is the quantity formed in the same way as $\Delta_{i_\theta,i_\phi,\dots,i_\omega}$, except that the suffixes i_κ which are not suffixes of Δ'

are *not* put equal to p_κ. And suppose that the quantities a, u satisfy the following conditions:—

(i.) All the quantities Δ' are positive;

(ii.) $\operatorname*{Lim}_{i_r=\infty} a_{i_1,\ldots,i_r,\ldots,i_n}=0$ $(i=1, 2, \ldots, n)$ *uniformly* for all values of $i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_n$;

(iii.) $\left|\sum_1^{i_1}\sum_1^{i_2}\ldots\sum_1^{i_n} u_{k_1, k_2, \ldots, k_n}\right|$ is less than a constant C for all values of $i_1, i_2, \ldots, i_n$.

Then *the series* $\sum_1^\infty\sum_1^\infty\ldots\sum_1^\infty a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n}$ *is convergent.*

It is evident that (ii.) implies that the s-ple limit obtained by keeping any $n-s$ suffixes constant and making the remaining s tend simultaneously to infinity is zero.

5. To prove the convergence of the series we have to show that however small be σ we can so choose M that

$$\left|\sum_{i_1=1}^{m_1+p_1}\sum_{i_2=1}^{m_1+p_2}\ldots\sum_{i_n=1}^{m_n+p_n} a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n} - \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\ldots\sum_{i_n=1}^{m_n} a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n}\right| < \sigma \quad (4)$$

for any values of $m_1, m_2, \ldots, m_n$ all $> M$ and all positive values of $p_1, p_2, \ldots, p_n$.

Now let us take any selection of the i's (including at least one) and form the sum $\Sigma\Sigma\ldots\Sigma a_{i_1, i_2, \ldots, i_n} u_{i_1, i_2, \ldots, i_n}$, in which the limits for i_ν are $m_\nu+1$ to $m_\nu+p_\nu$ if i_ν is selected, 1 to m_ν if not. We can form 2^n-1 such sums, and their sum is the quantity whose modulus figures in (4).

Consider, for example, the sum for which the selected i's are $i_1, i_2, \ldots, i_\mu$, and put

$$j_\nu = i_\nu - m_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$j_\nu = i_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$q_\nu = p_\nu \quad (\nu = 1, 2, \ldots, \mu),$$

$$q_\nu = m_\nu \quad (\nu = \mu+1, \ldots, n),$$

$$a'_{j_1, j_2, \ldots, j_n} = a_{j_1+m_1, j_2+m_2, \ldots, j_\mu+m_\mu, \ldots, j_n},$$

and similarly for the u's. Then the sum is

$$\sum_{j_1=1}^{q_1} \sum_{j_2=1}^{q_2} \cdots \sum_{j_n=1}^{q_n} a'_{j_1, j_2, \ldots, j_n} u'_{j_1, j_2, \ldots, j_n},$$

which, by (A′), $$= \Sigma[\Sigma\Delta(\Sigma\Sigma \ldots \Sigma u'_{k_1, k_2, \ldots, k_n})],$$

the Δ's being now formed from a' instead of a. The modulus of this $< C\Sigma[\Sigma\Delta]$. But to find $\Sigma[\Sigma\Delta]$ we have only to suppose that $u = 1$ if all its suffixes are $= 1$, and $= 0$ otherwise. This gives

$$a'_{1, 1, \ldots, 1} = \Sigma[\Sigma\Delta].$$

Thus the modulus of our sum is $< Ca'_{1, 1, \ldots, 1}$, *i.e.*, $< Ca_{m_1+1, \ldots, m_\mu+1, 1, \ldots, 1}$, and can therefore be made $< \sigma/2^n$ by choice of M.

Exactly the same argument applies to the other 2^n-2 partial sums; and (4) follows. Therefore the series is convergent.

6. The most interesting case is that in which

$$a_{i_1, i_2, \ldots, i_n} = \phi\left(\sum_{\nu=1}^{n} a_\nu i_\nu\right),$$

where $a_1, \ldots, a_n$ are positive, and $\phi(u)$ is a function which has 0 as its limit for $u = \infty$ and has continuous derivates $\phi'(u), \phi''(u), \ldots, \phi^{(n)}(u)$, such that $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi'''(u) < 0$, ..., and

$$u_{i_1, i_2, \ldots, i_n} = \exp\left\{\left(\sum_{\nu=1}^{n} i_\nu \theta_\nu\right)\sqrt{(-1)}\right\},$$

where $\theta_1, \ldots, \theta_n$ are any real quantities other than multiples of 2π. Then

$$(-)^s \Delta'_{i_1, i_2, \ldots, i_s} = a_1 a_2 \ldots a_s \int_{i_1}^{i_1+1} \int_{i_2}^{i_2+1} \cdots \int_{i_s}^{i_s+1} \phi^{(s)}(a_1 x_1 + \ldots + a_s x_s + a_{s+1} i_{s+1} + \ldots + a_n i_n)\, dx_1\, dx_2 \ldots dx_s,$$

which has the sign of $(-)^s$, so that $\Delta'_{i_1, i_2, \ldots, i_s} > 0$, and similarly for each Δ'. Thus (i.) of § 4 is satisfied. Evidently (ii.) is satisfied. Finally,

$$\left|\sum_1^{i_1} \sum_1^{i_2} \cdots \sum_1^{i_n} \exp\{(i_1\theta_1 + \ldots + i_n\theta_n)\sqrt{(-1)}\}\right|$$

$$= \left|\Pi \frac{e^{\theta_\nu\sqrt{(-1)}}\{1 - e^{i_\nu\theta_\nu\sqrt{(-1)}}\}}{1 - e^{\theta_\nu\sqrt{(-1)}}}\right| \leqslant \operatorname{cosec}\tfrac{1}{2}\theta_1 \operatorname{cosec}\tfrac{1}{2}\theta_2 \ldots \operatorname{cosec}\tfrac{1}{2}\theta_\nu;$$

so that (iii.) is satisfied.

In particular we may suppose

$$\phi(u) = \frac{1}{u^\rho} \quad (\rho > 0).$$

Hence *the series*

$$\sum_1^\infty \sum_1^\infty \dots \sum_1^\infty \frac{\begin{matrix}\cos\\ \sin\end{matrix}(i_1\theta_1+i_2\theta_2+\dots+i_n\theta_n)}{(a_1i_1+a_2i_2+\dots+a_ni_n)^\rho}$$

are convergent if $\rho > 0$.

It was with the object of proving the convergence of these series, which form the natural generalization of some of the simplest single series considered in the books, that I undertook the preceding investigation.

[*Note added October 4th,* 1903. — A very interesting question is whether the multiple series written above is also convergent for all complex values of ρ whose real part is positive. The argument of §§ 4–6 fails when ρ is complex, but it can be proved directly from (A′) that the series is convergent if the right values of the complex powers are taken. A further question is how far the restriction that the a's are to be real and positive is necessary.

These questions, however, belong rather to the theory of zeta and allied functions than to the elementary theory of series, and would probably be answered most easily by totally different methods depending on Cauchy's theorem.]

CORRECTIONS

p. 125, *lines* 13–14. For the 2nd term on the right read $a_{p_1,p_2,\dots,i_{n-1}+1,i_n}$. The sign before the last term should be $+$.

—— *line* 11 *up*. For (2) read (A').

p. 126, *line* 4. For $(i =$ read $(r =$.

—— *line* 10 *up*. For i^n read i_n.

p. 127, *line* 5. For u read u'.

COMMENTS

The theorem given in § 4 is an extension to multiple series of Dirichlet's test for convergence. The composite series is shown to be convergent in Pringsheim's sense,

but the analysis also shows that its partial sums are bounded. In an addendum (1905, 3) the conclusion is improved to *regular* (or *complete*) convergence. See 1917, 3 and the Comments on 1905, 3.

Condition (ii) may be replaced by

$$\text{(ii)}' \quad \lim_{i_r\to\infty} a_{1,\ldots,1,i_r,1,\ldots,1} = 0 \quad (r = 1, 2, \ldots, n).$$

This was proved for double series by Bromwich.† In view of the monotonic properties in (i), condition (ii)′ is equivalent to (ii), and also to

$$\text{(ii)}'' \quad \lim_{i_r\to\infty} a_{i_1,\ldots,i_r,\ldots,i_n} = 0 \quad (r = 1, 2, \ldots, n),$$

for all choices of $i_1,\ldots, i_{r-1}, i_{r+1},\ldots, i_n$.

On the other hand, condition (i) may be replaced by

$$\text{(i)}' \quad \sum_{i_1} \cdots \sum_{i_n} |\Delta_{i_1,\ldots,i_n}| < \infty,$$

if condition (ii) is replaced by (ii)″.

The theorem may be completed in the form: *Necessary and sufficient conditions for the series*

$$\sum_{i_1} \cdots \sum_{i_n} a_{i_1\ldots i_n} u_{i_1\ldots i_n}$$

to be convergent (or boundedly convergent), whenever (iii) *holds, are* (i)′ *and* (ii)″; see also the Comments on 1905, 3. A construction, which shows the necessity of (i)′, is given in Moore, § 1.14.

Necessary and sufficient conditions for single series were given by Dedekind‡ (sufficiency) and Hadamard§ (necessity), and for double series by Hardy (sufficiency) (in 1917, 3) and Kojima|| (necessity). Hamilton†† showed that the conditions of Hardy and Kojima are equivalent to (i)′ and (ii)″ with $n = 2$. See 1917, 3 and the Comments on 1917, 3.

In 1917, 3 Hardy remarks that an extension of Abel's partial summation formula to double series was also given by Krause. Hardy's general formula, § 3, has often been quoted by later writers, but the important pioneering result in § 4 seems to be virtually unknown. The paper is not included in any of the bibliographies mentioned in the Introduction.

Answers to the questions raised in the note added at the end of the paper are given, for double series, in 1905, 14 and 1917, 3. An analogous multiple integral is discussed in 1905, 11 (in Vol. V).

† 1st edn., p. 89; 2nd edn., p. 97.

‡ Dirichlet's *Vorlesungen über Zahlentheorie*, 2nd edn., revised (with supplements) by Dedekind (1871), Suppl. IX, § 143, pp. 376–7.

§ *Acta Math.* 27 (1903), 177–84.

|| *Tôhoku Math. J.* 17 (1920), 213–20.

†† *Bull. American Math. Soc.* 42 (1936), 275–83.

XIII. *On Differentiation and Integration of Divergent Series.*

By G. H. HARDY.

[*Received* 7 February 1902.]

GENERAL PRELIMINARIES.

§ 1. THERE is a class of problems—the double limit problems of the integral calculus—which are all particular cases of one or other of the two following general problems: to determine the conditions under which

$$(1)\quad \lim_{n=\infty} \int_a^A f(x, n)\, dx = \int_a^A \lim_{n=\infty} f(x, n)\, dx,$$

$$(2)\quad \lim_{\alpha=\alpha_0} \int_a^A f(x, \alpha)\, dx = \int_a^A \lim_{\alpha=\alpha_0} f(x, \alpha)\, dx,$$

n being a positive integer, and α a continuous variable. In a paper* entitled "On the continuity and discontinuity of definite integrals which contain a continuous parameter," I investigated general conditions for the truth of the second of these equations, and discussed various cases in which it does not hold. These were all cases in which the two sides of the equation are finite and determinate, but distinct.

Now let us suppose that the right hand of (1) and (2) is finite and determinate, but the left hand indeterminate. Under these circumstances we may *define* the value of the left hand as being equal to that of the right: that is to say we may use the otherwise meaningless expression on the left as a formal equivalent for the determinate expression of the right. The value of such conventions can of course be only formal and practical; but so long as they are consistent they are perfectly legitimate, and may, as has been abundantly proved, be extremely profitable †.

* *Quarterly Journal of Mathematics*, vol. XXXIV., pp. 28—52.

† For a general account of the possible uses of divergent series I may refer to Borel's writings on the subject.

SUM OF A DIVERGENT SERIES.

§ 2. Suppose, for instance, that in (1) $a=0$, $A=\infty$, and

$$f(x, n) = e^{-x} \sum_0^n u_n \frac{x^n}{n!}.$$

Then (1) becomes

$$\sum_0^\infty u_n \int_0^\infty e^{-x} \frac{x^n}{n!} dx = \int_0^\infty e^{-x} \sum_0^\infty u_n \frac{x^n}{n!} dx,$$

i.e.

$$(3) \quad \sum_0^\infty u_n = \int_0^\infty e^{-x} u(x)\, dx,$$

where

$$u(x) = \sum_0^\infty u_n \frac{x^n}{n!}.$$

When therefore Σu_n is divergent we may define its 'sum' as being equal to the integral

$$\int_0^\infty e^{-x} u(x)\, dx,$$

whenever this integral is convergent. This is the definition given by Borel, whose point of view is however different.

CONDITION OF CONSISTENCY.

§ 3. It is obvious that our definition will involve us in contradictions unless equation (3) is true whenever Σu_n is convergent. It is therefore essential to prove that this is the case.

Now, if Σu_n is convergent, $\Sigma e^{-x} u_n \frac{x^n}{n!}$ is uniformly convergent in $(0, X)$, however great be X. For, however small be the positive quantity σ, we can choose N so that

$$|u_n| < \sigma$$

if only $n \geqq N$. And then

$$\left| \sum_{N'}^{N'+p} e^{-x} u_n \frac{x^n}{n!} \right| < \sigma e^{-x} \sum_0^\infty \frac{x^n}{n!}$$
$$< \sigma,$$

for all values of x in $(0, X)$, all values of $N' \geqq N$, and all positive values of p.

Hence

$$(1) \quad \sum_0^\infty u_n \int_0^X e^{-x} \frac{x^n}{n!} dx = \int_0^X e^{-x} \sum_0^\infty u_n \frac{x^n}{n!} dx.$$

We shall be justified in replacing X by ∞ if only

$$\lim_{X=\infty} \sum_0^\infty u_n \int_X^\infty e^{-x} \frac{x^n}{n!} dx = 0.$$

But
$$\int_X^\infty e^{-x} x^n dx = e^{-X} \{X^n + nX^{n-1} + \ldots\ldots + n!\},$$

so that this series is

$$e^{-X} \sum_0^\infty u_n \sum_0^n \frac{X^i}{i!} = e^{-X} \sum_0^\infty \frac{X^n}{n!} \sum_n^\infty u_i.$$

For if
$$s_n = \sum_n^\infty u_i$$

$$u_0 + u_1(1+X) + u_2\left(1 + X + \frac{X^2}{2!}\right) \ldots\ldots + u_n\left(1 + X + \ldots\ldots + \frac{X^n}{n!}\right)$$

$$= s_0 + s_1 X + \ldots\ldots + s_n \frac{X^n}{n!} - s_{n+1}\left(1 + X + \ldots\ldots + \frac{X^n}{n!}\right),$$

and the limit of the last term for $n = \infty$ is zero.

Now let σ be any assigned positive quantity, and determine N so that if $n \geqq N$

$$|s_n| < \sigma.$$

Then
$$\left| e^{-X} \sum_{N+1}^\infty s_n \frac{X^n}{n!} \right| < \sigma e^{-X} \sum_0^\infty \frac{X^n}{n!}$$
$$< \sigma,$$

whatever be the value of X. Now we can choose a value of X' so great that

$$\left| e^{-X} \sum_0^N s_n \frac{X^n}{n!} \right| < \sigma$$

for all values of $X \geqq X'$. And then

$$\left| e^{-X} \sum_0^\infty s_n \frac{X^n}{n!} \right| < 2\sigma$$

if only $X \geqq X'$. Thus

$$\lim_{X=\infty} \sum_0^\infty u_n \int_X^\infty e^{-x} \frac{x^n}{n!} dx = 0.$$

Therefore we may replace X by ∞ in (1), and so

$$\sum_0^\infty u_n = \int_0^\infty e^{-x} \sum_0^\infty u_n \frac{x^n}{n!} dx,$$

whenever the series on the left is convergent*.

* Since writing this I have proved a more general theorem, viz. that if $\sigma_n = u_0 + u_1 + \ldots\ldots + u_n$

$$\lim \frac{\sigma_0 + \sigma_1 + \ldots\ldots + \sigma_n}{n} = \int_0^\infty e^{-x} \sum_0^\infty u_n \frac{x^n}{n!} dx,$$

if the left hand is determinate, and a certain other condition is satisfied.

It is to be observed that we have not proved that $\int_0^\infty e^{-x} u(x)\, dx$ will diverge to a definite infinity whenever Σu_n does so. This is true, however, when all the terms of the series are positive. For then, however great be G, we can determine N so that

$$\sum_0^N u_n > G.$$

That is

$$\int_0^\infty e^{-x} \sum_0^N u_n \frac{x^n}{n!}\, dx > G;$$

and *a fortiori*

$$\int_0^\infty e^{-x} u(x)\, dx > G.$$

But this can only be the case if the last integral diverges to $+\infty$.

§ 4. When the integral

$$\int_0^\infty e^{-x} u(x)\, dx$$

is convergent, I shall say, with Borel, that the series Σu_n is *summable*; and I shall denote its sum by

$$\mathop{\mathscr{S}}_0^\infty u_n.$$

It is not obvious that the sum of the series

$$0 + u_0 + u_1 + \ldots\ldots$$

is equal to that of

$$u_0 + u_1 + u_2 + \ldots\ldots$$

The sum of the first series (if it is summable) is

$$\int_0^\infty e^{-x} v(x)\, dx$$

where

$$v(x) = u_0 x + u_1 \frac{x^2}{2!} + \ldots\ldots$$

$$= \int_0^x u(x)\, dx,$$

provided $u(x)$ be uniformly convergent. And

$$\int_0^X e^{-x} \int_0^x u(x)\, dx = -\left[e^{-x} \int_0^x u(x)\, dx\right]_0^X + \int_0^X e^{-x} u(x)\, dx.$$

Hence $0 + u_0 + u_1 \ldots\ldots$ will be summable, and its sum equal to $u_0 + u_1 + u_2 \ldots\ldots$ if $u(x)$ is uniformly convergent and

$$\lim_{X=\infty} e^{-X} \int_0^X u(x)\, dx = 0.$$

And under these circumstances

$$a + u_0 + u_1 \ldots\ldots = a + (u_0 + u_1 \ldots\ldots).$$

We can deal similarly with the series

$$a + b + \ldots\ldots + k + u_0 + u_1 \ldots\ldots$$

But I shall not enter in detail into these points at present*.

* See Borel's *Leçons sur les Séries Divergentes*. Borel confines himself to *absolutely* summable series. I may remark that there is no difficulty in seeing that we may prefix any number of zero terms to the trigonometrical series considered later.

Uniform Summability.

§ 5. Now let us suppose that the terms of the series are functions of a variable α.

Let
$$u(x, \alpha) = \sum_0^\infty u_n(\alpha) \frac{x^n}{n!}$$

We shall say that $\mathop{\mathscr{S}}\limits_0^\infty u_n(\alpha)$ is *uniformly summable in* (β, γ) *if*
$$\int_0^\infty e^{-x} u(x, \alpha)\, dx$$
is uniformly convergent in (β, γ).

Continuity of the sum of a Divergent Series.

§ 6. *Theorem I. If all the terms* $u_n(\alpha)$ *are continuous functions of* α, *and*
$$\mathop{\mathscr{S}}\limits_0^\infty u_n(\alpha)$$
is uniformly summable, and
$$\sum_0^\infty u_n(\alpha) \frac{x^n}{n!}$$
uniformly convergent for any finite value of x, *in an interval* (β, γ), *the sum of the first series is a continuous function of* α *throughout the interval.*

In the first place, $e^{-x} u(x, \alpha)$ is a uniformly continuous function of α throughout the domain
$$(0, X, \beta, \gamma),$$
however great be X.

To prove this we observe that, since $\Sigma\, u_n(\alpha) \frac{X^n}{n!}$ is uniformly convergent in (β, γ), we can determine a value of N, corresponding to any assigned positive quantity σ, so that
$$\left| u_n(\alpha) \frac{X^n}{n!} \right| < \sigma,$$
for all values of α in (β, γ) and all values of $n \geqq N$.

If then X_1 is any positive quantity $< X$, and $0 < x \leqq X_1$,
$$\left| \sum_{N'}^{N'+p} u_n(\alpha) \frac{x^n}{n!} \right| < \sigma \sum_0^\infty \left(\frac{x}{X}\right)^n < \frac{\sigma X}{X - X_1},$$
for all values of α in (β, γ), all values of x in $(0, X_1)$, all values of $N' \geqq N$, and all positive values of p.

Hence the series $\Sigma\, u_n(\alpha) \frac{x^n}{n!}$ is uniformly convergent throughout $(0, X_1, \beta, \gamma)$; and as X is arbitrarily great, so is X_1. And so $e^{-x} u(x, \alpha)$ is a uniformly continuous function of α throughout any such region.

Moreover the integral

$$\int_0^\infty e^{-x} u(x, \alpha)\, dx$$

is uniformly convergent in (β, γ). Hence it is a continuous function of α.

§ 7. Suppose for instance that

$$u_n(\alpha) = \alpha^n \cos n\theta.$$

Then if $-1 < \alpha < 1$, $\sum\limits_0^\infty u_n(\alpha)$ is convergent and equal to

$$\frac{1 - \alpha \cos\theta}{1 - 2\alpha\cos\theta + \alpha^2}.$$

If $\mathscr{S}_0^\infty \alpha^n \cos n\theta$ is continuous at the extremities of $(-1, 1)$ we may make $\alpha = -1$ and 1, and so obtain

$$(1) \quad \mathscr{S}_0^\infty (-1)^n \cos n\theta = 1 - \cos\theta + \cos 2\theta \ldots\ldots = \frac{1}{2},$$

$$(2) \quad \mathscr{S}_0^\infty \cos n\theta = 1 + \cos\theta + \cos 2\theta \ldots\ldots = \frac{1}{2}.$$

Let us see whether the conditions of I are satisfied. All the terms are continuous, and $\Sigma\, \alpha^n \cos n\theta \dfrac{x^n}{n!}$ is evidently uniformly convergent for any finite value of x. It only remains to show that

$$\int_0^\infty e^{-x} u(x, \alpha)\, dx = \int_0^\infty e^{-x(1-\alpha\cos\theta)} \cos(\alpha x \sin\theta)\, dx$$

is uniformly convergent in $(-1, 1)$. This is so provided θ is not a multiple of π, in which case uniform convergence ceases at one or other of the ends of the interval. If for instance $0 < \theta < \pi$

$$\left| \int_X^\infty e^{-x(1-\alpha\cos\theta)} \cos \alpha x \sin\theta\, dx \right|$$

$$< \int_X^\infty e^{-x(1-\alpha\cos\theta)}\, dx$$

$$< \frac{1}{1 - \alpha\cos\theta} e^{-X(1-\alpha\cos\theta)},$$

for all values of α in $(-1, 1)$, and this can be made as small as we please by choice of X alone. But if $\theta = 0$ uniform convergence ceases for $\alpha = 1$; if $\theta = \pi$, for $\alpha = -1$. And in fact in these cases (1) and (2) each give

$$1 + 1 + 1 \ldots\ldots = \frac{1}{2},$$

a result in contradiction with § 3.

Elementary Trigonometrical Series.

§ 8. If $\theta \neq (2n+1)\pi$,

$$\overset{\infty}{\underset{0}{\mathscr{S}}}(-)^n \cos n\theta = \int_0^\infty e^{-x(1+\cos\theta)} \cos(x\sin\theta)\,dx = \frac{1}{2}.$$

Similarly we deduce all the following series:

$$(1)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}(-)^n \cos n\theta = 1 - \cos\theta + \cos 2\theta \ldots\ldots = \frac{1}{2},$$

$$(2)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}\cos n\theta = 1 + \cos\theta + \cos 2\theta \ldots\ldots = \frac{1}{2},$$

$$(3)\quad \overset{\infty}{\underset{1}{\mathscr{S}}}(-)^{n-1} \sin n\theta = \sin\theta - \sin 2\theta \ldots\ldots = \frac{1}{2}\tan\frac{1}{2}\theta,$$

$$(4)\quad \overset{\infty}{\underset{1}{\mathscr{S}}}\sin n\theta = \sin\theta + \sin 2\theta \ldots\ldots = \frac{1}{2}\cot\frac{1}{2}\theta,$$

$$(5)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}(-)^n \cos(2n+1)\theta = \cos\theta - \cos 3\theta \ldots\ldots = \frac{1}{2}\sec\theta,$$

$$(6)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}\cos(2n+1)\theta = \cos\theta + \cos 3\theta \ldots\ldots = 0,$$

$$(7)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}(-)^n \sin(2n+1)\theta = \sin\theta - \sin 3\theta \ldots\ldots = 0,$$

$$(8)\quad \overset{\infty}{\underset{0}{\mathscr{S}}}\sin(2n+1)\theta = \sin\theta + \sin 3\theta \ldots\ldots = \frac{1}{2}\operatorname{cosec}\theta.$$

In these series θ may have any value except those for which the series takes the form

$$1 + 1 + 1 \ldots\ldots;$$

in the case of (3), (4), (5) and (8) these are the values for which the function which represents the sum of the series becomes infinite. And it is easy to see that any one of the series is uniformly summable in any interval of values of θ which does not include any of these exceptional values.

By writing $\theta+\phi$, $\theta-\phi$ instead of θ, and adding or subtracting the results we can obtain a number of more general formulae. It will be sufficient to give the following:

$$(9)\quad 1 \pm \cos\theta\cos\phi + \cos 2\theta\cos 2\phi \ldots\ldots = \frac{1}{2},$$

$$(10)\quad \sin\theta\sin\phi \pm \sin 2\theta\sin 2\phi \ldots\ldots = 0,$$

$$(11)\quad \cos\theta\sin\phi + \cos 2\theta\sin 2\phi \ldots\ldots = \frac{1}{2}\frac{\sin\phi}{\cos\theta - \cos\phi},$$

$$(12)\quad \cos\theta\sin\phi - \cos 2\theta\sin 2\phi \ldots\ldots = \frac{1}{2}\frac{\sin\phi}{\cos\theta + \cos\phi}.$$

These hold so long as $\theta+\phi$ and $\theta-\phi$ have not certain particular values easy to specify.

DIFFERENTIATION OF A DIVERGENT SERIES TERM BY TERM.

§ 9. Suppose that $u_n(\alpha)$, whatever be n, has a derivate $u_n'(\alpha)$ continuous throughout $(\alpha_0 - \xi, \alpha_0 + \xi)$, and that $\mathscr{S}_0^\infty u_n(\alpha)$ is summable throughout this interval, $u(\alpha)$ being its sum.

Theorem II. If

$$\mathscr{S}_0^\infty u_n'(\alpha)$$

is uniformly summable in $(\alpha_0 - \xi, \alpha_0 + \xi)$, *and*

$$\sum_0^\infty u_n'(\alpha) \frac{x^n}{n!}$$

uniformly convergent for any finite value of x; *the series* $\mathscr{S}_0^\infty u_n(\alpha)$ *may be differentiated term by term for* $\alpha = \alpha_0$.

For since $\Sigma u_n'(\alpha) \dfrac{x^n}{n!}$ is uniformly convergent, its sum is $\dfrac{\partial u(x, \alpha)}{\partial \alpha} = u'(x, \alpha)$.

Also we can prove, as in § 6, that $u'(x, \alpha)$ is a uniformly continuous function of α throughout $(0, X, \alpha_0 - \xi, \alpha_0 + \xi)$, however great be X. Finally, since $\mathscr{S}_0^\infty u_n'(\alpha)$ is uniformly summable,

$$\int_0^\infty e^{-x} u'(x, \alpha)\, dx$$

is uniformly convergent. Therefore

$$\begin{aligned} u'(\alpha) &= \frac{d}{d\alpha} \int_0^\infty e^{-x} u(x, \alpha)\, dx \\ &= \int_0^\infty e^{-x} u'(x, \alpha)\, dx \\ &= \int_0^\infty e^{-x} \sum_0^\infty u_n'(\alpha) \frac{x^n}{n!} dx \\ &= \mathscr{S}_0^\infty u_n'(\alpha), \end{aligned}$$

for $\alpha = \alpha_0$.

§ 10. Consider for instance the series

$$e^{i\alpha} + e^{2i\alpha} + \ldots\ldots = -\frac{1}{2} + \frac{i}{2} \cot \frac{1}{2} \alpha.$$

The result of differentiating the left hand p times is

$$i^p \mathscr{S}_0^\infty n^p e^{ni\alpha},$$

and the sum of this series, so long as $\alpha \neq 2m\pi$, is

$$i^p \int_0^\infty e^{-x} u_p(x, \alpha)\, dx;$$

where

$$u_p(x, \alpha) = \sum_1^\infty n^p e^{ni\alpha} \frac{x^n}{n!}.$$

This series is uniformly convergent in any interval of values of α, whatever be x. And in order to prove that we may differentiate $\mathscr{S} e^{ni\alpha}$ term by term any number of times, for $\alpha = \alpha_0$, we only need to be assured that, whatever be p, we can make

$$\left| \int_X^\infty e^{-x} u_p(x, \alpha)\, dx \right|$$

assignedly small by choice of X, for all values of α in a finite interval $(\alpha_0 - \xi,\ \alpha_0 + \xi)$.

Now $$i^p u_p(x, \alpha) = \left(\frac{d}{d\alpha}\right)^p \sum_1^\infty \frac{x^n e^{ni\alpha}}{n!} = \left(\frac{d}{d\alpha}\right)^p e^{xe^{i\alpha}}.$$

If we differentiate out and replace each term by its modulus we obtain a finite sum of the form

$$e^{x \cos \alpha} \sum_0^p a_\nu x^\nu.$$

Hence we need only prove that we can make

$$\int_X^\infty e^{-x(1-\cos\alpha)} x^\nu\, dx$$

assignedly small for all values of α in an interval $(\alpha_0 - \xi,\ \alpha_0 + \xi)$. Now if $\alpha \neq 2n\pi$ we can find an interval of this kind throughout which

$$1 - \cos \alpha > \gamma,$$

a positive constant. And then

$$\int_X^\infty e^{-x(1-\cos\alpha)} x^\nu\, dx < \int_X^\infty e^{-\gamma x} x^\nu\, dx,$$

which can certainly be made as small as we please by choice of X.

§ 11. It follows that we may differentiate (2) and (4) of § 8 as often as we please, so long as $\theta \neq 2n\pi$. The same is true of the other series of § 8, so long as θ has not certain special values.

Thus, for instance,

$$(1)\quad \mathscr{S}_1^\infty n^{2s} \cos n\theta = 0,$$

$$(2)\quad \mathscr{S}_1^\infty n^{2s+1} \sin n\theta = 0;$$

and

$$(3)\quad \mathscr{S}_1^\infty n^{2s} \sin n\theta = (-)^s \left(\frac{d}{d\theta}\right)^{2s} \frac{1}{2} \cot \frac{1}{2}\theta,$$

$$(4)\quad \mathscr{S}_1^\infty n^{2s+1} \cos n\theta = (-)^s \left(\frac{d}{d\theta}\right)^{2s+1} \frac{1}{2} \cot \frac{1}{2}\theta.$$

Similarly (5) $\mathop{\mathscr{S}}_1^\infty (-)^{n-1} n^{2s} \cos n\theta = 0,$

(6) $\mathop{\mathscr{S}}_1^\infty (-)^{n-1} n^{2s+1} \sin n\theta = 0,$

(7) $\mathop{\mathscr{S}}_1^\infty (-)^{n-1} n^{2s} \sin n\theta = (-)^s \left(\frac{d}{d\theta}\right)^{2s} \frac{1}{2} \tan \frac{1}{2}\theta,$

(8) $\mathop{\mathscr{S}}_1^\infty (-)^{n-1} n^{2s+1} \cos n\theta = (-)^s \left(\frac{d}{d\theta}\right)^{2s+1} \frac{1}{2} \tan \frac{1}{2}\theta.$

From (2) and (5) we deduce

(9) $1^{2s} - 2^{2s} + 3^{2s} \ldots\ldots = 0,$

(10) $1^{2s+1} - 3^{2s+1} + 5^{2s+1} \ldots\ldots = 0.$

Also since

$$\frac{1}{2}\tan\frac{1}{2}\theta = \frac{2^2-1}{2\,!} B_1\theta + \frac{2^4-1}{4\,!} B_2\theta^3 \ldots\ldots,$$

(11) $1^{2s+1} - 2^{2s+1} + 3^{2s+1} \ldots\ldots = (-)^s \frac{2^{2s+2}-1}{2s+2} B_{s+1}.$

To obtain $1^{2s} - 3^{2s} + 5^{2s} \ldots\ldots$ we differentiate (5) of § 8 $2s$ times and put $\theta = 0$. This gives

(12) $1^{2s} - 3^{2s} + \ldots\ldots = (-)^s \left[\left(\frac{d}{d\theta}\right)^{2s} \frac{1}{2} \sec\theta\right]_{\theta=0} = \frac{(-)^s}{2} E_s.$

§ 12. Now let us consider the more general series

$$\mathop{\mathscr{S}}_0^\infty p^n e^{ni\theta}.$$

The sum of this is

$$\int_0^\infty e^{-x} \sum_0^\infty e^{ni\theta} \frac{(px)^n}{n!} dx$$

$$= \int_0^\infty e^{-x(1-p\cos\theta - ip\sin\theta)} dx$$

$$= \frac{1}{1 - p\cos\theta - ip\sin\theta},$$

provided $1 - p\cos\theta > 0.$

That is to say

(1) $\mathop{\mathscr{S}}_0^\infty p^n \cos n\theta = \frac{1-p\cos\theta}{1-2p\cos\theta+p^2},$

(2) $\mathop{\mathscr{S}}_0^\infty p^n \sin n\theta = \frac{p\sin\theta}{1-2p\cos\theta+p^2},$

if $1 - p\cos\theta > 0$. If $-1 < p < 1$ the series are convergent.

Again, consider the series

$$\mathop{\mathscr{S}}_0^\infty h^n P_n(\cos\theta).$$

The sum of this is

$$\int_0^\infty e^{-x} \sum_0^\infty P_n \frac{(hx)^n}{n!} dx.$$

Now
$$P_n = \frac{1}{\pi} \int_0^\pi (\mu - i\nu \cos\phi)^n d\phi,$$

where
$$\mu = \cos\theta, \quad \nu = \sin\theta.$$

Thus the sum of our series is

$$\frac{1}{\pi} \int_0^\infty e^{-x} \sum_0^\infty \left(\int_0^\pi \frac{\{(\mu - i\nu\cos\phi)\, hx\}^n}{n!} d\phi \right) dx.$$

Now it may be shown (I shall not stop now to work out the proof in detail) that we are justified in writing this in the form

$$\frac{1}{\pi} \int_0^\pi d\phi \int_0^\infty \left(\sum_0^\infty \frac{\{(\mu - i\nu\cos\phi)\, hx\}^n}{n!} \right) e^{-x} dx$$

$$= \frac{1}{\pi} \int_0^\pi d\phi \int_0^\infty e^{-x\{1 - h(\mu - i\nu\cos\phi)\}} dx$$

$$= \frac{1}{\pi} \int_0^\pi \frac{d\phi}{1 - h\mu + ih\nu\cos\phi}$$

$$= \frac{1}{\sqrt{(1 - 2h\mu + h^2)}},$$

provided $1 - h\mu > 0$. If $-1 < h < 1$ the series is convergent, and this is its sum in the ordinary sense.

Integration of a Divergent Series term by term.

§ 13. Let us now consider the problem of the integration of a divergent series. Suppose in the first instance that the range of integration (β, γ) is finite. Then integration term by term is justified by the following series of transformations:

$$\int_\beta^\gamma \overset{\infty}{\underset{0}{\mathscr{S}}} u_n(\alpha)\, d\alpha$$

$$= \int_\beta^\gamma d\alpha \int_0^\infty e^{-x} u(x, \alpha)\, dx$$

$$= \int_0^\infty e^{-x} dx \int_\beta^\gamma u(x, \alpha)\, d\alpha$$

$$= \int_0^\infty e^{-x} dx \int_\beta^\gamma \sum_0^\infty u_n(\alpha) \frac{x^n}{n!} d\alpha$$

$$= \int_0^\infty e^{-x} dx \sum_0^\infty \frac{x^n}{n!} \int_\beta^\gamma u_n(\alpha)\, d\alpha$$

$$= \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

We have therefore to consider what assumptions are involved in this procedure. They are evidently (i) that the order of integrations in

$$\int_\beta^\gamma d\alpha \int_0^\infty e^{-x} u(x, \alpha)\, dx$$

may be changed, (ii) that the series

$$\sum_0^\infty u_n(\alpha) \frac{x^n}{n!}$$

may be integrated term by term over (β, γ), whatever be the value of x.

Now (ii) will certainly be true if (a) $u_n(\alpha)$ is finite and integrable throughout (β, γ), whatever be n, and (b) $\Sigma u_n(\alpha)\frac{x^n}{n!}$ is uniformly convergent in (β, γ), whatever be x. And (i) will be true if (c)

$$\int_\beta^\gamma e^{-x} u(x, \alpha)\, d\alpha$$

is uniformly convergent in $(0, X)$, however great be X, and (d)

$$\int_0^\infty e^{-x} u(x, \alpha)\, dx$$

is uniformly convergent in (β, γ)*. This last condition is equivalent to that of the uniform summability of $\mathscr{S} u_n(\alpha)$. And (c) will be certainly satisfied if $\Sigma u_n(\alpha)\frac{x^n}{n!}$ is uniformly convergent throughout the domain $(0, X, \beta, \gamma)$; for then $e^{-x} u(x, \alpha)$ is a continuous function of both variables throughout this domain.

We may therefore enunciate the following theorem. It is to be understood that each separate term $u_n(\alpha)$ is finite and integrable.

Theorem III. If

$$\mathop{\mathscr{S}}_0^\infty u_n(\alpha)$$

is uniformly summable in (β, γ), *and*

$$\sum_0^\infty u_n(\alpha) \frac{x^n}{n!}$$

uniformly convergent throughout the domain $(0, X, \beta, \gamma)$, *however great be* X: *the series* $\mathscr{S} u_n(\alpha)$ *may be integrated term by term over* (β, γ).

§ 14. All the trigonometrical series of § 8, to which I am devoting particular attention in this paper, cease to be uniformly summable at certain isolated points. It is important for my present purpose to obtain theorems which may be used in cases in which some of these exceptional points are included in the range of integration.

Theorem IV. If the conditions of III. are satisfied except that $\mathscr{S} u_n(\alpha)$ *is only uniformly summable in* $(\beta, \gamma - \epsilon)$, *however small be* ϵ, *while*

$$\mathop{\mathscr{S}}_0^\infty \int_\beta^\alpha u_n(\alpha)\, d\alpha$$

* Ch. de la Vallée Poussin, *Journal de Math.*, sér. 4, t. VIII.

is a continuous function of α *up to and including* $\alpha=\gamma$; *the series* $\mathscr{S}u_n(\alpha)$ *may be integrated term by term over* (β, γ).

For
$$\int_\beta^{\gamma-\epsilon}\overset{\infty}{\underset{0}{\mathscr{S}}}u_n(\alpha)\,d\alpha=\overset{\infty}{\underset{0}{\mathscr{S}}}\int_\beta^{\gamma-\epsilon}u_n(\alpha)\,d\alpha,$$
however small be ϵ, and the theorem follows on proceeding to the limit.

§ 15. We know, for instance, that

$$\sin\alpha+\sin 3\alpha+\ldots\ldots=\tfrac{1}{2}\operatorname{cosec}\alpha,$$
$$\sin 2\alpha+\sin 4\alpha+\ldots\ldots=\tfrac{1}{2}\cot\alpha.$$

Multiply by α and integrate term by term from $\alpha=0$ to $\alpha=\frac{1}{2}\pi$. Since

$$\int_0^{\frac{1}{2}\pi}\alpha\sin(2n+1)\,\alpha d\alpha=\frac{(-)^n}{(2n+1)^2},$$
$$\int_0^{\frac{1}{2}\pi}\alpha\sin 2n\alpha d\alpha=(-)^{n-1}\frac{\pi}{4n},$$

we obtain

$$(1)\quad \int_0^{\frac{1}{2}\pi}\frac{\alpha}{\sin\alpha}\,d\alpha=2\sum_0^\infty\frac{(-)^n}{(2n+1)^2},$$
$$(2)\quad \int_0^{\frac{1}{2}\pi}\alpha\cot\alpha d\alpha=\frac{1}{2}\pi\log 2,$$

provided the conditions of III. or IV. be satisfied. The condition which concerns $\Sigma u_n(\alpha)\frac{x^n}{n!}$ is evidently satisfied in each case, and $\mathscr{S}u_n(\alpha)$ is uniformly convergent in $(\epsilon, \frac{1}{2}\pi)$, however small be ϵ. Also

$$\int_0^\epsilon\alpha\sin(2n+1)\,\alpha d\alpha=\frac{\epsilon\cos(2n+1)\,\epsilon}{2n+1}+\frac{\sin(2n+1)\,\epsilon}{(2n+1)^2}.$$

It is clear that

$$\lim_{\epsilon=0}\sum_0^\infty\frac{\sin(2n+1)\,\epsilon}{(2n+1)^2}=0;$$

and
$$\lim_{\epsilon=0}\epsilon\sum_0^\infty\frac{\cos(2n+1)\,\epsilon}{2n+1}=\lim\frac{1}{2}\epsilon\log\cot\frac{1}{2}\epsilon$$
$$=0.$$

Hence the conditions of IV. are satisfied, and (1) and (2) are correct.

§ 16. Now consider the equation

$$\sin 2\alpha-\sin 4\alpha\ldots\ldots=\tfrac{1}{2}\tan\alpha.$$

Suppose that $\phi(\alpha)$ is a function which does not vanish for $\alpha=\frac{1}{2}\pi$, and has a derivate $\phi'(\alpha)$ which is continuous throughout $(0, \pi)$. Then

$$\int_0^\pi\tan\alpha\phi(\alpha)\,d\alpha$$

is indeterminate, but its *principal value**, which I denote by

$$P\int_0^{\pi} \tan \alpha\phi(\alpha)\, d\alpha,$$

is determinate, and it is naturally suggested that we may have

$$\frac{1}{2}P\int_0^{\pi} \tan \alpha\phi(\alpha)\, d\alpha = \int_0^{\pi} \sin 2\alpha\phi(\alpha)\, d\alpha - \int_0^{\pi} \sin 4\alpha\phi(\alpha)\, d\alpha \ldots\ldots$$

Such cases may be dealt with by means of the following theorem.

Theorem V. If the conditions of III. are satisfied except that the uniform summability of $\mathcal{S} u_n(\alpha)$ *ceases for a single value* ξ *of* α, *while*

$$\lim_{\epsilon=0} \overset{\infty}{\underset{0}{\mathcal{S}}} \int_{\xi-\epsilon}^{\xi+\epsilon} u_n(\alpha)\, d\alpha$$

is zero; then

$$P\int_\beta^\gamma \overset{\infty}{\underset{0}{\mathcal{S}}} u_n(\alpha)\, d\alpha = \overset{\infty}{\underset{0}{\mathcal{S}}} \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

For under these circumstances

$$\left(\int_\beta^{\xi-\epsilon} + \int_{\xi+\epsilon}^{\gamma}\right) \overset{\infty}{\underset{0}{\mathcal{S}}} u_n(\alpha)\, d\alpha = \overset{\infty}{\underset{0}{\mathcal{S}}} \left(\int_\beta^{\xi-\epsilon} + \int_{\xi+\epsilon}^{\gamma}\right) u_n(\alpha)\, d\alpha;$$

and the theorem follows on proceeding to the limit.

Another case which may occur is that in which some or all of the terms $u_n(\alpha)$ become infinite for $\alpha = \xi$ in such a way that $\int_\beta^\gamma u_n(\alpha)\, d\alpha$ is not determinate, although its principal value is. In this case we must substitute for the condition of V. the condition that

$$\lim_{\epsilon=0} \overset{\infty}{\underset{0}{\mathcal{S}}} P\int_{\xi-\epsilon}^{\xi+\epsilon} u_n(\alpha)\, d\alpha = 0;$$

and the final result will be

$$P\int_\beta^\gamma \overset{\infty}{\underset{0}{\mathcal{S}}} u_n(\alpha)\, d\alpha = \overset{\infty}{\underset{0}{\mathcal{S}}} P\int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

§ 17. Let us consider, for instance, the example at the beginning of § 16. We have to prove that

$$\lim_{\epsilon=0} \overset{\infty}{\underset{1}{\mathcal{S}}} (-)^{n-1} \int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} \sin 2n\alpha\phi(\alpha)\, d\alpha$$

is 0, or that

$$\lim_{\epsilon=0} \int_0^\infty e^{-x} dx \sum_1^\infty (-)^{n-1} \frac{x^n}{n!} \int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} \sin 2n\alpha\, \phi(\alpha)\, d\alpha$$

is 0.

Now we may sum under the sign of integration with respect to α. Thus we obtain

$$\int_0^\infty e^{-x}\Phi(x, \epsilon)\, dx,$$

* I have worked out the theory of 'principal values' in considerable detail in some papers in the London Mathematical Society's *Proceedings*.

where
$$\Phi(x, \epsilon) = \int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} \phi(\alpha)\, d\alpha \sum_1^\infty (-)^{n-1} \sin 2n\alpha \frac{x^n}{n!}$$
$$= R\left[\frac{1}{i}\int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} e^{2i\alpha - xe^{2i\alpha}} \phi(\alpha)\, d\alpha\right].$$

The last integral is
$$-\frac{1}{2ix}\left[e^{-xe^{2i\alpha}}\phi(\alpha)\right]_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} + \frac{1}{2ix}\int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} e^{-xe^{2i\alpha}}\phi'(\alpha)\, d\alpha;$$
and the second term of this is
$$\frac{1}{4x^2}\left[e^{-2i\alpha - xe^{2i\alpha}}\phi'(\alpha)\right]_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} - \frac{1}{4x^2}\int_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} e^{-xe^{2i\alpha}}\frac{d}{d\alpha}\{e^{-2i\alpha}\phi'(\alpha)\}\, d\alpha,$$
if $\phi''(\alpha)$ is also continuous. Thus
$$\Phi(x, \epsilon) = \frac{1}{2x}\left[e^{-x\cos 2\alpha}\cos(x\sin 2\alpha)\phi(\alpha)\right]_{\frac{1}{2}\pi-\epsilon}^{\frac{1}{2}\pi+\epsilon} + \frac{\psi(x, \epsilon)}{x^2},$$
where
$$|\psi(x, \epsilon)| < He^x,$$
(H a constant); i.e.
$$= \{\phi(\tfrac{1}{2}\pi+\epsilon) - \phi(\tfrac{1}{2}\pi-\epsilon)\}\frac{e^{x\cos 2\epsilon}\cos(x\sin 2\epsilon)}{2x} + \frac{\psi(x, \epsilon)}{x^2}.$$

Now
$$\int_0^\infty e^{-x}\Phi(x, \epsilon)\, dx = \int_0^X + \int_X^\infty.$$

We can choose X so that $\int_X^\infty \frac{e^{-x}\psi(x, \epsilon)}{x^2}\, dx$, which is numerically less than
$$\int_X^\infty \frac{H}{x^2}\, dx = \frac{H}{X},$$
is as small as we please. Also $\phi(\frac{1}{2}\pi+\epsilon) - \phi(\frac{1}{2}\pi-\epsilon)$ vanishes with ϵ like ϵ; and (if we suppose $X > 1$)
$$\left|\int_X^\infty e^{-x(1-\cos 2\epsilon)}\cos(x\sin 2\epsilon)\frac{dx}{x}\right|$$
$$< \int_0^\infty e^{-x(1-\cos 2\epsilon)}\, x^{-\frac{3}{4}}\, dx$$
$$= \frac{\Gamma(\frac{1}{4})}{(1-\cos 2\epsilon)^{\frac{1}{4}}},$$
and therefore becomes infinite for $\epsilon = 0$ at the most like $\epsilon^{-\frac{1}{2}}$; so that
$$\lim_{\epsilon=0}\{\phi(\tfrac{1}{2}\pi+\epsilon) - \phi(\tfrac{1}{2}\pi-\epsilon)\}\int_X^\infty e^{-x(1-\cos 2\epsilon)}\cos(x\sin 2\epsilon)\frac{dx}{x} = 0.$$

And we can also choose ϵ so small that $\int_0^X e^{-x}\Phi(x, \epsilon)\, dx$ is as small as we please. Consequently
$$\lim_{\epsilon=0}\int_0^\infty e^{-x}\Phi(x, \epsilon)\, dx = 0.$$

Hence*

$$\frac{1}{2} P \int_0^\pi \tan \alpha \phi(\alpha)\, d\alpha = \mathscr{S}_1^\infty (-)^{n-1} \int_0^\pi \sin 2n\alpha \phi(\alpha)\, d\alpha.$$

We can prove in the same way that we may multiply any of the series (3), (4), (5), (8), (11), (12) of § 8 by $\phi(\theta)$ and integrate term by term, provided that neither limit is a point at which the sum of the series becomes infinite, and that we insert the sign of the principal value whenever any such point is included in the range of integration.

§ 18. I shall give a few examples of the use of this theorem. Since

$$\frac{1}{2} \tan(\theta - \phi) = \mathscr{S}_0^\infty (-)^{n-1} \sin 2n(\theta - \phi),$$

$$\frac{1}{2} \cot(\theta - \phi) = \mathscr{S}_0^\infty \sin 2n(\theta - \phi),$$

$$P \int_0^{2\pi} \tan(\theta - \phi) \cos(2n+1)\theta\, d\theta = 0,$$

$$P \int_0^{2\pi} \tan(\theta - \phi) \sin(2n+1)\theta\, d\theta = 0,$$

and the two corresponding principal values containing $\cot(\theta - \phi)$ are also zero. And

$$P \int_0^{2\pi} \tan(\theta - \phi) \cos 2n\theta\, d\theta = (-)^n\, 2\pi \sin 2n\phi,$$

$$P \int_0^{2\pi} \tan(\theta - \phi) \sin 2n\theta\, d\theta = (-)^{n-1}\, 2\pi \cos 2n\phi,$$

$$P \int_0^{2\pi} \cot(\theta - \phi) \cos 2n\theta\, d\theta = 2\pi \sin 2n\phi,$$

$$P \int_0^{2\pi} \cot(\theta - \phi) \sin 2n\theta\, d\theta = 2\pi \cos 2n\phi.$$

These formulae are true so long as the subject of integration does not become infinite at either of the limits. Similarly

$$P \int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\theta - \cos\phi} = \pi \frac{\sin n\phi}{\sin\phi},$$

if $0 < \phi < \pi$; and so on.

§ 19. The question of the integration of (1), (2), (6), (7), (9), and (10) of § 8, is however of much greater importance; and it is plain that integration term by term is not always legitimate. If for instance we integrate

$$\cos\theta - \cos 2\theta + \ldots = \tfrac{1}{2}$$

from $\theta = 0$ to $\theta = \pi$, we obtain the obviously false result $0 = \tfrac{1}{2}\pi$.

Let us consider then whether the equation

$$1 + \cos(\alpha - \theta) + \cos 2(\alpha - \theta) \ldots = \tfrac{1}{2}$$

* I proved this formula by an entirely different method in the case in which the series on the right hand is convergent, in the *Proc. Lond. Math. Soc.*, XXXIV., p. 80.

may be multiplied by $\phi(\alpha)$ and integrated from β to γ, $\phi(\alpha)$ being a function whose first and second derivates are continuous.

In the first place, integration is permissible if (β, γ) does not contain any of the points $\alpha = 2n\pi + \theta$. Hence if $\theta - 2\pi < \beta < \theta$, and ϵ is a small positive quantity

$$\int_\beta^{\theta-\epsilon} \overset{\infty}{\underset{0}{\mathscr{S}}} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha = \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^{\theta-\epsilon} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha,$$

and therefore

$$\int_\beta^{\theta} \overset{\infty}{\underset{0}{\mathscr{S}}} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha$$

$$= \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^{\theta} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha - \lim_{\epsilon=0} \overset{\infty}{\underset{0}{\mathscr{S}}} \int_{\theta-\epsilon}^{\theta} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha,$$

provided that any two of these three terms be determinate. The left hand is simply

$$\frac{1}{2}\int_\beta^\theta \phi(\alpha)\,d\alpha,$$

while the second term on the right is the limit of

$$\int_0^\infty e^{-x}dx \sum_0^\infty \frac{x^n}{n!} \int_{\theta-\epsilon}^{\theta} \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha$$

$$= \int_0^\infty e^{-x}dx \int_{\theta-\epsilon}^{\theta} \phi(\alpha)\,d\alpha \sum_0^\infty \frac{x^n}{n!} \cos n(\alpha-\theta)$$

$$= R\left[\int_0^\infty e^{-x}dx \int_{\theta-\epsilon}^{\theta} e^{xe^{i(\alpha-\theta)}}\,\phi(\alpha)\,d\alpha\right]$$

$$= \int_0^\infty e^{-x}\Theta(x,\epsilon)\,dx,$$

where $$\Theta(x,\epsilon) = R\int_{\theta-\epsilon}^{\theta} e^{xe^{i(\alpha-\theta)}}\,\phi(\alpha)\,d\alpha.$$

Now this integral is

$$\int_0^\epsilon e^{xe^{-iu}}\psi(u)\,du = -\frac{1}{ix}\left[e^{iu+xe^{-iu}}\psi(u)\right]_0^\epsilon + \frac{1}{ix}\int_0^\epsilon e^{xe^{-iu}}\frac{d}{du}\{e^{iu}\psi(u)\}\,du,$$

where $\psi(u) = \phi(\theta - u)$. The second term is

$$\frac{1}{x^2}\left[e^{iu+xe^{-iu}}\frac{d}{du}\{e^{iu}\psi(u)\}\right]_0^\epsilon - \frac{1}{x^2}\int_0^\epsilon e^{xe^{-iu}}\left(\frac{d}{du}e^{iu}\right)^2\psi(u)\,du.$$

Hence

$$\Theta(x,\epsilon) = -\frac{1}{x}e^{x\cos\epsilon}\sin(\epsilon - x\sin\epsilon)\,\psi(\epsilon) + \frac{\chi(x,\epsilon)}{x^2},$$

$$|\chi(x,\epsilon)| < He^x,$$

H being a constant.

Now if ξ be any positive quantity

$$\lim_{\epsilon=0}\int_0^\infty e^{-x}\,\Theta(x,\epsilon)\,dx = \lim_{\epsilon=0}\int_\xi^\infty e^{-x}\,\Theta(x,\epsilon)\,dx\,;$$

and it follows as in the last paragraph that we may neglect the term χ/x^2 in the expression of $\Theta(x, \epsilon)$ found above. Hence our limit is the same as

$$-\lim_{\epsilon=0} \psi(\epsilon) \int_{\xi}^{\infty} e^{-x(1-\cos\epsilon)} \sin(\epsilon - x\sin\epsilon) \frac{dx}{x}.$$

Now
$$\lim_{\epsilon=0} \left[\sin\epsilon \int_{\xi}^{\infty} e^{-x(1-\cos\epsilon)} \cos(x\sin\epsilon) \frac{dx}{x}\right] = 0;$$

for we may suppose $\xi > 1$, and then

$$\left|\int_{\xi}^{\infty}\right| < \int_0^{\infty} e^{-x(1-\cos\epsilon)} x^{-\frac{3}{4}} dx$$
$$< \frac{\Gamma(\frac{1}{4})}{(1-\cos\epsilon)^{\frac{1}{4}}}.$$

And
$$\lim_{\epsilon=0} \left[\cos\epsilon \int_{\xi}^{\infty} e^{-x(1-\cos\epsilon)} \sin(x\sin\epsilon) \frac{dx}{x}\right]$$
$$= \lim_{\epsilon=0} \int_0^{\infty} e^{-x(1-\cos\epsilon)} \sin(x\sin\epsilon) \frac{dx}{x}$$
$$= \frac{1}{2}\pi.$$

Hence finally
$$\lim_{\epsilon=0} \int_0^{\infty} e^{-x} \Theta(x, \epsilon)\, dx = \frac{1}{2}\pi\psi(0)$$
$$= \frac{1}{2}\pi\phi(\theta).$$

Consequently
$$\frac{1}{2}\int_{\beta}^{\theta} \phi(\alpha)\, d\alpha = \mathop{\mathscr{S}}_{0}^{\infty} \int_{\beta}^{\theta} \cos n(\alpha-\theta)\, \phi(\alpha)\, d\alpha - \frac{1}{2}\pi\phi(\theta).$$

That is to say
$$\phi(\theta) = \frac{2}{\pi}\left\{\frac{1}{2}\int_{\beta}^{\theta} \phi(\alpha)\, d\alpha + \mathop{\mathscr{S}}_{1}^{\infty} \int_{\beta}^{\theta} \cos n(\alpha-\theta)\, \phi(\alpha)\, d\alpha\right\}.$$

Similarly, if $\theta < \gamma < 2\pi + \theta$,
$$\phi(\theta) = \frac{2}{\pi}\left\{\frac{1}{2}\int_{\theta}^{\gamma} \phi(\alpha)\, d\alpha + \mathop{\mathscr{S}}_{1}^{\infty} \int_{\theta}^{\gamma} \cos n(\alpha-\theta)\, \phi(\alpha)\, d\alpha\right\}.$$

And generally
$$\frac{1}{2}\int_{\beta}^{\gamma} \phi(\alpha)\, d\alpha + \mathop{\mathscr{S}}_{1}^{\infty} \int_{\beta}^{\gamma} \cos n(\alpha-\theta)\, \phi(\alpha)\, d\alpha = \pi \Sigma\, \epsilon_n \phi(2n\pi + \theta),$$

the summation extending to all values of n such that $2n\pi + \theta$ lies in (β, γ), and ϵ being $= 1$ in general, but $= \frac{1}{2}$ if $2n\pi + \theta = \beta$ or $= \gamma$. This is a form of Fourier's Theorem. Of course the conditions which we have imposed upon ϕ are much narrower than they need be*. I shall not, however, attempt to generalise them, beyond

* The ordinary proofs of Fourier's theorem show that as a matter of fact the series on the left is not only summable but convergent. I need hardly say that my object is not to give a proof of Fourier's theorem equal in generality to the accepted proofs, but to show how naturally one is led to it from a point of view quite different from those usually adopted.

It is the inversion of integrations in § 13 which is not legitimate in the case of the series $\cos\theta - \cos 2\theta + \dots$, when $\beta = 0$, $\gamma = \pi$.

When we integrate term by term we assume that

$$\int_0^{\infty} dx \int_0^{\pi} F(x, \alpha)\, d\alpha = \int_0^{\pi} d\alpha \int_0^{\infty} F(x, \alpha)\, dx,$$

when $F(x, \alpha) = e^{-x(1+\cos\alpha)} \cos(\alpha - x\sin\alpha)$, and this is untrue, the left hand being $= 0$ and the right hand $= \frac{1}{2}\pi$, as is easily seen on working out the integrations.

remarking that ϕ may have a finite number of ordinary discontinuities in (β, γ). In this case, if any one of the points $2n\pi+\theta$ be one of them, we must substitute

$$\tfrac{1}{2}\{\phi(2n\pi+\theta-0)+\phi(2n\pi+\theta+0)\}$$

for $\phi(2n\pi+\theta)$.

Integration over an Infinite Range.

§ 20. I shall now suppose that the range of integration is infinite. Let us assume that, however great be γ,

$$\int_\beta^\gamma \overset{\infty}{\underset{0}{\mathscr{S}}} u_n(\alpha)\,d\alpha = \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\gamma u_n(\alpha)\,d\alpha.$$

Then if $\lim_{\gamma=\infty} \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\gamma u_n(\alpha)\,d\alpha$ is determinate and equal to

$$\overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\infty u_n(\alpha)\,d\alpha,$$

this equation passes over in the limit into

$$\int_\beta^\infty \overset{\infty}{\underset{0}{\mathscr{S}}} u_n(\alpha)\,d\alpha = \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\infty u_n(\alpha)\,d\alpha.$$

The additional condition which must be satisfied is therefore that

$$\lim_{\gamma=\infty} \overset{\infty}{\underset{0}{\mathscr{S}}} \int_\gamma^\infty u_n(\alpha)\,d\alpha = 0,$$

or

$$\lim_{\gamma=\infty} \int_0^\infty e^{-x}dx \sum_0^\infty \frac{x^n}{n!} \int_\gamma^\infty u_n(\alpha)\,d\alpha = 0.$$

Now let us assume that

$$\sum_0^\infty \frac{x^n}{n!} \int_\beta^\alpha u_n(\alpha)\,d\alpha$$

converges uniformly up to and including $\alpha=\infty$. Then it may be integrated term by term over (β, ∞), and our condition becomes

$$\lim_{\gamma=\infty} \int_0^\infty e^{-x}dx \int_\gamma^\infty u(x, \alpha)\,d\alpha = 0.$$

This will certainly be satisfied if

$$\int_0^\infty e^{-x}dx \int_\gamma^\alpha u(x, \alpha)\,d\alpha$$

is uniformly convergent up to and including $\alpha=\infty$; i.e. if

$$\overset{\infty}{\underset{0}{\mathscr{S}}} \int_\gamma^\alpha u_n(\alpha)\,d\alpha$$

is uniformly summable up to and including $\alpha=\infty$.

Theorem VI. If the conditions of III. are satisfied for any finite value of γ, however great, and

$$\overset{\infty}{\underset{0}{\mathscr{S}}} \int_\beta^\alpha u_n(\alpha)\,d\alpha$$

is uniformly summable in (β, ∞), *and*

$$\sum_0^\infty \frac{x^n}{n!}\int_\beta^\alpha u_n(\alpha)\,d\alpha$$

uniformly convergent in (β, ∞), *for any finite value of* x, *the series*

$$\mathscr{S}_0^\infty u_n(\alpha)$$

may be integrated term by term over (β, ∞).

§ 21. Theorem IV. was designed to meet the case in which integration term by term is permissible, although the original series ceases to be uniformly summable at a number of isolated points. In the corresponding case in which the range of integration is infinite we need the following theorem.

Theorem VII. If the series $\mathscr{S} u_n(\alpha)$ *may be integrated term by term over* (β, γ), *for any finite value of* γ, *however great; and the integral series*

$$\mathscr{S}_0^\infty \int_\beta^\alpha u_n(\alpha)\,d\alpha$$

converges and represents a continuous function of α *for* $\alpha = \infty$; *the series* $\mathscr{S} u_n(\alpha)$ *may be integrated term by term over* (β, ∞).

I need not delay over a formal proof of this proposition.

§ 22. We have next to consider how to extend V. in a similar way. Let us suppose that $\mathscr{S} u_n(\alpha)$ is uniformly summable over any finite interval which does not include any one of a set of isolated points γ_i, near which it behaves as in V. No new point arises if the number of these points is finite, so we shall suppose it infinite; also $\gamma_i < \gamma_{i+1}$, and $\lim_{i=\infty} \gamma_i = \infty$. Then for any finite value of γ, distinct from any γ_i,

$$(1)\quad P\int_\beta^\gamma \mathscr{S}_0^\infty u_n(\alpha)\,d\alpha = \mathscr{S}_0^\infty \int_\beta^\gamma u_n(\alpha)\,d\alpha.$$

Now let us suppose that

$$\mathscr{S}_0^\infty \int_\beta^\infty u_n(\alpha)\,d\alpha$$

is summable, and that when γ tends to ∞, in a manner subject to certain restrictions (one of which must obviously be that of never taking any of the values γ_i), the right hand of (1) tends to a limit equal to the sum of this series. Then

$$\lim P\int_\beta^\gamma \mathscr{S}_0^\infty u_n(\alpha)\,d\alpha = \mathscr{S}_0^\infty \int_\beta^\infty u_n(\alpha)\,d\alpha.$$

I shall write this in the form

$$P\int_\beta^\infty \mathscr{S}_0^\infty u_n(\alpha)\,d\alpha = \mathscr{S}_0^\infty \int_\beta^\infty u_n(\alpha)\,d\alpha.$$

For a detailed discussion of the definition of principal values such as that on the left I must refer to the papers on "The Theory of Cauchy's Principal Values" already mentioned.

Theorem VIII. If

$$P\int_\beta^\gamma \overset{\infty}{\underset{0}{\mathcal{S}}}\, u_n(\alpha)\,d\alpha = \overset{\infty}{\underset{0}{\mathcal{S}}}\int_\beta^\gamma u_n(\alpha)\,d\alpha,$$

for every finite value of γ *distinct from any of a certain set of values* γ_i; *and if, when* γ *tends to* ∞ *in a manner subject to certain restrictions,* $\overset{\infty}{\underset{0}{\mathcal{S}}}\int_\beta^\gamma u_n(\alpha)\,d\alpha$ *tends to a finite limit equal to*

$$\overset{\infty}{\underset{0}{\mathcal{S}}}\int_\beta^\infty u_n(\alpha)\,d\alpha\,;$$

then

$$P\int_\beta^\infty \overset{\infty}{\underset{0}{\mathcal{S}}}\, u_n(\alpha)\,d\alpha = \overset{\infty}{\underset{0}{\mathcal{S}}}\int_\beta^\infty u_n(\alpha)\,d\alpha.$$

§ 23. We saw in § 17 that if $\phi(\alpha)$ satisfies certain conditions

$$\frac{1}{2}P\int_0^\pi \tan\alpha\,\phi(\alpha)\,d\alpha = \overset{\infty}{\underset{1}{\mathcal{S}}}(-)^{n-1}\int_0^\pi \sin 2n\alpha\,\phi(\alpha)\,d\alpha.$$

And if these conditions are satisfied throughout any finite interval of values of α

$$\frac{1}{2}P\int_0^{N\pi} \tan\alpha\,\phi(\alpha)\,d\alpha = \overset{\infty}{\underset{1}{\mathcal{S}}}(-)^{n-1}\int_0^{N\pi} \sin 2n\alpha\,\phi(\alpha)\,d\alpha,$$

for any finite value of N.

Can we replace the upper limit by ∞?

Let us suppose, in the first place, that the series

$$\overset{\infty}{\underset{1}{\mathcal{S}}}(-)^{n-1}\int_0^\infty \sin 2n\alpha\,\phi(\alpha)\,d\alpha$$

is summable; and let us consider the series

$$\begin{aligned}&\overset{\infty}{\underset{1}{\mathcal{S}}}(-)^{n-1}\int_{N\pi}^\infty \sin 2n\alpha\,\phi(\alpha)\,d\alpha\\ &\quad=\int_0^\infty e^{-x}\,dx \sum_1^\infty (-)^{n-1}\frac{x^n}{n\,!}\int_{N\pi}^\infty \sin 2n\alpha\,\phi(\alpha)\,d\alpha\\ &\quad=\int_0^\infty e^{-x}\,\Phi(x, N)\,dx,\end{aligned}$$

where

$$\Phi(x, N) = R\left[\frac{1}{i}\int_{N\pi}^\infty e^{2i\alpha - xe^{2i\alpha}}\,\phi(\alpha)\,d\alpha\right].$$

The last integral is

$$-\frac{1}{2ix}\left[e^{-xe^{2i\alpha}}\,\phi(\alpha)\right]_{N\pi}^\infty + \frac{1}{2ix}\int_{N\pi}^\infty e^{-xe^{2i\alpha}}\,\phi'(\alpha)\,d\alpha\,;$$

and the second term of this is

$$\frac{1}{4x^2}\left[e^{-2i\alpha - xe^{2i\alpha}}\,\phi'(\alpha)\right]_{N\pi}^\infty - \frac{1}{4x^2}\int_{N\pi}^\infty e^{-xe^{2i\alpha}}\,\frac{d}{d\alpha}\{e^{-2i\alpha}\,\phi'(\alpha)\}\,d\alpha,$$

provided this double integration by parts be justified. This will certainly be the case if, from some finite value of α, $\phi(\alpha)$, $\phi'(\alpha)$ tend steadily to zero, as then

$$\int^{\infty} \phi'(\alpha)\, d\alpha, \quad \int^{\infty} \phi''(\alpha)\, d\alpha$$

are absolutely convergent.

Hence
$$\Phi(x, N) = -\frac{e^{-x}}{2x}\phi(N\pi) + \frac{\psi(x, N)}{x^2},$$

where
$$|\psi(x, N)| < He^{x},$$

H being a constant.

Now
$$\int_0^{\infty} e^{-x}\Phi(x, N)\, dx = \int_0^{X} + \int_X^{\infty}.$$

We can choose X so that $\int_X^{\infty} e^{-x}\frac{\psi}{x^2}\, dx$, which is numerically less than

$$\int_X^{\infty} \frac{H}{x^2}\, dx = \frac{H}{X},$$

is as small as we please; and the same is evidently true of

$$-\phi(N\pi)\int_X^{\infty} \frac{e^{-x}}{2x}\, dx.$$

And we can also choose N so great that $\left|\int_0^X\right|$ is as small as we please. Hence

$$\lim_{N=\infty} \int_0^{\infty} e^{-x}\Phi(x, N)\, dx = 0.$$

Consequently

$$\lim \frac{1}{2} P\int_0^{N\pi} \tan\alpha\, \phi(\alpha)\, d\alpha = \mathop{\mathscr{S}}_1^{\infty} (-)^{n-1}\int_0^{\infty} \sin 2n\alpha\, \phi(\alpha)\, d\alpha.$$

The left hand of this equation is

$$\frac{1}{2} P\int_0^{\infty} \tan\alpha\, \phi(\alpha)\, d\alpha,$$

according to the definition in the paper quoted above. It is the same as

$$\lim_{\alpha'=\infty} \frac{1}{2} P\int_0^{\alpha'} \tan\alpha\, \phi(\alpha)\, d\alpha,$$

where α' tends to ∞ through any series of values differing from any odd multiple of $\frac{1}{2}\pi$ by more than an arbitrarily small, but fixed, quantity δ.

In the same way we arrive at the general conclusion that we may multiply any of the series (3), (4), (5), (8), (11), (12) of § 8 by $\phi(\theta)$ and integrate term by term over a suitably chosen interval whose upper limit is ∞, provided that $\phi(\theta)$ satisfy the conditions imposed upon $\phi(\alpha)$ in the preceding argument, and that we insert the sign of the principal value before the integral of the sum of the series.

We may also multiply by $\cos a\theta\, \phi(\theta)$ and integrate. To prove this we only need to modify the preceding argument very slightly.

§ 24. Suppose for instance that we multiply the two equations

$$\tfrac{1}{2}\tan\alpha = \sin 2\alpha - \sin 4\alpha + \ldots\ldots,$$

$$\tfrac{1}{2}\cot\alpha = \sin 2\alpha + \sin 4\alpha + \ldots\ldots,$$

by
$$\cos a\alpha \frac{\alpha}{\alpha^2+\gamma^2}, \qquad (0<\gamma,\ 0<a<2),$$

and integrate term by term. We obtain

$$\frac{1}{2} P\int_0^\infty \cos a\alpha \tan\alpha \frac{\alpha d\alpha}{\alpha^2+\gamma^2}$$

$$= \mathcal{S}_1^\infty (-)^{n-1} \int_0^\infty \frac{\alpha\cos a\alpha \sin 2n\alpha}{\alpha^2+\gamma^2} d\alpha$$

$$= \frac{1}{4}\pi \sum_1^\infty (-)^{n-1} \{e^{-(2n+a)\gamma} + e^{-(2n-a)\gamma}\},$$

i.e.
$$P\int_0^\infty \cos a\alpha \tan\alpha \frac{\alpha d\alpha}{\alpha^2+\gamma^2} = \frac{\pi\cosh a\gamma}{e^{2\gamma}+1};$$

and similarly*
$$P\int_0^\infty \cos a\alpha \cot\alpha \frac{\alpha d\alpha}{\alpha^2+\gamma^2} = \frac{\pi\cosh a\gamma}{e^{2\gamma}-1}.$$

In exactly the same way we may prove Cauchy's formulae

$$P\int_0^\infty \frac{\cos a\alpha}{\cos b\alpha}\frac{d\alpha}{1+\alpha^2} = \frac{1}{2}\pi\frac{\cosh a}{\cosh b},$$

$$P\int_0^\infty \frac{\sin a\alpha}{\sin b\alpha}\frac{d\alpha}{1+\alpha^2} = \frac{1}{2}\pi\frac{\sinh a}{\sinh b},$$

$(0<a<b)$, and many others of the same kind. We can also find the corresponding formulae when $a>b$.

§ 25. In these examples the series of integrals is convergent, but all that is essential to our theorems is that it should be summable.

Suppose, for instance, that we multiply by

$$\cos a\alpha \frac{\alpha}{\alpha^2-\gamma^2}.$$

Then, so long as γ is not an odd multiple of $\tfrac{1}{2}\pi$,

$$\frac{1}{2} P\int_0^\infty \cos a\alpha \tan\alpha \frac{\alpha d\alpha}{\alpha^2-\gamma^2} = \mathcal{S}_1^\infty (-)^{n-1} P\int_0^\infty \frac{\alpha\cos a\alpha \sin 2n\alpha}{\alpha^2-\gamma^2} d\alpha.$$

To prove this we have only to observe that, by § 23,

$$\frac{1}{2} P\int_{N\pi}^\infty \cos a\alpha \tan\alpha \frac{\alpha d\alpha}{\alpha^2-\gamma^2} = \mathcal{S}_1^\infty (-)^{n-1}\int_{N\pi}^\infty,$$

if $N\pi$ is any multiple of $\pi>\gamma$; and, by § 16,

$$\frac{1}{2} P\int_0^{N\pi} \cos a\alpha \tan\alpha \frac{\alpha d\alpha}{\alpha^2-\gamma^2} = \mathcal{S}_1^\infty (-)^{n-1}\int_0^{N\pi}.$$

* Two other proofs of these formulae will be found in the *Proc. Lond. Math. Soc.* xxxiv. pp. 61—65 and 83—84, and a fourth in the *Quarterly Journal*, 1900, p. 120.

Now $P\int_0^\infty \frac{\alpha\cos a\alpha \sin 2n\alpha}{\alpha^2-\gamma^2}\,d\alpha = \frac{\pi}{4}\{\cos(2n+a)\gamma+\cos(2n-a)\gamma\} = \frac{1}{2}\pi\cos a\gamma\cos 2n\gamma$;

and so $$P\int_0^\infty \cos a\alpha \tan\alpha\,\frac{\alpha d\alpha}{\alpha^2-\gamma^2} = \pi\cos a\gamma \mathop{\mathscr{S}}_1^\infty (-)^{n-1}\cos 2n\gamma = \frac{1}{2}\pi\cos a\gamma.$$

Similarly $$P\int_0^\infty \cos a\alpha \cot\alpha\,\frac{\alpha d\alpha}{\alpha^2-\gamma^2} = -\frac{1}{2}\pi\cos a\gamma,$$

so long as γ is not a multiple of π.

In the same way we may prove that

$$P\int_0^\infty \frac{\cos a\alpha}{\cos b\alpha}\,\frac{d\alpha}{1-\alpha^2} = 0,$$

$$P\int_0^\infty \frac{\sin a\alpha}{\sin b\alpha}\,\frac{d\alpha}{1-\alpha^2} = 0,$$

if $0 < a < b$ (so long as b has not any one of certain exceptional values).

§ 26. I pass now to the corresponding investigation connected with § 19. We found there that, if $\phi(\alpha)$ satisfies certain conditions

$$(1)\quad \int_\beta^\gamma \mathop{\mathscr{S}}_0^\infty \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha = \mathop{\mathscr{S}}_0^\infty \int_\beta^\gamma \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha - \pi\Sigma\epsilon_n\phi(2n\pi+\theta),$$

the summation extending to all values of $2n\pi+\theta$ which fall in (β, γ), and ϵ being $=\frac{1}{2}$ if $2n\pi+\theta=\beta$ or γ, $=1$ otherwise.

Now let us suppose that $\phi(\alpha)$ tends steadily to zero as α increases from some finite value to ∞, and that

$$\int_\beta^\infty \phi(\alpha)\,d\alpha$$

is convergent. If then we make γ tend to ∞, the left hand of (1) tends to

$$\frac{1}{2}\int_\beta^\infty \phi(\alpha)\,d\alpha$$

and the second term on the right to

$$\pi\sum^\infty \epsilon_n\phi(2n\pi+\theta).$$

Let us assume that

$$\mathop{\mathscr{S}}_0^\infty \int_\gamma^\infty \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha$$

is summable, and seek its limit for $\gamma=\infty$. The sum of the series is as before proved to be

$$\int_0^\infty e^{-x}\,\Theta(x,\gamma)\,dx,$$

where $$\Theta(x,\gamma) = R\int_\gamma^\infty e^{xe^{i(\alpha-\theta)}}\,\phi(\alpha)\,d\alpha.$$

We may suppose that $\gamma = (2N+1)\pi + \theta$, N being a positive integer which tends to ∞. Integrating by parts we obtain, instead of the last integral

$$-\frac{1}{ix}\left[e^{iu+xe^{-iu}}\psi(u)\right]_{-\infty}^{-(2N+1)\pi} + \frac{1}{x^2}\left[e^{iu+xe^{-iu}}\frac{d}{du}\{e^{iu}\psi(u)\}\right]_{-\infty}^{-(2N+1)\pi}$$
$$-\frac{1}{x^2}\int_{-\infty}^{-(2N+1)\pi} e^{xe^{-iu}}\left(\frac{d}{du}e^{iu}\right)^2\psi(u)\,du,$$

where $\psi(u) = \phi(\theta - u)$. We suppose as in § 23 that $\phi'(\alpha)$ as well as $\phi(\alpha)$ tends steadily to 0 for $\alpha = \infty$.

The first term vanishes, and to the other two we may apply the same argument as before, which shows that

$$\lim_{\gamma=\infty}\int_0^\infty e^{-x}\,\Theta(x,\gamma)\,dx = 0.$$

Thus $$\frac{1}{2}\int_\beta^\infty \phi(\alpha)\,d\alpha + \mathscr{S}_1^\infty\int_\beta^\infty \cos n(\alpha-\theta)\,\phi(\alpha)\,d\alpha = \pi\sum^\infty \epsilon_n\phi(2n\pi+\theta).$$

The applications of this formula which we obtain by making

$$\phi(\alpha) = \frac{1}{1+\alpha^2},\ e^{-\alpha},\ e^{-\alpha^2},\ \ldots\ldots,$$

are so well known that I need hardly give any.

COMMENTS

The general principle formulated in § 1 is known as 'Hardy's principle'; see Bromwich; 1st edn., pp. 267–8. It is further elaborated in 1904, 4, § 2. See also D.S., pp. 89–91.

In D.S., pp. 80 and 83, summability by *Borel's integral* is called *summability* (B'); summability by his *exponential mean* is called *summability* (B). The (B) mean was introduced by Borel† before the (B') integral as a particular case of a general mean, in which the role of e^x is played by an *integral function* $\sum c_n x^n$, $c_n \geqslant 0$. In D.S., pp. 79–80, summability by this mean is called the J *method*.

Hardy remarks in 1904, 4, § 3 that Borel had stated without proof the *consistency* (*regularity*) of the (B') method. Borel originally made the statement for the J method,‡ and repeated it for the (B) method.§ Borel paper (2), pp. 107–9 introduced the (B') integral of the series $u_0+u_1+\ldots$ as a formula for the (B) limit of the

† Borel (1), *Comptes rendus* 121 (1895), 1125–7.
‡ Borel (2), *J. de math. pure et appl.* (5), 2 (1896), 103–22.
§ Borel (3), *Ann. de l'École norm. sup.* (3), 16 (1899), 9–131.

sequence $(0, u_0, u_0+u_1, \ldots)$, i.e. the (B) sum of the series $0+u_0+u_1+\ldots$. Hence his statement also applies to the (B') method; compare D.S., Theorem 126. Borel (paper (2), p. 106) proved that if s_n ultimately lies between two numbers p and q, then the (B) limit, *if it exists*, also lies between p and q. He added: 'Il en résulte que, si les s_n ont une limite, sa valeur coincide avec celle de la limite généralisée, qu'on s'assure aisément exister toujours dans ce cas.' But he did not give a *proof* that the generalized limit exists whenever the ordinary one does.

The formula in § 4, connecting the Borel integrals of the series (1) $0+u_0+u_1+\ldots$ and (2) $u_0+u_1+\ldots$, shows that if either of (1) or (2) is summable (B'), then so is the other *if and only if* $e^{-X}\int_0^X u(x)\,dx$ *tends to a limit*, which can only be zero.† In 1904, 4, § 4, Hardy proves that the (B') summability of (2) *itself implies that*

$$e^{-X}\int_0^X u(x)\,dx \to 0,$$

and hence also that (1) is summable (B'). See D.S., Theorems 123–5.

The result partly stated in the footnote to § 3 is proved in 1904, 4, § 6. The extra condition is

$$(\sigma_0+\ldots+\sigma_n)/(n+1) = s+o(1/\sqrt{n}).$$

The investigation of the continuity and term-by-term differentiation of Borel summable series continues similar investigations for convergent and principal-value integrals, in Vol. V; see the Introduction to Vol. V, topics A and D. Similar results for integrals are given in 1908, 3.

† The uniform convergence of the power series $u(x)$ in $(0, X)$ follows from its convergence in $(0, \infty)$.

RESEARCHES IN THE THEORY OF DIVERGENT SERIES AND DIVERGENT INTEGRALS.

By G. H. HARDY, Trinity College, Cambridge.

Introduction.

§ 1. SINCE the appearance of M. Borel's original memoirs on Divergent Series, few branches of analysis have aroused more interest among mathematicians. The applications of divergent series to the theory of functions of a complex variable, and especially to the problems of analytic continuation and the asymptotic expansion, have given rise to extensive researches; I need only refer to the writings of M. Borel himself, of MM. Servant and Le Roy, and of Mr. Barnes. The 'arithmetic' theory of divergent series, as it was originally presented by M. Borel, has received less attention.

In the first few sections of this paper I discuss a few of the questions which arise in the early part of the theory; this part of the paper may be regarded as supplementary to Chap. III. of M. Borel's *Leçons sur les Séries Divergentes*, and to my paper 'On differentiation and integration of Divergent Series.'* I discuss in turn (i) the relation of convergence and absolute summability, (ii) the removal of terms from, or addition of terms to, a divergent series, (iii) the 'condition of consistency' for definitions other than M. Borel's original definition, (iv) the relation of 'generalised limits' and 'mean values,' and (v) the multiplication of divergent series. In the latter sections I suggest the outlines of a similar theory of divergent integrals.

PART I.

SOME POINTS IN THE THEORY OF DIVERGENT SERIES.

General Preliminaries.

§ 2. In the first paragraph of the paper referred to, I indicated very briefly a general principle which may often guide us in our choice of a convention as to the value to be attributed to an otherwise meaningless expression. It is that

* *Cambridge Philosophical Transactions*, XIX., p. 297.

when a number of limit operations, performed in a definite order on a function of several variables, lead to a definite result, but do not do so when performed in another definite order, we are to agree that the expression which is the result of the formal carrying out of the second sequence of operations MEANS *the result of the first sequence.*

I pointed out that Borel's definition of the sum of a divergent series was an application of this principle; and the same is true of the definitions given by M. Le Roy* and Mr. Barnes.† It is at first sight a little difficult to imagine how we can gain any advantage by writing X when we mean Y. Results, however, prove conclusively that we can. And if we consider an example, it is easy to see how.

Few analytical processes occur more frequently than that of the evaluation of a definite integral by expansion of the subject of integration, and integration term by term. Our work in this case depends on an equation of the form

$$(1)\quad \int_\beta^\gamma \sum_0^\infty u_n(\alpha)\, d\alpha = \sum_0^\infty \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

Now, if $\Sigma u_n(\alpha)$ and $\Sigma \int_\beta^\gamma u_n(\alpha)\, d\alpha$ are convergent,

$$\sum_0^\infty u_n(\alpha) = \int_0^\infty e^{-x} dx \sum_0^\infty \frac{x^n}{n!} u_n(\alpha)$$

and
$$\sum_0^\infty \int_\beta^\gamma u_n(\alpha)\, d\alpha = \int_0^\infty e^{-x} dx \sum_0^\infty \frac{x^n}{n!} \int_\beta^\gamma u_n(\alpha)\, d\alpha.$$

Thus the simple transformation (1) which may be represented symbolically by

$$\int_{(1)} \Sigma = \Sigma \int_{(1)}$$

may be replaced by a more elaborate permutation of limit operations, viz.

$$(2)\quad \int_{(1)} \int_{(2)} \Sigma = \int_{(2)} \int_{(1)} \Sigma = \int_{(2)} \Sigma \int_{(1)},$$

the first and last expressions being equivalent to the two members of (1).

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* *Annales de la Faculté des Sciences de Toulouse*, 1900, pp. 317–431.
† *Phil. Trans.* (A.), 199, pp. 411–500.

Now (2) may be true when neither or only one side of (1) is determinate. In this case, if we adopt Borel's definition of the sum of a divergent series, and the notation of my former paper,* (2) may be written in the form

$$(3)\ \int_{(1)} S = \left[\int_{(2)} \int_{(1)} \Sigma = \right] = S \int_{(1)} .$$

In practice we omit the intermediate step, so that the transformation is formally the same as that expressed by (1); and so our work gains immensely in quickness and compactness, to say nothing of the probability that but for the formal resemblance of (1) and (3) we should never have thought of the transformation (2). Thus in this, and in a greater degree in more complicated cases, divergent series provide us with a suggestive and easily manipulated shorthand representation of elaborate and often very difficult analytical processes.

They do not, however, exempt us from the labour of examining the legitimacy of these processes. Fortunately this is not very difficult in most of the cases which ordinarily occur. In the theory of divergent power series in particular, we are not troubled seriously by cases of exception. In the 'arithmetic' theory, the part of the theory which is concerned with the application of divergent series to the theory of functions of real variables, the fundamental difficulties of the subject are naturally more prominent.†

§ 3. I shall now consider one or two of the questions which arise in the beginning of the theory.

Convergence and Absolute Summability.

In my former paper I proved that what I called the 'Condition of consistency' was satisfied by M. Borel's definition; that is to say that

$$\overset{\infty}{\underset{0}{S}}\, u_n = \sum_{0}^{\infty} u_n$$

whenever the latter series is convergent. That this is so was stated by M. Borel in his first memoir, *Fondements de la théorie des séries divergentes summables*,‡ and again in his *Memoire sur les séries divergentes*§; but I am unable to find

* I use S instead of the symbol used there.
† See *e.g.*, §§ 19–26 of my paper referred to above.
‡ *J. de Math.*, 1896, pp. 103–122.
§ *Annales de l'École Normal Supérieure*, 1899, pp. 1–131.

any proof in any of his writings. In his *Leçons sur les séries divergentes*, M. Borel makes the further assertion that all convergent series are 'evidently' *absolutely* summable; that is, that if

$$\Sigma\, u_n$$

is convergent, and

$$u(x) = \Sigma \frac{x^n}{n!} u_n,$$

all the integrals

$$\int_0^\infty e^{-x} \left| \frac{d^\lambda}{dx^\lambda} u(x) \right| dx \quad (\lambda \geqq 0)$$

are convergent. This, it seems to me, is untrue; unless the analysis which follows is inaccurate, there are convergent series which are *not* absolutely summable.*

Consider the series

$$0 - 1 + 0 + 0 + \tfrac{1}{2} + 0 + 0 + 0 + 0$$
$$- \tfrac{1}{3} + 0 + 0 + 0 + 0 + 0 + 0 + \tfrac{1}{4} + \ldots,$$

in which

$$u_n = \frac{(-)^{\sqrt{n}}}{\sqrt{n}},$$

if n is a square, and $= 0$ otherwise. The series is evidently convergent.

I shall prove that if

$$u(x) = \sum_1^\infty \frac{(-)^i}{i} \frac{x^{i^2}}{i^2!},$$

$$| e^{-x} u(x) | > \frac{K}{x}$$

throughout a certain infinite series of intervals. I may remark that this example was suggested to me by an investigation of M. Borel's† concerning the series

$$1 + x + x^4 + x^9 + \ldots,$$

and that I adopt a very convenient notation used by him elsewhere.‡ I denote by K a constant entering into an

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* I give this investigation more because it shows how we can determine the manner of growth of a very interesting type of function, than because the question of the truth or falsity of M. Borel's statement is particularly important.

† *J. de Math.*, 1896, p. 447.

‡ *Comptes Rendus*, 11 Mai 1896.

equality or inequality, not necessarily the same in all inequalities, but always lying between, say, 0·00001 and 10000. Also when I say that one quantity is approximately another, I mean that their ratio differs from unity by less than σ, a very small positive quantity fixed throughout the investigation.

We can choose I so that

$$i^2! = i^{2i^2+1} e^{-i^2} \sqrt{(2\pi)}\,(1+\rho),$$

$$|\rho| < \sigma,$$

for all values of $i > I$. When this has been done we may evidently neglect the terms of $u(x)$ up to $i = I$, as for sufficiently large values of x,

$$e^{-x} \left| \sum_1^I \frac{(-)^i}{i} \frac{x^{i^2}}{i^2!} \right|$$

is less than any power of $\frac{1}{x}$.

Let
$$v_i = \frac{1}{i} \frac{x^{i^2}}{i^2!},$$

$$[\sqrt{x}] = X,$$

$$\sqrt{x} = X + f,$$

and suppose $0 < f < \frac{1}{4}$. Then it can be shown that $|u(x)|$ is of the same order of greatness as its X^{th} term.

In the first place, neglecting factors which are approximately unity,

$$\frac{v_i}{v_{i+1}} = (ex)^{-2i-1} \frac{(i+1)^{2i^2+4i+3}}{i^{2i^2+1}}$$

$$= (ex)^{-2i-1} (i+1)^{4i+2} \left(1 + \frac{1}{i}\right)^{2i^2+1}$$

$$= e^{-2} \left(\frac{i+1}{X+f}\right)^{4i+2},$$

as
$$\left(1 + \frac{1}{i}\right)^{2i^2} = e^{2i^2 \log\left(1+\frac{1}{i}\right)}$$

$$= e^{2i-1+\cdots}.$$

Hence if $i \leqq X-1$,

$$\frac{v_i}{v_{i+1}} < \tfrac{1}{4}.$$

Hence

$$\sum_{I+1}^{X} (-)^i v_i = (-)^X \{(v_X - v_{X-1}) + (v_{X-2} - v_{X-3}) + \ldots \pm v_{I+1}\}$$

has the sign of $(-)^X$, and is numerically greater than

$$\tfrac{3}{4} v_X.$$

If $i = X$,

$$\frac{v_X}{v_{X+1}} = e^{-2}\left(1 + \frac{1-f}{X+f}\right)^{4X+2}$$
$$= e^{2-4f}$$

approximately, *i.e.* $> e > 2$.

If $i > X$ it is evident that the ratio of v_i to v_{i+1} is greater still. Hence

$$\sum_{X+1}^{\infty} (-)^i v_i = (-)^{X+1} \{v_{X+1} - (v_{X+2} - v_{X+3}) - \ldots\}$$

is numerically less than $\tfrac{1}{2} v_X$.

Hence $$\left| \sum_{I+1}^{\infty} (-)^i v_i \right| > \tfrac{1}{4} v_X.$$

Now v_X is approximately

$$\frac{1}{X} \frac{(ex)^{X^2}}{X^{2X^2+1}\sqrt{(2\pi)}}$$
$$= \frac{K}{X^2} e^{X^2} \left(1 + \frac{f}{X}\right)^{2X^2}$$
$$= \frac{K}{X^2} e^{X^2 + 2fX},$$

and so

$$e^{-x} v_X = \frac{K}{X^2} = \frac{K}{x}.$$

It follows that we can find a number M, such that

$$| e^{-x} u(x) | > \frac{K}{x}$$

throughout the intervals

$$m^2,\ (m+\tfrac{1}{4})^2 \quad (m > M).$$

But
$$\Sigma \int_{m^2}^{(m+\frac{1}{4})^2} \frac{dx}{x} = 2\Sigma \int_{m}^{m+\frac{1}{4}} \frac{du}{u}$$

is evidently divergent. Hence

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is not convergent.

It is evident that we can apply the preceding analysis to a whole class of functions

$$u(x) = \sum_1^\infty \frac{(-)^i}{\phi(i)} \frac{x^{i^2}}{i^2\,!}$$

for which

$$\lim_{i=\infty} \frac{\phi(i+1)}{\phi(i)} = 1.$$

The behaviour of $u(x)$ for large values of x is dominated by that of its greatest term, and its modulus is generally comparable with

$$\frac{K}{\sqrt{x}\,\phi(\sqrt{x})}.$$

Taking, *e.g.*, $\phi(i) = i^\mu$ $(\mu > 0)$, the series

$$0 - 1 + 0 + 0 + \frac{1}{2^\mu} + 0 + 0 + 0 + 0 - \frac{1}{3^\mu} + \ldots$$

is not absolutely summable unless $\frac{1}{2}(\mu + 1) > 1$ or $\mu > 1$, in which case the series is absolutely convergent.

It is easy to see that an absolutely convergent series must be absolutely summable; for

$$\mid u(x) \mid < \Sigma \frac{x^n}{n\,!} \mid u_n \mid,$$

and $\mid u_n \mid$ is convergent, so that

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is so. And as

$$u_\lambda + u_{\lambda+1} + \ldots$$

is absolutely convergent, the same is true of

$$\int_0^\infty e^{-x} \left| \frac{d^\lambda}{dx^\lambda} u(x) \right| dx.$$

For a similar reason, if Σu_n is convergent, the integrals

$$\int_0^\infty e^{-x} \frac{d^\lambda}{dx^\lambda} u(x)\, dx$$

are convergent, even when

$$\int_0^\infty e^{-x} | u(x) | \, dx$$

is not.

Removal of terms from, and addition of terms to, a divergent series.

§ 4. M. Borel's alternative definition of the sum s of a divergent series $u_0 + u_1 + u_2 + \ldots$ is

$$s = \lim_{x=\infty} e^{-x} \sum_0^\infty \frac{s_n x^n}{n!}$$

where $$s_n = u_0 + u_1 + \ldots + u_n.$$

He proves that if the limit on the right is determinate

$$s - u_0 = \int_0^\infty e^{-x} \frac{d}{dx} u(x)\, dx$$

$$= u_1 + u_2 + \ldots,$$

according to the integral definition.

It does not follow that

$$s = u_0 + u_1 + u_2 + \ldots = \int_0^\infty e^{-x} u(x)\, dx.$$

We are thus led to consider the relation of the series

$$(1)\quad s = u_0 + u_1 + u_2 + \ldots;$$

$$(2)\quad s' = u_1 + u_2 + u_3 + \ldots,$$

and as M. Borel's discussion* of this question does not seem to me altogether satisfactory, I shall treat it in some detail.

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* *Leçons*, pp. 100–103

M. Borel's analysis is affected by the erroneous assumption that if

$$\int_0^\infty e^{-x} \mid u(x) \mid dx$$

is convergent, $e^{-x} u(x)$ has necessarily the limit 0 for $x = \infty$. And it seems to me that the notion of absolute summability, on which M. Borel lays considerable stress, does not, here at any rate, give us any real assistance.

I assume that $u(x)$ is an integral function. Then the sums of the series (1) and (2) are defined as

$$\int_0^\infty e^{-x} u(x)\, dx, \quad \int_0^\infty e^{-x} \frac{d}{dx} u(x)\, dx$$

respectively. Since

$$\int_0^X e^{-x} u(x)\, dx = -\left[e^{-x} u(x)\right]_0^X + \int_0^X e^{-x} u'(x)\, dx,$$

it follows that if

$$\lim_{x=\infty} e^{-x} u(x) = 0,$$

the summability of either (1) or (2) involves that of the other, and the relation

$$(3) \quad s = u_0 + s'.$$

Again, if both are summable, $e^{-x} u(x)$ has a limit for $x = \infty$, which can only be zero; so that (3) must be true.

But it can be shown that if (2) is summable, (1) must be so. The converse is not true; if, for instance

$$u_n = 2^n \sum_{\nu=0}^{\infty} \frac{(-)^\nu (\nu+1)^n}{2\nu+1\,!} = R\left[\frac{1}{i} \sum_{p=0}^{\infty} \frac{i^p (p+1)^n}{p\,!}\right],$$

$$\begin{aligned} u(x) &= R\left[\frac{1}{i} \sum_{n=0}^{\infty} \frac{x^n}{n\,!} \sum_{p=0}^{\infty} \frac{i^p (p+1)^n}{p\,!}\right] \\ &= R\left[\frac{1}{i} \sum_{p=0}^{\infty} \frac{i^p}{p\,!} e^{(p+1)x}\right] \\ &= e^x \sin e^x, \end{aligned}$$

$$\begin{aligned} \overset{\infty}{\underset{0}{S}} u_n &= \int_0^\infty \sin e^x\, dx \\ &= \int_1^\infty \frac{\sin y}{y}\, dy, \end{aligned}$$

while $$\int_0^\infty e^{-x} \frac{d}{dx} \{e^x \sin e^x\} \, dx$$

is divergent.

I shall now prove that the summability of (1) follows from that of (2). In the first place I may remark that it is easy to give a valid form to M. Borel's proof of his proposition, that the absolute summability of (1) follows from that of (2). For, as M. Borel shows,* it is enough to prove this on the assumption that $u(x)$, $u'(x)$ are positive.

This being so, we can choose X so that

$$\int_X^{X'} e^{-x} u'(x) \, dx < \sigma,$$

however small be σ, for all values of $X' > X$. *A fortiori*

$$e^{-X'} \int_X^{X'} u'(x) \, dx < \sigma,$$

i.e. $$e^{-X'} \{u(X') - u(X)\} < \sigma.$$

But, X being fixed, we can choose $X_1 > X$ so that

$$e^{-X'} u(X) < \sigma$$

for all values of $X' > X_1$. Hence

$$e^{-X'} u(X') < 2\sigma \quad (X' > X_1).$$

Therefore

$$\lim_{x=\infty} e^{-x} u(x) = 0,$$

and the desired conclusion follows.

It is, however, almost equally easy to prove the more general theorem. For suppose that (2) is summable, and (1) not. Let

$$e^{-x} u(x) = \phi(x).$$

Then $$e^{-x} u'(x) = \phi(x) + \phi'(x).$$

Thus $$\int_0^\infty \{\phi(x) + \phi'(x)\} \, dx$$

is convergent, while $\int_0^\infty \phi(x) \, dx$ is not. Hence $\int_0^\infty \phi(x) \, dx$

* *l.c.*, p. 101.

is not convergent, so that $\phi(x)$ does not tend to any finite limit for $x = \infty$.

Now $\phi(x)$ either has or has not infinitely many zeroes in $(0, \infty)$. In the latter case it is ultimately of constant sign, say positive.

Then we can determine a positive quantity H, and an infinite series of intervals

$$x_\nu, \ x_\nu + \delta_\nu, \quad (x_{\nu+1} > x_\nu + \delta_\nu),$$

so that

$$(4) \quad \int_{x_\nu}^{x_\nu + \delta_\nu} \phi'(x)\, dx > H,$$

$(\nu = 1, 2, \ldots)$. This is obvious if

$$\overline{\lim_{x=\infty}}\, \phi(x) = \infty.$$

If this is not so, let the upper and lower limits of indetermination of $\phi(x)$ for $x = \infty$ be U, L. However small be the positive quantity σ, we can find two infinite series of quantities

$$\alpha_\nu, \beta_\nu \ (\alpha_{\nu+1} > \alpha_\nu, \ \beta_{\nu+1} > \beta_\nu, \ \lim_{\nu=\infty} \alpha_\nu = \infty, \ \lim_{\nu=\infty} \beta_\nu = \infty),$$

such that

$$\phi(\alpha_\nu) < L + \sigma,$$
$$\phi(\beta_\nu) > U - \sigma$$

$(\nu = 1, 2, \ldots)$. Hence we can find an infinite series of intervals $(x_\nu, x_\nu + \delta_\nu)$ as required, taking $H = U - L - 2\sigma$, and choosing the lower limits from the α's, and the upper limits from the β's.

A fortiori, as $\phi(x) > 0$,

$$\int_{x_\nu}^{x_\nu + \delta_\nu} \{\phi(x) + \phi'(x)\}\, dx > H.$$

This contradicts the hypothesis that

$$\int_0^\infty \{\phi(x) + \phi'(x)\}\, dx$$

is convergent.

There remains the possibility that $\phi(x)$ has infinitely many

zeroes. One of its limits of indetermination must differ from 0; suppose that

$$\overline{\lim_{x=\infty}}\ \phi(x) = U > 0$$

(U may be $+\infty$). We can determine an infinite series of ascending quantities β_ν, whose limit is ∞, so that, if U is finite,

$$\phi(\beta_\nu) \geqq U - \sigma,$$

where σ is any small fixed positive quantity, or, if U is infinite,

$$\phi(\beta_\nu) \geqq G,$$

where G is any fixed positive quantity, however large. With each β_ν we associate the largest zero $< \beta_\nu$. Some zeroes will be associated with several of the points β; in this case we disregard all save the one which is nearest to the zero. We have thus an infinity of associated pairs of zeroes and points β; and it is evident that we can satisfy (4) by taking H to be $U - \sigma$ or G, x_ν to be one of the selected zeroes, and $x_\nu + \delta_\nu$ to be the associated point β. Moreover, $\phi(x)$ is positive throughout the intervals $(x_\nu, x_\nu + \delta_\nu)$, so that, as before,

$$\int_{x_\nu}^{x_\nu+\delta_\nu} \{\phi(x) + \phi'(x)\}\, dx > H.$$

It is therefore impossible that $\int_0^\infty \phi(x)\, dx$ should be divergent, and $\int_0^\infty \{\phi(x) + \phi'(x)\}\, dx$ convergent, Hence if (2) is summable, (1) is so too.

We conclude then that *if any one of the series*

$$\begin{array}{l}
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
0 + 0 + 0 + u_0 + u_1 + \ldots, \\
0 + 0 + u_0 + u_1 + u_2 + \ldots, \\
0 + u_0 + u_1 + u_2 + u_3 + \ldots, \\
u_0 + u_1 + u_2 + u_3 + u_4 + \ldots, \\
u_1 + u_2 + u_3 + u_4 + u_5 + \ldots, \\
u_2 + u_3 + u_4 + u_5 + u_6 + \ldots, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

is summable, all those above it are summable, and their sums are related as if they were convergent.

I may remark, before leaving this part of the subject, that non-absolutely summable series are by no means 'artificial monstrosities.' Not only are the convergent series considered in § 4 of a natural and simple type, but so simple a divergent series as

$$\overset{\infty}{\underset{0}{S}} \frac{(1+i)^{n+1}}{n+1}$$

is non-absolutely summable.

Condition of consistency for other definitions.

§ 5. I shall now prove that the other definitions proposed by MM. Borel and Le Roy all satisfy the 'condition of consistency.'

In the first place let us consider M. Borel's alternative definition mentioned at the beginning of the last paragraph. If $u(x)$ is an integral function

$$e^{-x} \sum_0^\infty \frac{s_n x^n}{n!} - u_0 = \int_0^x \frac{d}{dx} \left\{ e^{-x} \sum_0^\infty \frac{s_n x^n}{n!} \right\} dx$$

$$= \int_0^x e^{-x} \sum_0^\infty \frac{u_{n+1} x^n}{n!} dx.$$

Hence, $\lim\limits_{x=\infty} . e^{-x} \Sigma \frac{s_n x^n}{n!}$ is finite and determinate if, and only if,

$$u_1 + u_2 + \dots$$

is summable. If this is so

$$u_0 + u_1 + u_2 + \dots$$

is summable. The latter series may be summable without $\lim . e^{-x} \Sigma \frac{s_n x^n}{n!}$ existing. But if it is convergent, $u_1 + u_2 + \dots$ is convergent too, and

$$\lim_{x=\infty} . e^{-x} \Sigma \frac{s_n x^n}{n!} = u_0 + u_1 + \dots,$$

as may easily be proved directly.

M. Borel's generalised method of exponential summation

presents no special difficulty. He defines the sum of the series $u_0 + u_1 + u_2 + \ldots$ as

$$\int_0^\infty e^{-x} u_k(x)\, dx,$$

$$u_k(x) = u_0 + u_1 + \ldots + u_{k-1} + (u_k + \ldots + u_{2k-1}) \frac{x}{1\,!} + \ldots.$$

According to this method, we associate the terms of the series in groups, and apply the ordinary exponential method to the series of groups; and as the terms of a convergent series may be grouped in any manner, the proof that the ordinary method satisfies the condition of consistency applies to this method also.*

M. le Roy suggests two generalisations of the exponential definition. According to one, we define $u_0 + u_1 + \ldots$ as

$$\frac{1}{p} \int_0^\infty e^{-x^{\frac{1}{p}}} x^{\frac{1}{p}-1} u_p(x)\, dx,$$

where $p > 0$, and

$$u_p(x) = \sum_0^\infty \frac{u_n x^n}{\Gamma(np+1)}.$$

This integral is equal to

$$\int_0^\infty e^{-y} \sum_0^\infty \frac{u_n y^{np}}{\Gamma(np+1)}\, dy,$$

if the latter is convergent. If p is an integer (the only case of much interest) the condition of consistency is certainly satisfied. For the integral is the sum of

$$u_0 + 0 + 0 + \ldots + u_1 + 0 + \ldots$$
$$+ u_2 + \ldots,$$

in which $p - 1$ zero terms separate two consecutive u's, calculated by the ordinary method, and is therefore convergent and equal to the sum of this series, if

$$u_0 + u_1 + u_2 + \ldots$$

is convergent.

---

* The generalised exponential method is *not* always consistent with the ordinary method. If *e.g.* $k = 2$, it gives 0 as the sum of $1 - 1 + 1 - \ldots$. In fact we obtain $\frac{1}{2}$ or 0 according as k is odd or even.

According to M. le Roy's other definition

$$u_0 + u_1 + \ldots = \lim_{t=1} \sum_0^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$$

$(0 < t < 1)$, if this limit is determinate. It may be expressed in the form

$$\lim_{t=1} \int_0^\infty e^{-x} \sum_0^\infty \frac{u_n x^{nt}}{n!} dx,$$

for, as I proved in § 3 of my former paper, integration term by term is in this case legitimate whenever the resulting series is convergent. Hence our series is defined as

$$\lim_{t=1} \int_0^\infty e^{-x} u(x^t)\, dx,$$

or

$$\lim_{t=1} \frac{1}{t} \int_0^\infty e^{-x^{\frac{1}{t}}} x^{\frac{1}{t}-1} u(x)\, dx,$$

or

$$(1) \quad \lim_{\alpha=0} \int_0^\infty e^{-x^{1+\alpha}} x^\alpha u(x)\, dx.$$

Now it can be shown that if

$$(2) \quad \int_0^\infty e^{-x} u(x)\, dx$$

is convergent, the integral in (1) is convergent for values of α in an interval $(0, \alpha_1)$, and that it tends to (2) for $\alpha = 0$.

For we can determine X so that

$$(3) \quad \left| \int_X^{X'} e^{-x} u(x)\, dx \right| < \sigma$$

$(X' > X)$. We can also suppose X so great* that $e^{x-x^{1+\alpha}} x^\alpha$ decreases steadily as x increases from X to ∞, for all values of α in a small interval $(0, \alpha')$. Then, by the second theorem of the mean

$$\int_X^{X_1} e^{-x^{1+\alpha}} x^\alpha u(x)\, dx$$

$$= e^{X - X^{1+\alpha}} X^\alpha \int_X^\xi e^{-x} u(x)\, dx$$

$$+ e^{X_1 - X_1^{1+\alpha}} X_1^\alpha \int_\xi^{X_1} e^{-x} u(x)\, dx$$

$(X \leqq \xi \leqq X_1)$, which is numerically less than 3σ.

* For this we need, in fact, only suppose $X \geqq 1$. Thus $e^{x-x^{1+\alpha}} x^\alpha \leqq 1$, $(x \geqq 1)$.

Hence

$$(4) \quad \int_0^\infty e^{-x^{1+\alpha}} x^\alpha u(x)\, dx$$

is convergent. And it is equal to

$$\int_0^{X_2} + \int_{X_2}^\infty, \quad (X_2 > X).$$

The second part is equal to

$$e^{X_2 - X_2^{1+\alpha}} X_2^{\alpha} \int_{X_2}^{X_3} e^{-x} u(x)\, dx \quad (X_3 \geqq X_2),$$

which is numerically $< 2\sigma$. And we can choose α_1 so small that the first part differs from

$$\int_0^{X_2} e^{-x} u(x)\, dx$$

by less than σ, for all values of α in $(0, \alpha_1)$. Then, for such values of α, (4) differs from (2) by less than 5σ, so that

$$\lim_{\alpha=0} \int_0^\infty e^{-x^{1+\alpha}} x^\alpha dx = \int_0^\infty e^{-x} u(x)\, dx.$$

It follows that if $u_0 + u_1 + u_2 + \ldots$ is summable by the exponential method, it is summable also by M. le Roy's method, and that the two sums are the same. Hence M. le Roy's definition satisfies the condition of consistency.

Divergent Series and Mean Values.

§ 6. The 'mean value' of an infinite series of quantities $s_0, s_1, s_2, \ldots$ is

$$(1) \quad \lim_{n=\infty} \frac{s_0 + s_1 + \ldots + s_n}{n+1},$$

if this limit exists. It is well known that if $\lim. s_n$ exists and $= s$, the mean value also exists and $= s$. As the converse is not true, the expression (1) provides us with a generalisation of a notion of a limit or of the sum of a series from which interesting conclusions have been deduced.*

* Frobenius, *Crelle*, 89, p. 262; Hölder, *Math. Ann.*, 20, p. 535; Césaro, *Bull. des Sc. Math.*, 1890, p. 114.

The question which I shall discuss in this paragraph is whether the existence of a mean value for

$$s_0, s_1, s_2, \ldots$$

involves the summability of the series

$$u_0 + u_1 + u_2 + \ldots.$$

If the mean value exists we may put

$$s_0 + s_1 + \ldots + s_n = (n+1)(s + \rho_n),$$

$$\lim_{n=\infty} \rho_n = 0.$$

Then

$$\sum_0^\infty \frac{s_n x^n}{n!} = \sum_0^\infty (s_0 + s_1 + \ldots + s_n)\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$

$$= \sum_0^\infty (n+1)(s+\rho_n)\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$

$$= se^x + \sum_0^\infty (n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right).$$

Since $\lim. \rho_n = 0$, we can choose a series of descending positive quantities ϵ_n such that

$$|\rho_{n'}| < \epsilon_n \quad (n' \geqq n),$$

and $\lim. \epsilon_n = 0$. Also if $\xi = [x]$

$$\sum_0^\infty (n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)$$

$$= \sum_0^{X-1} + \sum_X^{\xi-1} + \sum_\xi^\infty \quad (1 < X < \xi).$$

All the coefficients of the quantities ρ_n in the first two sums are negative, except the last, which may be zero; and all in the third are positive. Hence

$$\left|\sum_0^{X-1}(n+1)\rho_n\left(\frac{x^n}{n!} - \frac{x^{n+1}}{n+1!}\right)\right|$$

$$< K\sum_0^{X-1}(n+1)\left(\frac{x^{n+1}}{n+1!} - \frac{x^n}{n!}\right)$$

$$< K\left\{\frac{Xx^X}{X!} - \left(1 + x + \ldots + \frac{x^{X-1}}{X-1!}\right)\right\}$$

$$< KX\frac{x^X}{X!},$$

where K is a constant.

Again

$$\left|\sum_{X}^{\xi-1}(n+1)\rho_n\left(\frac{x^n}{n!}-\frac{x^{n+1}}{n+1!}\right)\right|$$

$$<\epsilon_X\sum_{X}^{\xi-1}(n+1)\left(\frac{x^{n+1}}{n+1!}-\frac{x^n}{n!}\right)$$

$$<\epsilon_X\left\{\frac{\xi x^\xi}{\xi!}-\frac{x^{X+1}}{X+1!}-\dots-\frac{x^{\xi-1}}{(\xi-1)!}-(X+1)\frac{x^X}{X!}\right\}$$

$$<\epsilon_X\frac{\xi x^\xi}{\xi!}.$$

Finally

$$\left|\sum_{\xi}^{\infty}(n+1)\rho_n\left(\frac{x^n}{n!}-\frac{x^{n+1}}{n+1!}\right)\right|$$

$$<\epsilon_\xi\sum_{\xi}^{\infty}(n+1)\left(\frac{x^n}{n!}-\frac{x^{n+1}}{n+1!}\right)$$

$$<\epsilon_\xi\left\{(\xi+1)\frac{x^\xi}{\xi!}+\sum_{\xi}^{\infty}\frac{x^{n+1}}{n+1!}\right\}$$

$$<\epsilon_\xi\left\{e^x+(\xi+1)\frac{x^\xi}{\xi!}\right\}.$$

Hence

$$(1)\quad \left|\sum_{0}^{\infty}(n+1)\rho_n\left(\frac{x^n}{n!}-\frac{x^{n+1}}{n+1!}\right)\right|$$

$$<KX\frac{x^X}{X!}+\epsilon_X\frac{\xi x^\xi}{\xi!}$$

$$+\epsilon_\xi\left\{e^x+(\xi+1)\frac{x^\xi}{\xi!}\right\}.$$

Now let $x=\xi+f$. Then when x is very large

$$\frac{x^\xi}{\xi!}=\frac{\xi^\xi\left(1+\frac{f}{\xi}\right)^\xi}{\xi^{\xi+\frac{1}{2}}e^{-\xi}\sqrt{2\pi}}(1+\alpha)=\frac{e^x}{\sqrt{(2\pi\xi)}}(1+\beta),$$

where α, β are very small. Hence the last term of (1) is equal to

$$\epsilon_\xi e^x\sqrt{\frac{\xi}{2\pi}}(1+\gamma),$$

where γ is very small; and its product by e^{-x} will tend to zero for $x=\infty$, if

$$(2) \quad \lim_{\xi=\infty} \epsilon_\xi \sqrt{\xi} = 0;$$

and similarly the product of the second term by e^{-x} will vanish for $x=\infty$ if

$$(3) \quad \lim_{\xi=\infty} e_X \sqrt{\xi} = 0.$$

Finally, the same will be true of the first term if

$$(4) \quad \lim_{x=\infty} e^{-x} X \frac{x^X}{X!} = 0.$$

Suppose (2) satisfied, and $X=\frac{1}{2}\xi$, or $\frac{1}{2}(\xi+1)$, according as ξ is even or odd. Then, supposing ξ even, (3) will be satisfied, and as

$$\lim_{x=\infty} e^{-(\xi+f)} \left(\tfrac{1}{2}\xi\right) \frac{\xi^{\frac{1}{2}\xi}\left(1+\dfrac{f}{\xi}\right)^{\frac{1}{2}\xi}}{\left(\frac{1}{2}\xi\right)^{\frac{1}{2}\xi+\frac{1}{2}} e^{-\frac{1}{2}\xi}\sqrt{2\pi}}$$

$$= \lim_{x=\infty} \tfrac{1}{2} e^{-\frac{1}{2}f} \sqrt{\left(\frac{\xi}{\pi}\right)} \left(\frac{2}{e}\right)^{\frac{1}{2}\xi} = 0,$$

(4) is also satisfied.

Hence *if*

$$\frac{s_0+s_1+\ldots+s_n}{n+1} = s+\rho_n,$$

and

$$\lim_{n=\infty} \rho_n \sqrt{n} = 0,$$

then

$$\lim_{x=\infty} e^{-x} \sum_0^\infty \frac{s_n x^n}{n!} = s.$$

Thus the series $u_1+u_2+\ldots$ and *a fortiori* the series

$$u_0+u_1+u_2+\ldots$$

is summable.

I may remark that the condition $\lim. \rho_n \sqrt{n} = 0$ is by no means *necessary*. In fact it is easy to construct *convergent* series for which it is not satisfied.

Suppose, *e.g.* $s_0 = 1$, and

$$s_n = 1 + \frac{1}{n^\epsilon} \quad (0<\epsilon<\tfrac{1}{2}).$$

Then $$\rho_n = \frac{\sum_1^n i^{-\epsilon}}{n+1},$$

and for $n=\infty$, ρ_n is of order

$$n^{-\epsilon},$$

so that $$\lim. \rho_n \sqrt{n} = \infty.$$

On the other hand the result would not be true if the condition were omitted. For we can choose the quantities s_n so that ρ_n is infinitely small of order $n^{-\frac{1}{2}}$ for $n=\infty$, and yet

$$\lim. e^{-x} \sum_0^\infty \frac{s_n x^n}{n!}$$

does not exist.

Suppose, *e.g.*

$$s_0 = 0,\ s_1 = -1,\ s_2 = s_3 = 0,\ s_4 = 2,$$
$$s_5 = \ldots = s_8 = 0,\ s_9 = -3,\ \ldots,$$
$$s_{n^2} = (-)^n n,\ \ldots.$$

Then we can prove as in § 3 that

$$e^{-x} \sum_0^\infty \frac{s_n x^n}{n!}$$

takes, as x increases to ∞, an infinite series of values, positive and negative, all numerically greater than a constant K. In fact, in the notation used near the end of § 3,

$$\frac{K}{\sqrt{x}\phi(\sqrt{x})} = K.$$

Now if $x = p^2 + q$ (p being the largest square contained in n)

$$s_0 + s_1 + \ldots + s_n = -1 + 2 - 3 + \ldots + (-)^p p$$
$$= \tfrac{1}{2}p \text{ or } -\tfrac{1}{2}(p+1),$$

according as p is even or odd.

Hence $$\left|\frac{s_0 + s_1 + \ldots + s_n}{n+1}\right| = \frac{\frac{1}{2}p}{p^2+q+1},$$
$$\text{or } \frac{\frac{1}{2}(p+1)}{p^2+q+1},$$

i.e. is infinitely small for $n=\infty$ like

$$\frac{1}{p} \text{ or } \frac{1}{\sqrt{n}}.$$

The condition therefore corresponds to a real limitation, and has not been introduced by the method employed.

I shall not attempt now to prove any similar theorems concerning the more general mean values introduced by Hölder and Césaro. I will only remark that a similar line of argument may be applied to the more general question whether, if

$$\lim. \frac{s_0+s_1+\ldots+s_n}{n+1}$$

exists,

$$\lim_{x=\infty}. \frac{a_0s_0+a_1s_1x+a_2s_2x^2+\ldots}{a_0+a_1x+a_2x^2+\ldots}$$

will exist and be equal to it,

$$a_0+a_1x+\ldots=\phi(x)$$

being an integral function.

For

$$\sum_0^\infty a_ns_nx^n = \sum_0^\infty (s_0+s_1+\ldots+s_n)(a_nx^n-a_{n+1}x^{n+1})$$

$$=\sum_0^\infty (n+1)(s+\rho_n)(a_nx^n-a_{n+1}x^{n+1})$$

$$=s\phi(x)+\sum_0^\infty (n+1)\rho_n(a_nx^n-a_{n+1}x^{n+1}).$$

Thus

$$\frac{\sum_0^\infty a_ns_nx^n}{\sum_0^\infty a_nx^n}=s+\frac{1}{\phi(x)}\sum_0^\infty (n+1)\rho_n(a_nx^n-a_{n+1}x^{n+1}).$$

We have then to prove that the limit of the last term for $x=\infty$ is zero, by a method analogous to that employed for the case in which $a_n=\frac{1}{n!}$, $\phi(x)=e^x$.

The multiplication of divergent series.

§ 7. M. Borel has proved that if two series are absolutely summable, they may be multiplied according to the ordinary rule, and the validity of his proof is not affected by the considerations of §§ 3—4.

I showed in those paragraphs that in the treatment of certain questions fundamental in the theory we gained nothing by the introduction of the idea of absolute summability. This is, however, no longer true when we come to the question of the multiplication of series; I am, for instance, unable to prove that if two series Su_n, Sv_n are such that

$$u_\lambda + u_{\lambda+1} + \ldots,$$
$$v_\lambda + v_{\lambda+1} + \ldots$$

summable, whatever be λ, we may multiply them as if they were convergent; and indeed, if we consider the analogy of convergent series, it seems unlikely that this should be true.

There are, however, two interesting resemblances between the theories of the multiplication of convergent and divergent series. I may remark that in this connnection I consider *only* series which satisfy the condition mentioned above, for if a divergent series does not resemble a convergent one in this first and most elementary point, it is not of much interest to inquire whether it resembles one in other points more difficult to investigate.

This being understood, we can prove (1) that it is enough to suppose *one* series absolutely summable, and (2) that if the product series is summable, its sum is the product of the sums of the original series.

To prove (1) we have only to make a slight alteration in M. Borel's proof for the case in which both series are absolutely summable. The whole question turns* on whether we may assume that

$$(1) \int_0^\infty e^{-x} u(x)\,dx \times \int_0^\infty e^{-y} v(y)\,dy$$
$$= \int_0^\infty e^{-t} \left[\int_{-t}^{t} u\left(\frac{t+w}{2}\right) u\left(\frac{t-w}{2}\right) \frac{dw}{2}\right] dt.$$

Now the left-hand side is the limit for $H=\infty$ of

$$(2) \int_0^H \int_0^H e^{-x-y} u(x)\, v(y)\, dx\, dy,$$

* *l.c.*, p. 104.

and the right-hand side is the limit of

$$(2') \quad \iint e^{-x-y} u(x)\, v(y)\, dx\, dy$$

over the triangle bounded by

$$x=0, \ y=0, \text{ and } x+y=H.$$

Suppose that Su_n is absolutely summable. Then the difference of (2) and (2′) is

$$\int_0^H e^{-x} u(x)\, dx \int_{H-x}^H e^{-y} v(y)\, dy.$$

Now $$\left| \int_0^H e^{-x} u(x)\, dx \int_{H-x}^H e^{-y} v(y)\, dy \right|$$

$$\leqq \int_0^H \left\{ |\, e^{-x} u(x)\, | \left| \int_{H-x}^H e^{-y} v(y)\, dy \right| \right\} dx$$

$$\leqq \int_0^{\frac{1}{2}H} + \int_{\frac{1}{2}H}^H$$

$$\leqq \left| \int_{H-K}^H e^{-y} v(y)\, dy \right| \int_0^{\frac{1}{2}H} |\, e^{-x} u(x)\, |\, dx$$

$$+ \left| \int_{H-K'}^H e^{-y} v(y)\, dy \right| \int_{\frac{1}{2}H}^H |\, e^{-x} u(x)\, |\, dx,$$

where $$0 \leqq K \leqq \tfrac{1}{2}H, \quad \tfrac{1}{2}H \leqq K' \leqq H,$$

so that $\frac{1}{2}H \leqq H-K \leqq H$; and the limit of this for $H=\infty$ is 0.

The proof may then be completed as by M. Borel. It need only be observed that we do not prove that the product series is *absolutely* summable.

The second proposition (2) is most easily proved indirectly.* I shall first prove the following lemma.

If Su_n is summable, $Su_n t^n$ is absolutely summable for all positive values of t less than unity, and

$$\lim_{t=1} Su_n t^n = Su_n.$$

For† $$\lim_{x=0} e^{-x} \left(\frac{d}{dx} \right)^{\lambda} u(x) = 0 \quad (\lambda \geqq 0),$$

* The same is of course true of the corresponding theorem for convergent series

† On account of the limitation made at the beginning of this paragraph.

so that
$$e^{-\frac{x}{t}}\left(\frac{d}{dx}\right)^{\lambda}u(x) \quad (0 < t < 1)$$

vanishes exponentially for $x = \infty$. Hence

$$\int_0^\infty e^{-x}\left(\frac{d}{dx}\right)^{\lambda} u(tx)\,dx = t^{\lambda-1}\int_0^\infty e^{-\frac{x}{t}}\left(\frac{d}{dx}\right)^{\lambda}u(x)\,dx$$

is absolutely convergent.

Also
$$\lim_{t=1} \frac{1}{t}\int_0^\infty e^{-\frac{x}{t}}u(x)\,dx = \int_0^\infty e^{-x}u(x)\,dx$$

if the latter integral is convergent. This has been proved by Dirichlet.* The lemma follows.

I may remark that it is easy to prove in the same way a proposition analogous to Abel's well known theorem regarding series on their circle of convergence; viz. that *if a series* $Su_n x^n$ *is summable at any point* P *on the boundary of its polygon of summability, and* $f(P)$ *is its sum, the function* $f(x)$ *represented by the series within the polygon tends to* $f(P)$ *when* x *approaches* P *along the line* OP.

Now suppose that Su_n, Sv_n, and Sw_n or

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \ldots$$

are all summable. Then Su_nt^n, Sv_nt^n, and Sw_nt^n are all absolutely summable if $0 < t < 1$, and

$$Su_nt_u \times Sv_nt^n = Sw_ut^n.$$

Hence in the limit

$$Su_n \times Sv_n = Sw_n.$$

§ 8. Before leaving this part of the subject I shall indicate another method by which the question of the multiplication of divergent series can be attacked. It is capable of more general application than is M. Borel's method, which depends upon a particular property of the exponential function, although, in the particular case of series whose sums are defined by the exponential method, it does not lead to quite such general conditions.

* v. Stolz, *Grundzüge*, I., p. 447; and my paper 'On the continuity and discontinuity of definite integrals,' etc. (*Q.J.*, 1902, p. 28).

Suppose that the sum of the series

$$u_0 + u_1 + u_2 + \ldots$$

is defined as

$$\int_0^\infty \phi(x) \sum_0^\infty a_n u_n x^n \, dx,$$

where

$$\int_0^\infty \phi(x) x^n \, dx = \frac{1}{a_n}.$$

Then the multiplication of the series Su_n, Sv_n may be justified by the following series of transformations:

$$Su_n \times Sv_n = Su_n \times \int_0^\infty \phi(y) \sum_0^\infty a_n v_n y^n \, dy$$

$$= \int_0^\infty \phi(y) \, dy \, [a_0 v_0 (u_0 + u_1 + u_2 + \ldots)$$

$$+ a_1 v_1 y \; (0 + u_0 + u_1 + \ldots)$$

$$+ a_2 v_2 y^2 (0 + 0 + u_0 + \ldots)$$

$$+ \ldots]$$

$$= \int_0^\infty \phi(y) \, dy \sum_0^\infty a_n v_n y^n \int_0^\infty \phi(x) \, dx \sum_0^\infty a_{n+m} u \; x^{n+m}$$

$$= \int_0^\infty \phi(y) \, dy \int_0^\infty \phi(x) \, dx \sum_0^\infty a_n v_n y^n \sum_0^\infty a_{n+m} u_m x^{n+m}$$

$$= \int_0^\infty \phi(x) \, dx \int_0^\infty \phi(y) \, dy \sum_0^\infty a_n v_n y^n \sum_0^\infty a_{n+m} u_m x^{n+m}$$

$$= \int_0^\infty \phi(x) \, dx \int_0^\infty \phi(y) \, dy \sum_0^\infty a_p x^p \sum_0^p a_i u_{p-i} v_i y^i$$

$$= \int_0^\infty \phi(x) \, dx \sum_0^\infty a_p x^p \int_0^\infty \phi(y) \, dy \sum_0^p a_i u_{p-i} v_i y^i$$

$$= \int_0^\infty \phi(x) \, dx \sum_0^\infty a_p w_p x^p$$

$$= Sw_p.$$

Suppose, *e.g.*, that $a_n = \dfrac{1}{n!}$, $\phi(x) = e^{-x}$, and that we can find quantities H, ϵ, ($\epsilon < 1$), such that

$$|u(x)| < He^{\epsilon x},$$
$$|v(y)| < He^{\epsilon y}.$$

Then it is easy to see that

$$\left| \sum_0^\infty \frac{u_m x^{m+n}}{m+n!} \right| = \left| \left(\int_0^x dx \right)^n u(x) \right| < H\epsilon^{-n} e^{\epsilon x},$$

and hence that all the transformations are legitimate.

Part II.

DIVERGENT INTEGRALS.

Definitions of the value of a divergent integral.

§ 9. The various methods which I have considered in the preceding part of this paper enable us to attach a meaning to

$$\lim_{n=\infty} s_n, \quad u_0 + u_1 + \ldots + u_n + \ldots$$

in a number of cases in which they are otherwise meaningless. I shall now consider the possibility of framing similar conventions for the interpretation of the expressions

$$\lim_{x=\infty} \Phi(x), \quad \int^\infty \phi(x)\, dx.$$

Let us recall the process* which leads to M. Borel's definition of the sum of a divergent series. We take a doubly infinite sequence

$$a_n^{(p)},$$

such that $a_n^{(p)}$ increases with p for any fixed value of n, and decreases as n increases for any fixed value of p, and define the generalised limit of s_n as

$$\lim_{p=\infty} \frac{\sum_{n=0}^\infty a_n^{(p)} s_n}{\sum_{n=0}^\infty a_n^{(p)}}.$$

---

* Borel, *l.c.*, p. 95.

In particular we take

$$a_n^{(p)} = \frac{p^n}{n!}.$$

The analogous method of defining lim. $\Phi(x)$ is evidently to define it as

$$(1)\ \lim_{t=\infty} \frac{\int_0^\infty \theta(x, t)\,\Phi(x)\,dx}{\int_0^\infty \theta(x, t)\,dx},$$

$\theta(x, t)$ being a function of x and t which increases with t for for any fixed value of x (or at any rate for all values greater than a certain value), but decreases, when x increases and t is fixed, sufficiently rapidly to ensure the convergence of the integral in the numerator.

Further progress depends on our choice of a function $\theta(x, t)$.

§ 10. A natural assumption is

$$\theta(x, t) = p\left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p},$$

where p is any positive quantity. Then

$$\int_0^\infty \theta(x, t)\,dx = pt\int_0^\infty e^{-u^p}u^{p-1}du$$

$$= t,$$

so that (1) becomes

$$\lim_{t=\infty} \frac{p}{t}\int_0^\infty \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p}\,\Phi(x)\,dx$$

$$= \lim_{t=\infty} \int_0^\infty e^{-u}\,\Phi(t\sqrt[p]{u})\,du.$$

From this we easily deduce a definition of a divergent integral

$$\int_0^\infty \phi(x)\,dx.$$

For if $$\Phi(x) = \int_0^x \phi(x)\,dx,$$

we have defined $\lim\limits_{n=\infty} \Phi(x)$ as

$$\lim_{t=\infty} \frac{p}{t} \int_0^\infty \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p} \Phi(x)\, dx$$

$$= \lim_{t=\infty} \int_0^\infty e^{-\left(\frac{x}{t}\right)^p} \phi(x)\, dx,$$

provided $\lim\limits_{x=\infty} \left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p} \Phi(x) = 0.$

The simplest case is that in which $p = 1$; and this assumption enables us to deal with many of the most interesting integrals which present themselves naturally in analysis. I shall therefore leave the general question on one side for the present, while I consider some consequences of the hypothesis

$$\theta(x, t) = e^{-\frac{x}{t}}.$$

I shall suppose that $\phi(x)$ is continuous for all finite values of x, and that

$$\lim_{x=\infty} e^{-\tau x} \phi(x) = 0$$

for any positive value of τ. And I define the *generalised limit* $L\Phi(x)$ by the equation

$$(1) \qquad \underset{x=\infty}{L}\, \Phi(x) = \lim_{t=\infty} \int_0^\infty e^{-x} \Phi(tx)\, dx,$$

and the *divergent integral*

$$G \int_0^\infty \phi(x)\, dx$$

by the equation

$$(2) \qquad G \int_0^\infty \phi(x)\, dx = \lim_{t=\infty} \int_0^\infty e^{-\frac{x}{t}} \phi(x)\, dx,$$

it being always understood that

$$\Phi(x) = \int^x \phi(x)\, dx.$$

Our hypotheses ensure that these definitions are consistent;

and we may (as is easily seen) replace the right-hand side of (1) or (2) by the repeated integral

$$\int_0^\infty dt \int_0^\infty x e^{-tx} \phi(x)\, dx.$$

And it is evident that these definitions (and the more general definitions for which $p \neq 1$) are governed by the general principle of § 2.

Condition of consistency.

§ 11. It is easy to prove that if $\lim_{x=\infty} \Phi(x)$ is determinate

$$\underset{x=\infty}{L}\, \Phi(x) = \lim_{x=\infty} \Phi(x).$$

For $$L\Phi(x) = \lim_{t=\infty} \int_0^\infty e^{-x} \Phi(tx)\, dx$$

$$= \lim_{\tau=0} \left(\tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx \right)$$

$$= \lim_{\tau=0} \tau \left(\int_0^X + \int_X^\infty \right),$$

if the last limit is determinate.

Now we can choose X so great that

$$| \Phi(x) - \Phi(\infty) | < \sigma$$

for all values of $x \geqq X$, however small be σ. Then

$$\tau \int_X^\infty e^{-\tau x} \Phi(x)\, dx = e^{-\tau X} \Phi(\infty) + \rho,$$

where $$|\rho| < \sigma e^{-\tau X} < \sigma.$$

We can then choose τ_0 so that

$$\left| \tau \int_0^X \right| < \sigma$$

for all values of $\tau \leqq \tau_0$.

Thus $$L\Phi(x) = \lim_{\tau=0} e^{-\tau X} \phi(\infty)$$

$$= \phi(\infty).$$

We may also prove directly that the condition of consistency is satisfied by our definition of

$$G\int_0^\infty \phi(x)\,dx$$

by using Dirichlet's theorem, mentioned in § 7, that

$$\lim_{\tau=0} \int_0^\infty e^{-\tau x}\phi(x)\,dx = \int_0^\infty \phi(x)\,dx$$

whenever the last integral is convergent.

It is perhaps worth while at this point to state explicitly that $L\Phi(x)$ is *not* generally equal to

$$\lim_{x=\infty} \Phi(x_n),$$

($x_0 < x_1 < x_2 < \ldots$, $\lim. x_n = \infty$) when the latter limit is determinate, in the ordinary sense, or in M. Borel's sense. Nor is it generally true that

$$G\int_0^\infty \phi(x)\,dx = \overset{\infty}{\underset{0}{S}} \int_{x_n}^{x_{n+1}} \phi(x)\,dx$$

(supposing $x_0 = 0$).

Elementary properties of the generalised limit and the divergent integral.

§ 12. I. *If* $L\Phi_1(x)$, $L\Phi_2(x)$, ..., $L\Phi_n(x)$ *are determinate*

$$L\{a_1\Phi_1(x) + a_2\Phi_2(x) + \ldots + a_n\Phi_n(x)\}$$
$$= a_1 L\Phi_1(x) + a_2 L\Phi_2(x) + \ldots + a_n L\Phi_n(x).$$

This follows at once from the definition. It is obvious that a similar theorem holds for the divergent integral. On the other hand it is not generally true that

$$L\Phi(x)\Psi(x) = L\Phi(x)\,L\Psi(x).$$

when the two latter generalised limits exist.

II. *If a and b are constants, and a positive*

$$L\Phi(ax+b) = L\Phi(x)$$

if the latter is determinate.

In the first place it follows immediately from the definition that

$$(1) \quad L\Phi\,(ax) = L\Phi\,(x).$$

Again
$$L\Phi\,(x+b) = \lim_{t=\infty} \int_0^\infty e^{-x}\,\Phi\,(tx+b)\,dx$$

$$= \lim_{t=\infty} e^{\frac{b}{t}} \int_{\frac{b}{t}}^\infty e^{-u}\,\Phi\,(tu)\,du.$$

But
$$\left| e^{\frac{b}{t}} \int_0^{\frac{b}{t}} e^{-u}\,\Phi\,(u)\,du \right| < K \int_0^{\frac{b}{t}} e^{-u}\,du,$$

and the limit of this for $t=\infty$ is plainly zero. Hence

$$(2) \quad L\Phi\,(x+b) = L\Phi\,(x).$$

From (1) and (2) the theorem follows.

It is easy to see that the corresponding theorem for divergent integrals is

$$G\int_0^\infty \phi\,(x)\,dx = a\,G\int_{-\frac{b}{a}}^\infty \phi\,(ax+b)\,dx,$$

$(a>0)$; *i.e. a divergent integral may be transformed by a linear substitution* $x = ay + b$ $(a>0)$ *according to the ordinary rule.*

Again from the equation

$$\int_0^x \{\phi\,(x)\,\psi'\,(x) + \phi'\,(x)\,\psi\,(x)\}\,dx = \phi\,(x)\,\psi\,(x) - \phi\,(0)\,\psi\,(0),$$

and Theorem I., it follows at once that

III. *A divergent integral may be transformed by integration by parts, if the process leads to a determinate result.*

For

$$G\int_0^\infty \{\phi\,(x)\,\psi'\,(x)+\phi'\,(x)\,\psi\,(x)\}\,dx = \underset{x=\infty}{L}\,\phi\,(x)\,\psi\,(x) - \phi\,(0)\,\psi\,(0),$$

if either of the two sides is determinate.

IV. *If*

$$G\int_0^\infty \phi(x)\,dx$$

is determinate,

$$L\phi(x) = 0.$$

For $$L\phi(x) = \lim_{\tau=0} \tau\int_0^\infty e^{-\tau x}\phi(x)\,dx = 0.$$

Generalised limits and mean values.

§ 13. The result of § 11 is capable of an extension naturally suggested by § 6.

We may define the *mean value of* $\Phi(x)$ *for* $x=\infty$ as

$$\lim_{x=\infty} \frac{1}{x}\int_0^x \Phi(x)\,dx,$$

if this limit exists. In § 6 I proved that under certain conditions the fact that s_n possesses a mean value ensures that it possesses a generalised limit in M. Borel's sense. And it is easy to state conditions under which the fact that $\Phi(x)$ possesses a mean value ensures that it possesses a generalised limit in the sense of § 10.

In the first place, if $\Phi(x)$ possesses a mean value k, we can find constants X, K such that

$$\left|\int_0^x \Phi(x)\,dx\right| < Kx$$

for all values of $x \geqq X$. For if this were not so we could find two ascending sequences $X_1, X_2, \ldots, K_1, K_2, \ldots$ such that

$$\lim_{n=\infty} X_n = \infty, \quad \lim_{n=\infty} K_n = \infty,$$

and $$\left|\int_0^{X_n} \Phi(x)\,dx\right| > K_n X_n;$$

and then $\Phi(x)$ could not have a mean value.

Now $$\lim_{x=\infty} e^{-\tau x}\Phi(x) = 0,$$

however small τ may be. Hence

$$\tau\int_0^\infty e^{-\tau x}\Phi(x)\,dx$$

is convergent and equal to

$$\tau^2 \int_0^\infty \left(e^{-\tau x} \int_0^x \Phi(x)\, dx\right) dx.$$

Now we can choose $X_1 > X$ so that, if $x \geqq X_1$,

$$\int_0^x \Phi(x)\, dx = (k+\rho)\, x, \quad (|\rho| < \sigma),$$

however small be σ.

Then

$$\tau^2 \int_{X_1}^\infty e^{-\tau x} (k+\rho)\, x\, dx = k e^{-\tau X_1} (1 + \tau X_1) + R,$$

where $$|R| < \sigma e^{-\tau X_1} (1 + \tau X_1) < \sigma.$$

But

$$\left| \tau^2 \int_0^{X_1} \left(e^{-\tau x} \int_0^x \Phi(x)\, dx\right) dx \right| = \left| \tau^2 \left(\int_0^X + \int_X^{X_1}\right) \right|$$
$$< \tau^2 L X + \tau^2 K X_1^2,$$

where L is the maximum of $\left| e^{-\tau x} \int_0^x \Phi(x)\, dx \right|$ in $(0, X)$.
Hence

$$\left| \tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx - k \right|$$
$$< k\{1 - e^{-\tau X_1}(1 + \tau X_1)\} + \sigma + \tau^2 L X + \tau^2 K X_1^2,$$

and we can choose τ_0 so small that this is less than 2σ for all values of $\tau \leqq \tau_0$. Hence

$$\lim_{\tau=0} \tau \int_0^\infty e^{-\tau x} \Phi(x)\, dx = k.$$

Thus, *e.g.* if $\Phi(x) = \cos x$, $k = 0$, and so

$$\underset{x=\infty}{L} \cos x = 0.$$

It is easy to prove that if $\Phi(\infty)$ is determinate

$$\lim_{x=\infty} \frac{1}{x} \int_0^x \Phi(x)\, dx = \Phi(\infty);$$

hence this result includes the result of § 11.

If $\Phi(x)$ has no mean value it may still happen that

$$\frac{1}{x}\int_0^x \Phi(x)\,dx$$

has one; in this case we may define the mean value of $\Phi(x)$ as being equal to it; and so on generally.

If *e.g.* $$\Phi(x) = x\sin x,$$

$$\frac{1}{x}\int_0^x \Phi(x)\,dx = -\cos x + \frac{\sin x}{x},$$

and so the mean value of $\Phi(x)$ is 0. These extensions are analogous to those given by Hölder and Césaro to the notion of the mean value of s_n.

Some particular generalised limits and divergent integrals.

§ 14. I shall now find the value of

$$Le^{axi} f(\sin^2 x),$$

$f(u)$ being any function of u continuous from $u=0$ to $u=1$, inclusive, and a being real.

By definition

$$Le^{axi} f(\sin x) = \lim_{t=\infty} \int_0^\infty e^{-x(1-ati)} f(\sin tx)\,dx$$

$$= \lim_{\tau=0} \tau \int_0^\infty e^{-(\tau-ai)x} f(\sin^2 x)\,dx$$

$$= \lim_{\tau=0} \tau \sum_0^\infty \int_{n\pi}^{(n+1)\pi}$$

$$= \lim_{\tau=0} \tau \sum_0^\infty e^{-(\tau-ai)n\pi} \times \lim_{\tau=0} \int_0^\pi e^{-(\tau-ai)x} f(\sin^2 x)\,dx,$$

$$(1) \quad = \lim_{\tau=0} \frac{\tau}{1-e^{-(\tau-ai)\pi}} \int_0^\pi e^{axi} f(\sin x)\,dx,$$

$$(2) \quad = \frac{1}{\pi}\int_0^\pi \cos ax\, f(\sin^2 x)\,dx,$$

or $$= 0,$$

according as a is or is not an even integer. Thus

$$L \sin ax\, f(\sin^2 x)$$

is always zero.

Again

$$G\int_0^\infty e^{axi}f(\sin^2 x)\,dx$$

$$=\lim_{\tau=0}\int_0^\infty e^{-(\tau-ai)x}f(\sin^2 x)\,dx$$

$$=\lim_{\tau=0}\frac{1}{1-e^{-(\tau-ai)\pi}}\int_0^\pi e^{-(\tau-ai)x}f(\sin^2 x)\,dx.$$

If a is not an even integer this is

$$\frac{1}{1-e^{a\pi i}}\int_0^\pi e^{axi}f(\sin^2 x)\,dx=\frac{i}{\sin\frac{1}{2}a\pi}\int_0^{\frac{1}{2}\pi}\cos ax\,f(\cos^2 x)\,dx.$$

Thus

(3) $$G\int_0^\infty \cos ax\,f(\sin^2 x)\,dx=0,$$

(4) $$G\int_0^\infty \sin ax\,f(\sin^2 x)\,dx=\frac{1}{\sin\frac{1}{2}a\pi}\int_0^{\frac{1}{2}\pi}\cos ax\,f(\cos^2 x)\,dx,$$

On the other hand, if a is an even integer the first of these divergent integrals is not in general determinate, as

$$L\cos ax\,f(\sin^2 x)$$

is not in general zero. If, however,

$$\int_0^\pi \cos ax\,f(\sin^2 x)\,dx=0,$$

we obtain

$$G\int_0^\infty \cos ax\,f(\sin^2 x)\,dx$$

$$=-\frac{1}{\pi}\int_0^\pi \cos ax\,f(\sin^2 x)\,x\,dx$$

$$=-\frac{1}{\pi}\int_0^\pi \cos ax\,f(\sin^2 x)\,(\pi-x)\,dx$$

$$=-\tfrac{1}{2}\int_0^\pi \cos ax\,f(\sin^2 x)\,dx$$

$$=0.$$

But

$$G\int_0^\infty \sin ax\, f(\sin^2 x)\,dx$$

$$= \lim_{\tau=0} \frac{i}{1-e^{-\tau\pi}} \int_0^\pi e^{-\tau x} \sin ax\, f(\sin^2 x)\,dx$$

$$(5) \qquad = -\frac{2}{\pi}(-)^{\frac{1}{2}a} \int_0^{\frac{1}{2}\pi} x \sin ax\, f(\cos^2 x)\,dx.$$

It may be pointed out, in connection with the remark at the end of § 11, that

$$G\int_0^\infty e^{axi} f(\sin^2 x)\,dx = \overset{\infty}{\underset{0}{S}} \int_{n\pi}^{(n+1)\pi} e^{axi} f(\sin^2 x)\,dx.$$

For the series is equal to

$$(1 + e^{a\pi i} + e^{2a\pi i} + \ldots) \int_0^\pi e^{axi} f(\sin^2 x)\,dx$$

$$= \frac{1}{1-e^{a\pi i}} \int_0^\pi e^{axi} f(\sin^2 x)\,dx,$$

provided a is not an even integer.

Some particular cases of the formulæ (1)—(5) are interesting. Thus

$$L \cos ax = L \sin ax = 0,$$

$$L(\cos^2 ax)^{\frac{1}{2}m} = L(\sin^2 ax)^{\frac{1}{2}m} = \frac{1}{\pi}\int_0^\pi (\cos^2 x)^{\frac{1}{2}m}\,dx$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}+1\right)},$$

if $m > 0$; and if $2n$ is a positive integer

$$L(\cos^2 x)^n = L(\sin^2 x)^n$$

$$= \frac{2}{\pi}\,\frac{2.4\ldots 2k}{3.5\ldots 2k+1}$$

or

$$= \frac{1.3\ldots 2k-1}{2.4\ldots 2k},$$

according as $2n = 2k+1$ or $= 2k$. But

$$L(\cos x)^{2k+1} = L(\sin x)^{2k+1} = 0.$$

Some of these results may be easily deduced from first principles. Thus, *e.g.*, if $L\cos x$ is determinate, it must, by II., be equal to

$$L\cos(x+\pi) = -L\cos x,$$

and therefore $= 0$.

Again

$$G\int_0^\infty \cos ax\, dx = 0,$$

$$G\int_0^\infty \sin ax\, dx = \frac{1}{a},$$

$$G\int_0^\infty (\cos x)^{2k+1}\, dx = 0,$$

$$G\int_0^\infty (\sin x)^{2k+1}\, dx = \int_0^{\frac{1}{2}\pi} (\cos x)^{2k+1}\, dx$$

$$= \frac{2.4\ldots 2k}{3.5\ldots 2k+1}.$$

Again

$$G\int_0^\infty \cos ax\,(\cos x)^{2k}\, dx = 0,$$

$$G\int_0^\infty \sin ax\,(\sin x)^{2k}\, dx$$

$$= \frac{1}{\sin\frac{1}{2}a\pi}\int_0^{\frac{1}{2}\pi} \cos au\,(\cos u)^{2k}\, du$$

$$= \frac{\pi}{2^{2k+1}\sin\frac{1}{2}a\pi}\,\frac{\Gamma(2k+1)}{\Gamma(k+1-\frac{1}{2}a)\,\Gamma(k+1+\frac{1}{2}a)}$$

$$= \frac{2k!}{a(2^2-a^2)(4^2-a^2)\ldots(4k^2-a^2)},$$

provided a is not an even integer.

Again

$$L\frac{1}{1-2p\cos x+p^2} = L\frac{1}{1-2p\cos 2x+p^2}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{dx}{1-2p\cos 2x+p^2} = \frac{1}{1-p^2},$$

$$or = \frac{1}{p^2-1} \text{ (according as } p^2 < \text{ or } > 1),$$

$$L \log(1-2p\cos x + p^2) = \quad 0, \quad (p^2 < 1),$$
$$= 2\log p, \quad (p^2 > 1),$$

$$G\int_0^\infty \log(1-2p\cos x + p^2)\, dx = 0, \quad (p^2 < 1),$$

and so on. Again, since

$$\int_0^\infty e^{-\tau x} x^{p-1} \frac{\cos}{\sin} ax\, dx \quad (p > 1) = \frac{\Gamma(p)}{(\tau^2+a^2)^{\frac{1}{2}p}} \frac{\cos}{\sin} p \tan^{-1}\frac{a}{\tau},$$

$$G\int_0^\infty x^{p-1} \frac{\cos}{\sin} ax\, dx = \frac{\Gamma(p)}{a^p} \frac{\cos}{\sin} \tfrac{1}{2}p\pi.$$

If $p < 1$ these integrals are convergent in the ordinary sense. And $Lx^p \cos ax = Lx^p \sin ax = 0$, $(p > 0)$. But it is not necessary to multiply examples of the general formulæ.

A large number of these formulæ for divergent integrals were given by Raabe,* and are reproduced in Bierens de Haan's Tables. Raabe also gives 'limits' for $\cos x$, $\sin x$, etc., but not any of the general formulæ (1)—(5). He appears to think that he has proved, *e.g.*, that

$$\lim_{x=\infty} \cos x = 0$$ $x = \infty$

in the ordinary sense; his proof, which bears no resemblance to the methods of this paper, is of course wrong. It need hardly be said that at the time at which he wrote the idea of giving conventional definitions of such expressions as $\cos\infty$, $\sin\infty$, etc., had not occurred to anyone. It had of course been noticed that such expressions often presented themselves in analysis, and it had been found that there was generally one value which could be attributed to them without contradiction resulting; but there was as much confusion as to what was meant by such an equation as

$$\cos\infty = 0$$

as there was in the case of $1-1+1-\ldots = \frac{1}{2}$. And the same is true of divergent integrals; in fact it would be very difficult to say in what sense Raabe regarded

$$\int_0^\infty \cos x\, dx$$

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* *Integralrechnung*, I., pp. 230 *et seq.*

as convergent. His results however, so far as they go, agree with those found here.

Raabe* discusses a more general integral

$$G\int_0^\infty e^{axi} f(\sin^2\mu_1 x,\ \sin^2\mu_2 x,\ \dots\ \sin^2\mu_n x)\,dx,$$

in which f is a continuous function, and $\mu_1, \mu_2, \dots, \mu_n$ positive and rational. Raabe supposes $a=0$, and his analysis is invalid.

We may suppose that k is the L. C. M. of the denominators of $\mu_1, \dots, \mu_n$. If $m_1 = k\mu_1$, $\mu_2 = k\mu_2, \dots$, the integral is

$$kG\int_0^\infty e^{akxi} f(\sin^2 m_1 x,\ \dots,\ \sin^2 m_n x)\,dx,$$

where now $m_1, \dots, m_n$ are integers; this integral falls under the classes already treated. A similar reduction may be applied to

$$Le^{axi} f(\sin^2\mu_1 x,\ \dots,\ \sin^2\mu_n x).$$

§ 15. We have supposed so far that the functions $\Phi(x)$, $\phi(x)$ are continuous. But there is nothing to prevent us from attaching a sense to $L\Phi(x)$, even when $\Phi(x)$ becomes infinite for an infinity of indefinitely increasing values of x. We may agree that the definitions of $L\Phi(x)$ and

$$G\int_0^\infty \phi(x)\,dx$$

are to apply whenever the integrals which occur in them are convergent, or even if only their principal values are convergent. The formulæ worked out in § 14 are, in general, the same under these extended conditions.

Thus, *e.g.*,

$$L\log(\cos x - \cos\alpha)^2 \quad (0 < \alpha < \pi)$$
$$= \frac{1}{\pi}\int_0^\pi \log(\cos x - \cos\alpha)^2\,dx$$
$$= -2\log 2,$$

$$GP\int_0^\infty \frac{\sin x\,dx}{\cos x - \cos\alpha} = \log(4\sin^2\tfrac{1}{2}\alpha),$$

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* *l.c.*, p. 310, and *Crelle*, XV., p. 355.

a result which may be deduced either directly or from the fact that we must have

$$GP\int_0^\infty \frac{\sin x\,dx}{\cos x - \cos \alpha} = L\left[-\tfrac{1}{2}\log(\cos x - \cos\alpha)^2\right]_0^x.$$

Again $$L\frac{\cos ax}{\cos bx} = L\frac{\cos\frac{2a}{b}x}{\cos 2x}.$$

If $\frac{2a}{b}$ is not an even integer this is zero. If it is

$$L\frac{\cos\frac{2a}{b}x}{\cos 2x} = \frac{1}{\pi}P\int_0^\pi \frac{\cos\frac{2a}{b}x}{\cos 2x}dx$$

$$= 0, \quad \left(\frac{a}{b}\text{ even}\right)$$

$$= (-1)^k, \quad \left(\frac{a}{b} = 2k+1\right).$$

And $$GP\int_0^\infty \frac{\cos ax}{\cos bx}dx = \frac{2}{b}\,GP\int_0^\infty \frac{\cos\frac{2a}{b}x}{\cos 2x}dx = 0$$

unless $\frac{a}{b} = 2k+1$. In the case of

$$L\frac{\sin ax}{\sin bx}$$

we should have to modify our procedure, as

$$P\int_0^\pi \frac{\sin\frac{2a}{b}x}{\sin 2x}dx$$

is not generally convergent. But I shall not enter into this now, as the examples I have given will be sufficient to illustrate the different cases which may occur.

§ 16. The definitions of the previous sections are perhaps of most use in connection with double limit problems, such as differentiation under the integral sign. Their employment in

such problems raises questions which demand a detailed treatment which I must reserve for the present. But they enable us to a considerable extent to disregard questions of convergence or divergence. I may, perhaps, illustrate this by showing how they can be used to calculate two important definite integrals.

(i) If
$$u = \int_0^\infty \frac{\cos ax}{1+x^2}\,dx, \quad (a > 0),$$
$$\frac{du}{da} = -\int_0^\infty \frac{x \sin ax}{1+x^2}\,dx,$$
$$\frac{d^2u}{da^2} = -\,G\int_0^\infty \frac{x^2 \cos ax}{1+x^2}\,dx,$$
$$\frac{d^2u}{da^2} - u = G\int_0^\infty \cos ax\,dx$$
$$= 0,$$
$$u = Ae^a + Be^{-a},$$

and as $u = \frac{1}{2}\pi$ for $a = 0$, and does not ever exceed $\frac{1}{2}\pi$,
$$u = \tfrac{1}{2}\pi e^{-a}.$$

(ii) If
$$u = P\int_0^\infty \frac{\cos ax}{\cos bx}\,\frac{dx}{1+x^2},$$

and
$$|a| \leqq |b|,$$
$$\frac{d^2u}{da^2} - u = -\,GP\int_0^\infty \frac{\cos ax}{\cos bx}\,dx = 0,$$
$$u = Ae^a + Be^{-a},$$

and as $u = \frac{1}{2}\pi$ for $a = \pm b$,
$$A = B = \frac{\pi}{4\cosh b},$$
$$u = \frac{\pi}{2}\,\frac{\cosh a}{\cosh b}.$$

This investigation of the value of (i) is that given in some of the older English text-books. The use of the definitions of this paper enables us to justify what appears to be a quite invalid line of argument.

Other definitions.

§ 17. If we had taken

$$\theta(x, t) = p\left(\frac{x}{t}\right)^{p-1} e^{-\left(\frac{x}{t}\right)^p}, \quad (p > 1),$$

so that

$$L\Phi(x) = \lim_{t=\infty} \int_0^\infty e^{-u} \Phi(t \sqrt[p]{u})\, du,$$

and $$G\int_0^\infty \phi(x)\, dx = \lim_{\tau=0} \int_0^\infty e^{-(\tau x)^p} \phi(x)\, dx,$$

we should have found the same formulæ as we deduced in § 15 on the hypothesis that $p=1$, though tthe proofs would have been more difficult. But we should also have been able to attach meanings to limits and integrals which our actual definitions were not powerful enough to deal with.

Suppose for instance that $p=2$, and

$$\phi(x) = e^{(a+bi)x} \quad (a > 0).$$

Then

$$\lim_{\tau=0} \int_{-\infty}^0 e^{-(\tau x)^2 + (a+bi)x}\, dx = \int_{-\infty}^0 e^{(a+bi)x}\, dx = \frac{1}{a+bi},$$

so that

$$G\int_0^\infty \phi(x)\, dx = \lim_{\tau=0} \int_{-\infty}^\infty e^{-(\tau x)^2 + (a+bi)x}\, dx - \frac{1}{a+bi}.$$

But

$$\int_{-\infty}^\infty e^{-(\tau x)^2 + (a+bi)x}\, dx = e^{\left(\frac{a}{2\tau}\right)^2} \int_{-\infty}^\infty e^{-(\tau x)^2 + bxi}\, dx$$

$$= \frac{\sqrt{\pi}}{\tau} e^{\left(\frac{a}{2\tau}\right)^2 - \left(\frac{b}{2\tau}\right)^2}.$$

If $a^2 < b^2$, the limit of this for $\tau = 0$ is 0, and so

$$G\int_0^\infty e^{(a+bi)x}\, dx = -\frac{1}{a+bi},$$

and $$Le^{(a+bi)x} = 0.$$

It is of course clear that if we are to give any definition of $L\Phi(x)$ which is consistent with itself, it must give

$$Le^{(a+bi)x} = Le^{(a+bi)(x+\pi)}$$
$$= e^{(a+bi)\pi} Le^{(a+bi)x}$$

and so $$= 0.$$

And it is very desirable to find a convention which will give

$$\underset{x=\infty}{L} e^{zx} = 0,$$

$$G\int_0^\infty e^{zx}\,dz = -\frac{1}{z}$$

for as large a region of values of $z = a + bi$ as possible. This region is evidently that formed by all values of z other than real and positive values. For it is only when z is real and positive that

$$\int_0^\infty e^{zx}\,dx$$

is definitely infinite.

§ 18. Now M. le Roy has proved, in the memoir already cited, that if $t < 1$

$$\lim_{t=1} \int_0^\infty e^{-x+zx^t}\,dx = \frac{1}{1-z}$$

for all values of z except real values > 1. And by a similar method we can prove that

$$\lim_{\tau=0} \int_0^\infty e^{-\tau x \log x + zx}\,dx = -\frac{1}{z}$$

for all values of z except real and positive values.

In the first place, it is clear that

$$\lim_{\tau=0} \int_0^\infty e^{-\tau x \log x + zx}\,dx = -\frac{1}{z},$$

if $R(z) < 0$. But

$$\int_0^\infty e^{-\tau x \log x + zx}\,dx = \int_0^\infty \exp.\{-\tau\rho e^{i\phi}\log(\rho e^{i\phi}) + z\rho e^{i\phi} + i\phi\}\,d\rho,$$

if $0 < \phi < \frac{1}{2}\pi$, and the logarithm has its principal value. To prove this we integrate

$$\int e^{-\tau x \log x + zx}\,dx \quad (x = \xi + i\eta = \rho e^{i\theta})$$

round the contour bounded by the lines

$$\theta = 0,\ \theta = \phi,\ \rho = \varepsilon,\ \rho = R,$$

ε being very small and R very large. All that is necessary is to prove that the integral over the arc $\rho = R$ tends to zero for $R = \infty$. This follows from the fact that when ρ is large the subject of integration is of order

$$\exp.\ \rho\,\{-\tau \log \rho \cos\theta + \tau\theta \sin\theta + a\cos\theta - b\sin\theta\}.$$

Now

$$\lim_{\tau=0} \int_0^\infty \exp.\ \{-\tau\rho e^{i\phi} \log(\rho e^{i\phi}) + z\rho e^{i\phi} + i\phi\}\, d\rho$$

$$= \int_0^\infty \exp.\ \{z\rho e^{i\phi} + i\phi\}\, d\rho$$

$$= -\frac{1}{z},$$

provided the second integral is convergent. This will be the case if $a\cos\phi - b\sin\phi < 0$. The line $\xi\cos\phi = \eta\sin\phi$ is the line through 0, making an angle $\frac{1}{2}(\pi - \phi)$ with the real axis; and $a\cos\phi - b\sin\phi < 0$ if z lies to the left of this line. But ϕ may have any value between 0 and $\frac{1}{2}\pi$, and as ϕ varies between these limits the region to the left of the different positions of the line covers all the plane except the fourth quadrant. Thus (1) holds except when z lies in the fourth quadrant. By supposing ϕ negative we can prove that it holds except in the first quadrant. It therefore holds except when z is on the real axis.

Thus if we define

$$G \int_0^\infty \phi(x)\, dx,$$

as

$$\lim_{\tau=0} \int_0^\infty e^{-\tau x \log x} \phi(x)\, dx,$$

we find that

$$G \int_0^\infty e^{(a+bi)x} dx = -\frac{1}{a+bi}$$

except when $a > 0$, $b = 0$.

If we start from the equation

$$\lim_{t=0} \int_0^\infty e^{-x+(1+z)x^t}\, dx = -\frac{1}{z},$$

and put

$$x^t = u,\ \frac{1}{t} = 1 + \tau,$$

we obtain

$$\lim_{\tau=0} \int_0^\infty u^\tau e^{u-u^{1+\tau}+uz} du = -\frac{1}{z},$$

or
$$\lim_{\tau=0} \int_0^\infty (1+\ldots)\, e^{-\tau u \log u - \ldots}\, e^{uz} du = -\frac{1}{z}.$$

This shows the connection between the equation proved by M. le Roy and that proved here.

Suppose that $\Phi(0)=0$. Then the corresponding definition of $L\Phi(x)$ is

$$L\Phi(x) = \lim_{\tau=0} \int_0^\infty e^{-\tau x \log x}\, \Phi'(x)\, dx$$

$$= \lim_{\tau=0} \int_0^\infty \tau (\log x + 1)\, e^{-\tau x \log x}\, \Phi(x)\, dx,$$

if $\lim_{x=\infty} e^{-\tau x \log x}\, \Phi(x) \log x = 0$.

If we observe that

$$\int_0^\infty (\log x + 1)\, e^{-\tau x \log x}\, dx = \frac{1}{\tau},$$

we see that this definition agrees with that given by the general formula of § 9, if

$$\theta(x, t) = (\log x + 1)\, e^{-\frac{x}{t}\log x}$$

$$= t \frac{d}{dx}\left(e^{-\frac{x}{t}\log x}\right).$$

Much more general results are given by this choice of $\theta(x, t)$ than by the more simple one adopted in §§ 11–16. But the latter enables us to deal with the simplest and most obvious cases with greater facility.

At this point I shall, for the present, bring these investigations to a close. My object in the second part of this paper has been rather to discuss the general principles which must govern our conventions, and to illustrate the different range of different definitions, than to enter into great detail concerning any one of them. The subject is a large one, and I hope to return to it on some future occasion.

CORRECTIONS

p. 29, *lines* 7–6 *up*. The statement 'It does not follow that . . .' is false. See pp. 31–3, where the contrary is proved.

p. 41, *line* 7 *up*. For x read n, and for $(p$ read $(p^2$.

p. 43, *line* 3 *up*. For the 2nd u read v.

p. 44, *last line*. For $x = 0$ read $x = \infty$.

p. 45, *line* 11 *up*. For t_u read t^n.

p. 46, *line* 7 *up*. For u read u_m.

p. 49, *line* 1. For $n = \infty$ read $x = \infty$.

—— *line* 4. For $\lim(x/t)^{p-1}e^{-(x/t)^p}\Phi(x) = 0$ read $\lim e^{-(x/t)^p}\Phi(x) = 0$.

—— *line* 4 *up*. Read $e^{-x/t}$.

p. 50, *last* 2 *lines*. For ϕ read Φ (twice).

p. 51, *line* 9. For $x = \infty$ read $n = \infty$.

—— *line* 3 *up*. For 'and a positive' read 'and a is positive'.

p. 53, *line* 3 *up*. For $\lim e^{-\tau x}\Phi(x) = 0$ read $\lim e^{-\tau x} \int\limits_0^x \Phi(t)\,dt = 0$.

p. 55, *line* 10 *up* (twice) and *line* 6 *up*. For sin read $\sin^2$.

—— *lines* 6–5 *up*. The numbers (1) and (2) should be moved down one line.

p. 57, *line* 3. For i read 1.

—— *lines* 14–15 and *p*. 58, *lines* 9–10. Here $a \neq 0$.

p. 59, *lines* 6–7. The argument assumes $a > 0$.

—— *line* 16. For $x = 0$ read $x = \infty$.

p. 63, *lines* 5–4 *up*. A factor $\exp(iab/2\tau^2)$ is omitted in the 2nd and 3rd expressions.

COMMENTS

PART I

In § 2, 'Hardy's principle', formulated in 1904, 3, is further extended; see D.S., pp. 89–91. Systematic applications are made in 1911, 2 by Hardy and Chapman. The applications to infinite integrals, in Part II, are continued in 1908, 3.

The definitions of (B) and (B') summability are given in the Comments on 1904, 3. An *absolutely summable* series, in Borel's sense, is one for which the integrals $\int e^{-x}|u^{(\lambda)}(x)|\,dx$ converge for $\lambda = 0, 1,\ldots$, as in Borel (1st and 2nd edns.). In D.S., p. 184, Hardy uses the term *regularly summable*.

The result (§ 4) that the (B') summability of $u_1+u_2+\ldots$ implies that of $u_0+u_1+\ldots$ completes a result in 1904, 3, § 4. Hardy's proof is Mercerian in character (cf. 1912, 5). Another proof is given in Bromwich (1st edn., pp. 272–3), where it is reduced to proving that: $f+f' \to 0$ *implies* $f \to 0$. See D.S., Theorem 53, the proof of which

(attributed to Hobson) is substantially the same as Bromwich's. Another proof of Hardy's result was given by Perron.† The result that $e^{-x}u(x) \to 0$ if (2) is summable may also be obtained directly. For if $\epsilon > 0$, X_0 may be chosen so that, for $X > X_0$,

$$|u(X)-u(X_0)| = \left|\int\limits_{X_0}^{X} e^{x}.e^{-x}u'(x)\,dx\right| = e^{X}\left|\int\limits_{X'}^{X} e^{-x}u'(x)\,dx\right| < \epsilon e^{X};$$

cf. Hardy's proof, § 4, of Borel's analogous proposition.

The theorem proved in § 6 (partly stated in 1904, 3, § 3) is generalized by Hardy and Littlewood in 1916, 8, § 3, where they point out that a generalization stated by Hardy in 1913, 1 (footnote to p. 10) is incorrect. The generalization has been extended to Euler–Knopp summability by Knopp;‡ see D.S., Theorem 149.

The formal product of the Borel integrals of $u_0+u_1+...$ and $v_0+v_1+...$ may be written $\int\limits_0^\infty e^{-t}w(t)\,dt$, where

$$w(t) = \int\limits_0^t u(x)v(t-x)\,dx.$$

This is the Borel integral of $0+w_0+w_1+...$, where

$$w_n = u_0 v_n+...+u_n v_0;$$

see Borel,§ or D.S., Theorem 187. Borel's multiplication theorem and Propositions (1) and (2), § 7, correspond to the multiplication theorems of Cauchy, Mertens, and Abel for series. Analogues for ordinary infinite integrals of the multiplication theorems of Mertens, Abel, and Cesàro were given by Bohr|| and by Hardy, 1908, 2.

Hardy asks (§ 7) whether the summability of $u_0+u_1+...$ and $v_0+v_1+...$ necessarily involves the convergence of the product series. The analogue of Cesàro's multiplication theorem shows that it is summable $(C, 1)$. For ordinary infinite integrals, Bohr (loc. cit.) showed that the 'square' of the convergent integral $\int x^{-\frac{1}{2}}e^{ix}\,dx$ is not convergent. To answer Hardy's question, we may take a series whose Borel integral is approximately Bohr's integral. Let

$$u_n = v_n = (1+i)^n\Gamma(n+1)/\Gamma(n+\tfrac{3}{2}).$$

Then

$$u(x) = v(x) = \frac{1}{\Gamma(\frac{1}{2})}\int\limits_0^1 e^{(1+i)xt}(1-t)^{-\frac{1}{2}}\,dt = \left(\int\limits_{-\infty}^{1} - \int\limits_{-\infty}^{0}\right) = j_1+j_2,$$

where $j_1 = (1+i)^{-\frac{1}{2}}x^{-\frac{1}{2}}e^{(1+i)x}$ and $j_2 = O(x^{-\frac{1}{2}})$ in $(0, 1)$, $= O(x^{-1})$

† *Math. Zeit.* 6 (1920), 158–60; see also ibid., 286–310 (302–3).
‡ *Math. Zeit.* 18 (1923), 125–56.
§ 1st edn., pp. 103–7; 2nd edn., pp. 131–5.
|| *Oversigt over K. Danske Vidensk. Selsk. Förh.* (1908), 213–32; *Collected Works*, Vol. III.

in $(1,\infty)$; cf. 1905, 5 (in Vol. IV). Hence

$$w(x) = (j_1+j_2)*(j_1+j_2) = j_1*j_1+2j_1*j_2+j_2*j_2 = w_1+w_2+w_3,$$

where $\int e^{-x}w_r\,dx$ converges for $r = 2, 3$, by the Mertens and Cauchy analogues, and for $r = 1$ is $(1+i)^{-1}\pi \int e^{ix}\,dx$. Further, Hardy's extra conditions are also satisfied; cf. 1905, 5. A series such that all the series $u_\lambda+u_{\lambda+1}+\ldots$ are summable (B') (or equivalently all summable (B)) is called *normally summable* in D.S., p. 184.

The *Borel polygon* Γ of a power series, with positive radius of convergence, was defined by Borel†† in terms of the singularities of its sum function; see D.S., § 8.8. Borel showed that the interior $\prod$ of Γ is the domain of *absolute summability* in his sense (apart from points on the boundary Γ). Phragmén‡‡ showed that $\prod$ is also the domain of *summability* (B'), and so also of *summability* (B) and *normal summability*. Hardy's analogue of Abel's limit theorem § 7, is stated for points of *normal summability* on Γ, but the deduction from Dirichlet's analogue for infinite integrals holds for points of *summability* (B'). Further, if there is no polygon of summability, it holds along a segment joining the origin to any point of summability; see 1911, 8. Bromwich (1st edn., pp. 291–2) observed that Phragmén (loc. cit.) had shown that, if Borel's integral $\int e^{-t}u(tx)\,dt$ converges for $x = x_0$, then it converges uniformly on any segment $(\delta x_0, x_0)$, $\delta > 0$. Hardy's theorem was evidently independent of Phragmén's. Bromwich inadvertently stated that the uniformity holds on $(0, x_0)$, and Hardy assumes this in 1910, 1; but in 1911, 8, he proves that it is, in fact, true. The analogue for infinite integrals of Abel's limit theorem was stated by Bonnet§§ in 1849. Hardy quotes the result from Stolz,‖‖ who refers to a book by Meyer,††† stated in the sub-title to be 'zumeist nach Vorträgen von Lejeune-Dirichlet im Sommer 1858'.

Part II

In § 10, the modified condition

$$\lim e^{-(x/t)^p}\Phi(x) = 0$$

(see Corrections) plays the role of the condition $\lim e^{-x}u(x) = 0$ in § 4. In particular, if $p = 1$, necessary and sufficient conditions for $G\int \phi\,dx$ to exist are that $L\Phi(x)$ should exist and $\lim e^{-\tau x}\Phi(x) = 0$ for $\tau > 0$ (if the assumption $\lim e^{-\tau x}\phi(x) = 0$ is omitted).

The alternative definition at the end of § 10 should be interpreted as $\lim\limits_{\tau\to 0} \int_\tau^\infty \int_0^\infty$; see § 25 of 1908, 3.

The 'questions which demand a detailed treatment which I must reserve for the present', § 16, are discussed in 1908, 3.

†† 1st edn., pp. 125–8; 2nd edn., pp. 157–60.

‡‡ *Comptes rendus* 132 (1901), 1396–9.

§§ *J. de math. pures et appl.* (1), 14 (1849), 249–56.

‖‖ *Differential- und Integralrechnung*, Vol. 1 (1893), pp. 447–8.

††† *Vorlesungen über die Theorie der bestimmten Integrale zwischen reellen Grenzen* (1871).

SOME EXTENSIONS TO MULTIPLE SERIES OF ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

By T. J. I'A. Bromwich *and* G. H. Hardy.*

1.

The object of this paper is to investigate certain extensions to multiple and repeated series of the following well-known theorem due to Abel:—

If the series

(1) $$a_0+a_1+a_2+\dots$$

is convergent, the series

(2) $$a_0+a_1x+a_2x^2+\dots$$

is absolutely convergent for all values of x whose modulus is less than unity, and if $f(x)$ denotes the function represented by the series (2), the limit of $f(x)$ when x approaches 1 along the straight line (0, 1) is equal to the sum of the series (1).†

Notation and Terminology.

It will be found essential in dealing with these questions to lay down as definite and concise a notation and as unambiguous a terminology as is possible, since those usually employed are in some ways misleading.

Suppose that

$$s_{m_1, m_2, \dots, m_n} = \sum_{i_1=0}^{i_1=m_1} \sum_{i_2=0}^{i_2=m_2} \dots \sum_{i_n=0}^{i_n=m_n} a_{i_1, i_2, \dots, i_n};$$

then we denote by

$$\sum_{(1, 2, \dots, p)\,(p+1, \dots, q)\,\dots\,(r+1, \dots, n)} a$$

* Mr. Hardy communicated his share of the paper on February 11th, 1904, and discovered shortly afterwards that Prof. Bromwich had at an earlier date arrived independently at the results of §§ 1–5. § 6 and §§ 12–17 are due more particularly to Mr. Hardy, and §§ 7–11 to Prof. Bromwich. Some of the earlier results (those relating to double series summed by rows or columns) were also obtained by Mr. A. Brown, to whom the subject had been suggested by Prof. Bromwich for a dissertation. As regards the latter part of the paper, each of the authors had arrived by conjecture at the other's results, but had not worked out formal proofs at the time when it was decided to unite them in one paper.

† The theorem is still true if x approaches 1 by any path (in the complex-plane) which does not touch the circle of convergence; but it is not with extensions of this kind that we shall be concerned now.

1905, 2 (with T. J. I'A. Bromwich) *Proceedings of the London Mathematical Society* (2), 2, 161–89.

the result (if it be determinate) of making the suffixes $m_1, m_2, \ldots, m_n$ tend to infinity in groups, the group $m_{r+1}, \ldots, m_n$ being made first to tend *simultaneously* to infinity, and so on, the groups corresponding to the brackets written under the sign of summation. Thus, to take the simplest case—that of two integral parameters i_1, i_2—the expressions

$$\sum_{(1,\,2)} a, \quad \sum_{(1)(2)} a, \quad \sum_{(2)(1)} a$$

denote respectively the double series

$$\Sigma a_{i_1,\, i_2}$$

in Pringsheim's sense, and the two repeated series in which the sum is effected with respect to one parameter first. A similar notation will be used for limits. Thus,

$$\sum_{(1,\,2)} a = \lim_{(1,\,2)} s, \qquad \sum_{(1)(2)} a = \lim_{(1)(2)} s.$$

Where there is more than one bracket the operation of proceeding to the limit which corresponds to the bracket on the *right* is always to be performed first. The same notation applies to limits of functions of continuous variables. Thus, if $f(x_1, x_2)$ is a function of x_1 and x_2, both of these being positive and less than 1, $\lim\limits_{(1)(2)} f$ means $\lim\limits_{x_1=1} \, (\lim\limits_{x_2=1} f)$ and $\lim\limits_{(1,\,2)} f$ means the *double* limit $\lim\limits_{x_1=1,\, x_2=1} f$.

It is always to be understood that the limits of summation, unless the contrary is expressly stated, are zero and infinity, and the limiting value of every variable, which we shall always assume to be real and positive,* unless the contrary is expressly stated, is 1, and the term "double limit" will be used always as indicating that two variables (integral or continuous) are made to tend *simultaneously* to their limiting values. When there are several distinct passages to the limit the result is a *repeated* limit; thus,

$$\lim_{(1,\,2)(3,\,4)}$$

would denote a repeated limit—in this case the double limit of a double limit.

The expression $$\sum_{(1)} a$$

denotes the result of summing with respect to i_1 *only*, and so on. Also,

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* There is, of course, no such limitation on the value of a.

if b depends on $i_1, \ldots, i_n$,

$$\underset{(1)}{\Delta} b = b_{i_1+1, i_2, \ldots, i_n} - b_{i_1, i_2, \ldots, i_n},$$

$$\underset{(1,2)}{\Delta} b = \underset{(2,1)}{\Delta} b = \underset{(2)}{\Delta}\underset{(1)}{\Delta} b$$

$$= b_{i_1+1, i_2+1, i_3, \ldots, i_n} - b_{i_1, i_2+1, i_3, \ldots, i_n} - b_{i_1+1, i_2, i_3, \ldots, i_n} + b_{i_1, i_2, i_3, \ldots, i_n},$$

and so on.

Finally, all this notation may be generalised to denote, not limits, but maximum and minimum limits;* thus,

$$\sum_{(\overline{1})(\underline{2})} a$$

denotes the maximum limit for $i_1 = \infty$ of the minimum limit of s_{i_1, i_2} for $i_2 = \infty$, and

$$\sum_{(\overline{1, 2})} a$$

denotes the maximum limit of s_{i_1, i_2} when i_1 and i_2 tend together to infinity. And, again, exactly the same applies to such expressions as

$$\lim_{(\overline{1})(\underline{2})} f.$$

2. *Statement of the Analogue of Abel's Theorem for the General Series.*

If the simple series Σa_i is convergent, there is certainly a constant C, such that

$$| s_i | < C$$

for all values of i. We express this by saying that such a convergent series necessarily satisfies the *condition of finitude.* The same is not true for multiple series. This being so, we cannot affirm that, if, *e.g.*,

$$\sum_{(1, 2, \ldots, n)} a$$

is convergent, then

$$\sum_{(1, 2, \ldots, n)} a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

converges for values of $x_1, x_2, \ldots, x_n$ less than 1, and it is easy to see by examples that this is not necessarily the case.†

It is therefore essential to subject our series to some condition beyond that of mere convergence. We shall assume that it *does* satisfy the "condition of finitude," that is to say, that

(3) $$| s_{m_1, m_2, \ldots, m_n} | < C$$

* Sometimes called "upper and lower limits of indetermination."

† For instance, compare § 3, end.

for all values of $m_1, m_2, \ldots, m_n$. Doubtless this condition is unnecessarily narrow, but it is simple and fulfils all requirements.

The analogue of Abel's theorem is then as follows:—*If the condition of finitude is satisfied, and*

$$\sum_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} a \tag{4}$$

is convergent, then

$$\Sigma\, a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \tag{5}$$

is absolutely convergent for all values of $x_1, \ldots, x_n$ whose moduli are less than 1, *and if $f(x_1, \ldots, x_n)$ is the function represented by this series, then*

$$\lim_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} f \tag{6}$$

is determinate and equal to the sum of the series (4).

We shall prove this theorem first for double series and give some illustrations in which the series Σa has different sums when summed in different ways, so that f has different limits when we proceed to the limit in different ways.* We shall then consider some further extensions of a different kind connected with double series. Finally, we shall establish the general theorem by induction. In dealing with double series we shall use i, j, x, y for i_1, i_2, x_1, x_2 in order to avoid suffixes, and we shall write $\sum\limits_{(i)(j)}, \sum\limits_{(j)(i)}, \sum\limits_{(i,j)}, \lim\limits_{(x)(y)}, \lim\limits_{(y)(x)}, \lim\limits_{(x,y)}$ for $\sum\limits_{(1)(2)}, \ldots$.

3. *Double Series.*

Since
$$a_{i,j} = \Delta_{(i,j)}\, s_{i-1,j-1}$$
and
$$|s_{m,n}| < C,$$
it follows that
$$|a_{i,j}| < 4C, \tag{7}$$
and hence that
$$\Sigma\, a_{i,j} x^i y^j$$
is absolutely convergent. Let $f(x, y)$ denote its sum. Then

$$f(x, y) = \Sigma\, s_{i,j}(1-x)(1-y)x^i y^j, \tag{8}$$

* This course seems best because this simple case affords the clearest illustration of the ideas on which our extensions of Abel's theorem are based, and its treatment does not involve the algebraical difficulties which occur in proving the more general theorems.

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as is at once evident if we compare the coefficients and use condition (7).*

Now to say that $\sum_{(i,j)} a$ is convergent is the same as to say that there is a quantity s such that, however small be σ, we can determine M and N so that

$$| s_{m,n} - s | < \sigma,$$

if only $m \geqslant M$ and $n \geqslant N$. It is evident, moreover, that $| s | \leqslant C$.

Now, since

$$\Sigma (1-x)(1-y) x^i y^j = 1,$$

it follows that $\quad f(x, y) - s = \Sigma (s_{i,j} - s)(1-x)(1-y) x^i y^j$

and $|f(x, y) - s | \leqslant \left| \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \right| + \left| \sum_{i=0}^{M-1} \sum_{j=N}^{\infty} \right| + \left| \sum_{j=0}^{N-1} \sum_{i=M}^{\infty} \right| + \left| \sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \right|.$

But
$$\left| \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \right| < 2CMN(1-x)(1-y),$$

since $x < 1$, $y < 1$, and $| s_{i,j} - s | < 2C$; also

$$\left| \sum_{i=0}^{M-1} \sum_{j=N}^{\infty} \right| < 2CM(1-x) \sum_{j=0} y^j (1-y) < 2CM(1-x),$$

$$\left| \sum_{j=0}^{N-1} \sum_{i=M}^{\infty} \right| < 2CN(1-y),$$

and
$$\left| \sum_{i=M}^{\infty} \sum_{j=N}^{\infty} \right| < \sigma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^i y^j (1-x)(1-y) < \sigma.$$

Thus

$$| f(x, y) - s | < 2CMN(1-x)(1-y) + 2CM(1-x) + 2CN(1-y) + \sigma.$$

But when σ has been fixed M and N are fixed, and we can determine δ, ϵ, so that

$$| f(x, y) - s | < 2\sigma,$$

if $1-x < \delta$, $1-y < \epsilon$. Therefore

$$\lim_{(x,y)} f = s.$$

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* The transformation

$$a_0 + a_1 x + a_2 x^2 = \ldots = (1-x)(s_0 + s_1 x + s_2 x^2 + \ldots)$$

was given by Dirichlet and used as the basis of a proof of Abel's theorem identical in principle with the proof stated here of the corresponding theorem for double series, though (at any rate in the form in which he presents it) less simple than Abel's original proof. See Abel, *Œuvres*, Vol. I., p. 223; Dirichlet, *Werke*, Vol. II., p. 306; Pringsheim, *Münch. Ber.*, 1897, p. 344.

We may remark in passing that a similar proof applies to the general case when it is the convergence of the multiple series

$$\sum_{(1,\,2,\,\ldots,\,n)} a$$

which is given. The real difficulties begin when *repeated* limits are introduced.

We may further remark that the necessity of some such limitation as is implied by the condition of finitude becomes apparent when we consider that, for example, the double series defined by the scheme

$$\begin{array}{ccccc} a_0+b_0, & a_1-b_0, & a_2, & a_3, & \ldots, \\ -a_0+b_1, & -a_1-b_1, & -a_2, & -a_3, & \ldots, \\ b_2, & -b_2, & 0, & 0, & \ldots, \\ b_3, & -b_3, & 0, & 0, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

is convergent and has the sum 0 *whatever* be the quantities a, b; even if $a_\nu = b_\nu = \nu!$, in which case $\Sigma a_{i,j} x^i y^j$ is not convergent for any values of x and y except $x=0$, $y=0$ and $x=1$, $y=1$. If $a_\nu = b_\nu = 2^\nu$, the series is convergent and equal to $(1-y)/(1-2x)+(1-x)/(1-2y)$ if x and y are both $<\frac{1}{2}$, but divergent if $\frac{1}{2} \leqslant x < 1$ or $\frac{1}{2} \leqslant y < 1$.

4. *Repeated (Two-fold) Series.*

Now let us suppose that Σa is convergent when summed by *columns*, thus implying the convergence of every column, and that

$$\sum_{(i)(j)} a = s.$$

The series is of course absolutely convergent as before, in virtue of the condition of finitude. To illustrate the necessity of some such condition in this case we might suppose $a_{i,j}$ given by the scheme

$$\begin{array}{ccccc} 1, & 2, & 4, & 8, & \ldots, \\ -\frac{1}{2}, & -1, & -2, & -4, & \ldots, \\ -\frac{1}{4}, & -\frac{1}{2}, & -1, & -2, & \ldots, \\ -\frac{1}{8}, & -\frac{1}{4}, & -\frac{1}{2}, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

Then $\sum_{(i)(j)} a = 0$, but the power series does not converge for any value of y if $\frac{1}{2} \leqslant x < 1$.

Let
$$b_i = \sum_{(j)} a_{i,j}\,;$$

then, since
$$\left|\sum_{j=0}^{n} a_{i,j}\right| = \left|\underset{(i)}{\Delta}\, s_{i-1,\,n}\right| < 2C,$$
$$|\, b_i \,| \leqslant 2C$$

and

(9)
$$\sum_{(i)} b_i x^i$$

is absolutely convergent. Similarly,
$$\sum_{(i)} a_{i,j}\, x^i$$

is absolutely convergent. Further we can prove that

(10)
$$\sum_{(j)(i)} a_{i,j}\, x^i$$

is convergent, and its sum equal to that of (9).*

For, if we introduce the abbreviation
$$b_{i,j} = \sum_{l=0}^{j} a_{i,l} = \underset{(i)}{\Delta}\, s_{i-1,\,j}\,,$$

then the series (10) is equal to the limit
$$\lim_{j=\infty}\left(\sum_{i=0}^{\infty} b_{i,j}\, x^i\right),$$

provided that this limit exists.

Now, by the condition of finitude,
$$|\, b_{i,j} \,| = |\, \underset{(i)}{\Delta}\, s_{i-1,j} \,| < 2C\,;$$

so that $|\, b_{i,j} - b_i \,| < 4C$, for all values of i, j.

Hence, for all values of j,
$$\left|\sum_{i=M}^{\infty} (b_{i,j} - b_i)\, x^i\right| < 4C \sum_{i=M}^{\infty} x^i = 4Cx^M/(1-x).$$

Let M be chosen so as to make $4Cx^M/(1-x)$ less than an assigned positive number σ; M being now fixed, N can be chosen so as to give
$$|\, b_{i,j} - b_i \,| < \sigma(1-x)$$

* This is a kind of converse of Weierstrass's theorem concerning series of power series.

for every value of $j \geqslant N$ and for $i = 0, 1, 2, \ldots, M-1$, since

$$\lim_{j=\infty} b_{i,j} = b_i.$$

Then $$\left| \sum_{i=0}^{M-1} (b_{i,j} - b_i) x^i \right| < \sigma (1-x) \sum_{i=0}^{M-1} x^i < \sigma,$$

and hence $$\left| \sum_{i=0}^{\infty} (b_{i,j} - b_i) x^i \right| \leqslant \left| \sum_{i=0}^{M-1} \right| + \left| \sum_{i=M}^{\infty} \right| < 2\sigma, \quad \text{if } j \geqslant N.$$

Thus*

$$\lim_{j=\infty} \sum_{i=0}^{\infty} b_{i,j} x^i = \sum_{i=0}^{\infty} b_i x^i;$$

that is to say, the series (10) converges and has the same sum as (9).

5.

Hence $$\lim_{(x)(y)} f(x, y) = \lim_{(x)(y)} \sum_{(j)} y^i \sum_{(i)} a_{i,j} x^i = \lim_{(x)} \sum_{(j)} \sum_{(i)} a_{i,j} x^i$$

(by Abel's theorem)

$$= \lim \sum_{(i)} x^i \sum_{(j)} a_{i,j} \quad \text{(by § 4)}$$

$$= \sum_{(i)(j)} a_{i,j} \quad \text{(by Abel's theorem).}$$

An exactly similar proof applies to the case in which the convergence of $\sum_{(j)(i)} a$ is given. Hence, *if the condition of finitude is satisfied, and any one of the three series* $\sum_{(i,j)} a$, $\sum_{(i)(j)} a$, $\sum_{(j)(i)} a$ *is convergent, the corresponding one of the three limits* $\lim_{(x,y)} f$, $\lim_{(x)(y)} f$, $\lim_{(y)(x)} f$ *is determinate and equal to the sum of the series.*

By similar methods we can easily establish corresponding theorems, in case the series $\sum_{(i)(j)} a_{i,j}$, $\sum_{(j)(i)} a_{i,j}$, $\sum_{(i,j)} a_{i,j}$ do not converge, but oscillate.

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* An alternative proof of this equation can be found by writing each side as a repeated limit, in the form

$$\lim_{(j)(i)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right), \quad \lim_{(i)(j)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right).$$

The equality can be then obtained by using conditions given by Bromwich (*Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 184).

Thus we find

$$\sum_{\underline{(i)(j)}} a_{i,j} \leqslant \lim_{\underline{(x)(y)}} f(x, y) \leqslant \lim_{\overline{(x)(y)}} f(x, y) \leqslant \sum_{\overline{(i)(j)}} a_{i,j},$$

$$\sum_{\underline{(j)(i)}} a_{i,j} \leqslant \lim_{\underline{(y)(x)}} f(x, y) \leqslant \lim_{\overline{(x)(y)}} f(x, y) \leqslant \sum_{\overline{(i)(j)}} a_{i,j},$$

$$\sum_{\underline{(i,j)}} a_{i,j} \leqslant \lim_{\underline{(x,y)}} f(x, y) \leqslant \lim_{\overline{(x,y)}} f(x, y) \leqslant \sum_{\overline{(i,j)}} a_{i,j}.$$

These results may be summed up in the statement that *the maximum and minimum limits of $f(x, y)$, when x, y approach unity in any one of the three standard ways, are included between the maximum and minimum limits of $\Sigma a_{i,j}$, when i, j approach infinity in the same way as x, y approach unity.*

6.

Before proceeding to the general case we shall illustrate this result by some examples:—

(i.) Suppose $$a_{i,j} = \frac{i-j}{2^{i+j}} \frac{(i+j-1)!}{i!\,j!} \qquad (i, j > 0)$$

and $a_{0,j} = -2^{-j}$ $(j > 0)$, $a_{i,0} = 2^{-i}$ $(i > 0)$, $a_{0,0} = 0$. Then, if $j > 0$,

$$\sum_{i=0}^{\infty} a_{i,j} = -2^{-j} + \frac{2^{-j}}{j!} \sum_{1}^{\infty} \frac{(i+j-1)!}{(i-1)!} 2^{-i} - \frac{2^{-j}}{(j-1)!} \sum_{1}^{\infty} \frac{(i+j-1)!}{i!} 2^{-i}$$

$$= -2^{-j} + 2^{-j-1}(1-\tfrac{1}{2})^{-j-1} - 2^{-j}\{(1-\tfrac{1}{2})^{-j} - 1\} = 0;$$

but $$\sum_{0}^{\infty} a_{i,0} = \sum_{1}^{\infty} 2^{-i} = 1.$$

Hence $$\sum_{(j)(i)} a = 1$$

and, as $$a_{j,i} = -a_{i,j},$$

$$\sum_{(i)(j)} a = -1.$$

It follows by a well known theorem of Pringsheim's that the double series $\sum_{(i,j)} a$ is not convergent. Hence we infer (assuming for a moment that the condition of finitude is satisfied) that

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1,$$

and therefore (by the same theorem) $\lim_{(x,y)} f$ is not determinate. It is interesting to note that in such a case as this we can make this last *negative* inference. In the case of Abel's theorem *no* negative inference is possible.

To verify that, as a matter of fact, the condition of finitude is satisfied, we have only to observe that, if $m = n$,

$$s_{m,n} = 0,$$

while, if $m > n$, $\quad s_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} = \sum_{i=n+1}^{m} \sum_{j=0}^{n} a_{i,j}.$

In this last expression every term is positive, and

$$s_{m,n} < \sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j};$$

but, since $s_{n,n} = 0$,

$$\sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} a_{i,j} = 1.$$

Thus we may take $C = 1$.

It is easy to verify our conclusions, for

$$\Sigma a_{i,j} x^i y^j = \frac{x-y}{2-x-y} = \frac{(1-y)-(1-x)}{(1-y)+(1-x)},$$

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1.$$

(ii.) Suppose that

$$\sin \frac{1}{1-x} = a_0 + a_1 x + a_2 x^2 + \dots \qquad (0 < x < 1),$$

and consider the double series defined by the scheme

$$\begin{array}{ccccc} a_0+a_0, & a_1-a_0, & a_2, & a_3, & \dots, \\ a_1-a_0, & -a_1-a_1, & -a_2, & -a_3, & \dots, \\ a_2, & -a_2, & 0, & 0, & \dots, \\ a_3, & -a_3, & 0, & 0, & \dots, \\ \dots & \dots & \dots & \dots & \dots. \end{array}$$

Then, if $m \geqslant 2$, $n \geqslant 2$, $s_{m,n} = 0$; so that

$$\sum_{(i,j)} a_{i,j} = 0.$$

But neither repeated series is convergent, since $a_0 + a_1 + a_2 + \dots$ is not convergent.* In this case

$$f(x, y) = (1-x) \sin \frac{1}{1-y} + (1-y) \sin \frac{1}{1-x};$$

so that
$$\lim_{(x,y)} f = 0$$

while neither repeated limit exists.

---

* For, if it were, $\sin \frac{1}{1-x}$ would by Abel's theorem have a limit for $x = 1$, which is not the case.

It is true that we have not in this case verified the condition of finitude, and it is difficult to see exactly how this can be done, as a_ν is a complicated function of ν. But it is only necessary to observe that to remove this objection we may replace $\sin\dfrac{1}{1-x}$ by any function of x which satisfies the following conditions :—

(i.) $$f(x) = a_0+a_1x+\ldots \qquad (0<x<1);$$

(ii.) $$|\,a_0+a_1+\ldots+a_\nu\,| < C;$$

(iii.) $f(x)$ oscillates between finite limits of indetermination for $x=1$. Such functions certainly exist.*

7. *Statement of the Theorems of Frobenius and Hölder.*

Abel's theorem gives no information as to the behaviour near $x=1$ of the function $f(x)$, in case the series (1) is not convergent; but if the series oscillates it is quite possible that the limit

$$\lim_{x=1} f(x)$$

may be finite and determinate, in spite of the divergence of the series.† Frobenius‡ was the first to obtain a result giving information about this case; his theorem may be stated as follows :—

Let $$s_n = \sum_{j=0}^{n} a_j;$$

in case s_n *approaches no definite limit as* n *increases to infinity, it may*

---

* One may, in fact, be constructed as follows. Divide (0, 1) into the intervals

$$i_n = \left(1-\frac{1}{2^n},\ 1-\frac{1}{2^{n+1}}\right) \quad (n=0,1,2,\ldots).$$

Let σ be an assigned small positive quantity. Choose n_1 so that throughout i_{n_1}

$$|\,(1-2x)-(-1)\,| < \sigma.$$

Now choose p_2 so that throughout i_{n_1}

$$x^{p_2}(1+x+x^2+\ldots) < \sigma,$$

n_2 so that throughout i_{n_2} $$|\,1-2x+2x^{p_2}-(+1)\,| < \sigma,$$

p_3 so that throughout i_{n_2} $$x^{p_3}(1+x+x^2+\ldots) < \sigma,$$

and so on. Then it is easy to see that, if

$$f(x) = 1-2x^{p_1}+2x^{p_2}-2x^{p_3}+\ldots \quad (p_1=1),$$

$f(x)$ differs from -1 by less than 3σ in i_{n_1}, i_{n_3}, i_{n_5}, ..., and from $+1$ by less than 3σ in i_{n_2}, i''_{4}, i_{n_6}, ... The numbers $p_1, p_2, p_3, \ldots$ increase with very great rapidity.

† For example, let $f(x) = 1/(1+x) = 1-x+x^2-x^3+\ldots$; then $\lim\limits_{x=1} f(x)$ is equal to $\frac{1}{2}$, although $1-1+1-1+1-1+\ldots$ is oscillatory. But it has been proved that if $\lim\limits_{n=\infty} s_n = \infty$, then $\lim\limits_{x=1} f(x) = \infty$.

‡ *Crelle's Journal*, Bd. LXXXIX., 1880, p. 262.

happen that the arithmetic mean

$$s_n^{(1)} = \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n)$$

approaches a limit l; *then the limit*

$$\lim_{x=1} f(x)$$

exists and is equal to l.

It may be noticed incidentally that, if s_j does approach a definite limit l, then the arithmetic mean $s_n^{(1)}$ will approach the same limit. For an integer n can be chosen so that

$$|s_j - l| < \sigma,$$

if $j \geqslant n$; n being fixed, choose N so that

$$|s_0+s_1+s_2+\ldots+s_{n-1}-nl| < N\sigma.$$

Then

$$|s_j^{(1)} - l| = \frac{1}{j+1}\left|(s_0+s_1+\ldots+s_{n-1}-nl)+(s_n-l)+(s_{n+1}-l)+\ldots+(s_j-l)\right|$$
$$< \frac{1}{j+1}[(N\sigma)+(j-n+1)\sigma] < 2\sigma,$$

if $j \geqslant n$ and N; that is, $\lim_{j=\infty} s_j^{(1)} = l.$

A similar method can be used to prove that if s_n tends to infinity with n, then the same is true of $s_n^{(1)}$.

The theorem of Frobenius was extended further by Hölder,* so as to cover cases in which the first arithmetic mean has no definite limit.

Hölder writes

$$
\begin{aligned}
s_n^{(1)} &= \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n), \\
s_n^{(2)} &= \frac{1}{n+1}(s_0^{(1)}+s_1^{(1)}+s_2^{(1)}+\ldots+s_n^{(1)}), \qquad (11)\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots, \\
s_n^{(k)} &= \frac{1}{n+1}(s_0^{(k-1)}+s_1^{(k-1)}+s_2^{(k-1)}+\ldots+s_n^{(k-1)}).
\end{aligned}
$$

The extended theorem is then

$$\underline{\lim}_{(n)}\, s_n^{(k)} \leqslant \underline{\lim}_{(x)}\, f(x) \leqslant \overline{\lim}_{(x)}\, f(x) \leqslant \overline{\lim}_{(n)}\, s_n^{(k)},$$

provided that $|s_n^{(k)}| < C$ for all values of n.

* *Math. Annalen*, Bd. xx., 1882, p. 535.

8. *Extension of Frobenius's Theorem to Double Series.*

Let us write

(12) $$s_{m,n} = \sum_{i,j=0} a_{i,j}, \quad s^{(1)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j};$$

so that $s^{(1)}_{m,n}$ is an arithmetic mean amongst the sums s_{mn}. Then the theorem is:—

If $$\lim_{(m,n)} s^{(1)}_{m,n} = l,$$

then also $$\lim_{(x,y)} f(x, y) = l,$$

provided that

(13) $$| s^{(1)}_{m,n} | < C$$

for all values of m, n (the present form of the condition of finitude).

In virtue of equations (12), we have

$$\Delta_{(i,j)} [ij\, s^{(1)}_{i-1,j-1}] = s_{i,j}, \quad \Delta_{(i,j)} [s_{i-1,j-1}] = a_{i,j}.$$

Hence, using (13), we deduce

(14) $$| s_{i\,j} | < C[(i+1)(j+1)+i(j+1)+(i+1)j+ij] < 4C(i+1)(j+1)$$

and

(15) $$| a_{i,j} | < 16C(i+1)(j+1).$$

It follows at once, from (13), (14), and (15), that the three series

$$\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j, \quad \Sigma s_{i,j}x^i y^j, \quad \Sigma a_{i,j}x^i y^j$$

are all absolutely convergent, since their terms are less numerically than the corresponding terms in the series for

$$16C(1-x)^{-2}(1-y)^{-2}.$$

Further we find by direct multiplication that

$$(1-x)(1-y)\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j = \Sigma s_{i,j}x^i y^j.$$

Thus, using (8), it is clear that

(16) $$f(x, y) = (1-x)^2(1-y)^2\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j.$$

But, since the arithmetic means have the limiting value l, an integer N can be found such that

(17) $$| s^{(1)}_{i,j} - l | < \sigma, \text{ for } i, j \geqslant N,$$

however small the positive number σ may be; further, from (13), it follows that

$$| l | \leqslant C, \quad | s^{(1)}_{i,j} - l | < 2C, \text{ for all values of } i, j.$$

Thus, since $$(1-x)^2(1-y)^2\Sigma(i+1)(j+1)\,x^i y^j = 1,$$
it follows, from (16), that

$$\begin{aligned} f(x, y)-l &= (1-x)^2(1-y)^2\Sigma\,(i+1)(j+1)\,(s^{(1)}_{i,j}-l)\,x^i y^j \\ &= (1-x)^2(1-y)^2\left[\sum_{i,j=0}^{N-1} + \sum_{i=0}^{N-1}\sum_{j=N}^{\infty} + \sum_{j=0}^{N-1}\sum_{i=N}^{\infty} + \sum_{i,j=N}^{\infty}\right]. \end{aligned}$$

But, from (17), it is evident that

$$\left|\sum_{i,j=N}^{\infty}\right| < \sigma \sum_{i,j=N}^{\infty}(i+1)(j+1)\,x^i y^j < \sigma\,(1-x)^{-2}(1-y)^{-2};$$

also $$\left|\sum_{i,j=0}^{N-1}\right| < 2C\sum_{i,j=0}^{N-1}(i+1)(j+1) = 2C\left[\tfrac{1}{2}N(N+1)\right]^2,$$

$$\left|\sum_{i=0}^{N-1}\sum_{j=N}^{\infty}\right| < 2C\sum_{i=0}^{N-1}(i+1)\sum_{j=N}^{\infty}(j+1)\,y^j < N(N+1)\,C\,(1-y)^{-2}.$$

$$\left|\sum_{j=0}^{N-1}\sum_{i=N}^{\infty}\right| < 2C\sum_{j=0}^{N-1}(j+1)\sum_{i=N}^{\infty}(i+1)\,x^i < N(N+1)\,C\,(1-x)^{-2}.$$

Combining these four inequalities, we obtain

(18) $$|\,f(x, y)-l\,| < \sigma+N(N+1)\,C\left[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2\right].$$

Now choose δ so that

$$N(N+1)\,C\delta^2\left[2+\tfrac{1}{2}N(N+1)\,\delta^2\right] < \sigma,$$

which is possible, since N is now fixed.* Then plainly

$$N(N+1)\,C\left[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2\right] < \sigma,$$

if $1-x < \delta$, $1-y < \delta$; and so (18) leads to the result

$$|\,f(x, y)-l\,| < 2\sigma,$$

if $1-x < \delta$, $1-y < \delta$; that is,

(19) $$\lim_{(x,\,y)} f(x, y) = l,$$

which is the analogue of Frobenius's theorem.

It is easy to prove, by a similar method, that, in case $s^{(1)}_{i,j}$ does not approach a definite limit, but oscillates between a maximum limit and a minimum limit, then

$$\varliminf_{(i,\,j)} s^{(1)}_{i,j} \leqslant \varliminf_{(x,\,y)} f(x, y) \leqslant \varlimsup_{(x,\,y)} f(x, y) \leqslant \varlimsup_{(i,\,j)} s^{(1)}_{i,j}.$$

Before considering the case of *repeated* limits of the double series, we shall give an example of the result contained in equation (19).

---

* One way of doing it is to take for δ the smaller of the two values $[\sigma/4N(N+1)\,C]^{\frac{1}{2}}$, $[\sigma/N^2(N+1)^2C]^{\frac{1}{4}}$; the smaller will usually be the first.

9. *Lord Kelvin's Series.*

In Lord Kelvin's discussion of the electrical force between two equal conducting spheres in contact,* he employs the double series given by

$$a_{i,j} = (-1)^{i+j} ij/(i+j)^2 \quad (i, j = 1, 2, 3, \ldots),$$

the scheme for which is

$$\begin{array}{cccccc}
+\frac{1.1}{2^2}, & -\frac{2.1}{3^2}, & +\frac{3.1}{4^2}, & -\frac{4.1}{5^2}, & +\ldots, \\[6pt]
-\frac{1.2}{3^2}, & +\frac{2.2}{4^2}, & -\frac{3.2}{5^2}, & +\frac{4.2}{6^2}, & -\ldots, \\[6pt]
+\frac{1.3}{4^2}, & -\frac{2.3}{5^2}, & +\frac{3.3}{6^2}, & -\frac{4.3}{7^2}, & +\ldots, \\[6pt]
-\frac{1.4}{5^2}, & +\frac{2.4}{6^2}, & -\frac{3.4}{7^2}, & +\frac{4.4}{8^2}, & -\ldots, \\[6pt]
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots\,.
\end{array}$$

He shows that† $\sum\limits_{(i)(j)} a_{i,j} = \sum\limits_{(j)(i)} a_{i,j} = \frac{1}{6}(\log 2 - \frac{1}{4}) = l,$

say, the method employed being, essentially, the same as that used below.

Before proceeding to the general discussion, we shall evaluate $f(x, x)$; now here $|a_{ij}| \leqslant \frac{1}{4}$, so that the series for $f(x, y)$ is absolutely convergent. Thus, we may write

$$f(x, x) = \sum_{n=2}^{\infty} x^n \left(\sum_{i=1}^{n-1} a_{i, n-i} \right).$$

But $$\sum_{i=1}^{n-1} a_{i, n-i} = (-1)^n \sum_{i=1}^{n-1} i(n-i)/n^2 = (-1)^n \tfrac{1}{6}(n-1/n),$$

and thus

$$f(x, x) = \tfrac{1}{6} \sum_{n=2}^{\infty} (n-1/n)(-x)^n = \tfrac{1}{6} \sum_{n=1}^{\infty} (n-1/n)(-x)^n$$
$$= \tfrac{1}{6}\left[\log(1+x) - x/(1+x)\right].$$

From this equation it is plain that

$$\lim_{x=1} f(x, x) = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l,$$

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* *Phil. Mag.*, April and August, 1853; *Reprint of Electrical Papers*, No. VI., Art. 140.

† It is of some interest to observe that it is the *repeated* summation which gives the correct expression for *the force between the spheres*. But this is NOT *the force between the two sets of images*; in fact, the latter force can only be regarded as $\lim s_{ij}$, where i, j approach infinity in such a way that i/j tends to the limit unity; but, as will be seen below, $\lim s_{ij}$ is then *not determinate.*

a result which has sometimes been used to evaluate the sum of Kelvin's series.*

Next, to find the general value of $f(x, y)$, we write

$$f(x, y) = \sum_{n=2}^{\infty} \left(\sum_{i=1}^{n-1} a_{i,\,n-i} x^i y^{n-i} \right);$$

but

$$\sum_{i=1}^{n-1} a_{i,\,n-i} x^i y^{n-i} = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \, \partial y} \left(\sum_{i=0}^{n} x^i y^{n-i} \right) = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \, \partial y} \left(\frac{x^{n+1} - y^{n+1}}{x-y} \right)$$

$$= (-1)^n \frac{1}{n^2} \left[(n+1) \frac{x^n + y^n}{(x-y)^2} - 2 \frac{x^{n+1} - y^{n+1}}{(x-y)^3} \right].$$

It will be observed that this expression is identically zero for $n = 1$, and so the summation may be extended to include $n = 1$; then we have

$$(x-y)^3 f(x, y) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} [(n+1)(x-y)(x^n + y^n) - 2(x^{n+1} - y^{n+1})]$$

$$= \sum_{n=1} (-1)^{n-1} \frac{1}{n^2} [(x+y)(x^n - y^n) - n(x-y)(x^n + y^n)].$$

If we introduce the function $\phi(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n / n^2$, it is clear that

$$\phi'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}/n = \frac{1}{x} \log(1+x);$$

and then

$$(x-y)^3 f(x, y) = (x+y)[\phi(x) - \phi(y)] - (x-y)[x\phi'(x) + y\phi'(y)].$$

If we write, for the moment,

$$\xi = \tfrac{1}{2}(x+y), \qquad \eta = \tfrac{1}{2}(x-y),$$

it will be found (after some reductions which are tedious, but not difficult) that

$$f(x, y) = -\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R$$

where

$$|R| \leqslant (\tfrac{5}{2}|\xi| + 2|\eta|)\lambda < \tfrac{9}{2}\lambda,$$

λ being the greatest value of $|\phi^{\text{iv}}(\zeta)|$ when ζ takes all values from x to y, inclusive.

Thus
$$\lim_{(x,\,y)} f(x, y) = \lim_{\xi=1,\ \eta=0} [-\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R]$$

$$= -\tfrac{1}{6} \lim_{\xi=1} [\phi'''(\xi) + 3\phi''(\xi)]$$

$$= -\tfrac{1}{6}[(2\log 2 - \tfrac{5}{4}) + 3(\tfrac{1}{2} - \log 2)] = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l;$$

and it is clear that l is also the value of the two repeated limits

$$\lim_{(x)(y)} f(x, y) \qquad \text{and} \qquad \lim_{(y)(x)} f(x, y).$$

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* For example, by Prof. Tarleton, in his book on *Attractions* (Ex. 9, p. 279), where the result is obtained by processes which can hardly be justified.

Next we consider the value of $s_{m,n}$; and, to find this, use the theorem

$$(i+j)^{-2} = \int_0^\infty e^{-(i+j)t}\, t\,dt\,;$$

so that

$$a_{i,j} = (-1)^{i+j} \int_0^\infty ij\, e^{-(i+j)t}\, t\,dt.$$

Hence

$$s_{m,n} = \int_0^\infty t\,dt \left[\sum_{i,j=0}^{m,n} (-1)^{i+j} ij\, e^{-(i+j)t} \right] = \int_0^\infty \phi(m,t)\,\phi(n,t)\,\frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4},$$

where

$$\phi(m,t) = 1+(-1)^{m-1}\left\{(m+1)\,e^{-mt}+me^{-(m+1)t}\right\}.$$

Now

$$\int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty \frac{me^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty me^{-mt}\,t\,dt = \frac{1}{m},$$

and accordingly

$$\lim_{m=\infty} \int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} = 0 = \lim_{m=\infty} \int_0^\infty \frac{(m+1)e^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4}.$$

Similarly

$$\lim_{m=\infty} \int_0^\infty \frac{m(n+1)\,e^{-(m+n+3)t}\,t\,dt}{(1+e^{-t})^4} = 0,$$

and so on; and hence

$$\lim_{(n)(m)} s_{m,n} = \int_0^\infty \frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4} = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l,$$

the value of the integral being obtained by direct integration.* In the same way,

$$\lim_{(m)(n)} s_{m,n} = l.$$

We have thus obtained an illustration of part of the theorem given in § 5; for we have proved directly that

$$\sum_{(i)(j)} a_{i,j} = \lim_{(x)(y)} f, \qquad \sum_{(j)(i)} a_{i,j} = \lim_{(y)(x)} f.$$

However, the double series $\sum_{(i,j)} a$ is not convergent, in spite of the fact

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* The indefinite integral is

$$\tfrac{1}{6}\left[\frac{e^t}{(1+e^t)^2} - \frac{t(1+3e^t)}{(1+e^t)^3} - \log(1+e^{-t})\right].$$

This is the method employed by Kelvin, *loc. cit.*

that $\lim\limits_{(x,\,y)} f$ is perfectly determinate. For

$$\lim_{m=\infty}\int_0^\infty \frac{m^2e^{-mt}\,t\,dt}{(1+e^{-t})^4} = \lim_{m=\infty}\int_0^\infty \frac{e^{-x}\,x\,dx}{(1+e^{-x/m})^4} = \tfrac{1}{16}\int_0^\infty e^{-x}\,x\,dx = \tfrac{1}{16},^*$$

from which it easily follows that

$$\lim_{m=\infty} s_{m,\,m+1} = l-\tfrac{1}{16}, \qquad \lim_{m=\infty} s_{m,\,m} = l+\tfrac{1}{16}.$$

It is not difficult to prove that these are the general values of

$$\underline{\lim}_{(m,\,n)}\, s_{m,\,n} \qquad \text{and} \qquad \overline{\lim}_{(m,\,n)}\, s_{m,\,n}.$$

If m, n tend to infinity in such a way that $\lim (m/n) = 1$, $s_{m,\,n}$ oscillates between these values; if in such a way that $\lim (m/n) = 0$ or ∞, $s_{m,\,n}$ tends to the determinate limit l.

It will be seen that, in agreement with § 5,

$$\underline{\Sigma}_{(i,\,j)}\, a_{i,\,j} < \lim_{(x,\,y)} f < \overline{\Sigma}_{(i,\,j)}\, a_{i,\,j}.$$

Next, if we form the arithmetic mean of $s_{m,\,n}$, it will be found that

$$s^{(1)}_{m,\,n} = \int_0^\infty \psi(m,\,t)\,\psi(n,\,t)\,\frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4},$$

where $\psi(m,\,t) = 1+\dfrac{(-)^{m-1}\,me^{-(m+1)t}}{m+1}+\dfrac{2}{m+1}\;\dfrac{e^{-t}+(-)^{m-1}e^{-(m+1)t}}{1+e^{-t}}.$

This gives at once

$$\lim_{(m,\,n)} s^{(1)}_{m,\,n} = \int_0^\infty \frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4} = l = \lim_{(x,\,y)} f(x,\,y);$$

and, to verify the condition of finitude, we observe that, since $|\psi(m,\,t)| < 4$,

$$|s^{(1)}_{m,\,n}| < 16\int_0^\infty e^{-2t}\,t\,dt \text{ for all values of } m,\,n,$$

or

$$|s^{(1)}_{m,\,n}| < 4.$$

Thus the equation $\qquad \lim\limits_{(m,\,n)} s^{(1)}_{m,\,n} = \lim\limits_{(x,\,y)} f(x,\,y)$

is in complete agreement with the theorem proved in § 8.

From the preceding work it is clear that there is no justification for *assuming* the equation

$$\sum_{(i)(j)} a_{i,\,j} = \sum_{(j)(i)} a_{i,\,j} = \lim_{x=1} f(x,\,x),$$

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* It is easy to see that the conditions given by Bromwich (*l.c.*, p. 201) for this inversion of limits are satisfied.

until we have proved (i.) that the *repeated* sums $\sum\limits_{(i)(j)} a_{i,j}$, $\sum\limits_{(j)(i)} a_{i,j}$ are convergent; and (ii.) that the *double* limit $\lim\limits_{(m,n)} s^{(1)}_{m,n}$ is determinate, in addition to verifying the condition of finitude.

It follows that this method of evaluating the repeated sums is really far more complicated than Kelvin's direct method of summation; although, superficially, the former method appears to be the easier.

10. *Extension to Repeated (Two-fold) Series of the Theorems of Frobenius and Hölder.*

Returning to the notation of § 4, suppose that the limit

$$\lim_{j=\infty} b_{i,j}$$

does not exist; it may then happen that the arithmetic means of $b_{i,j}$, namely,

$$b^{(1)}_{i,j} = \frac{1}{j+1} \sum_{n=0}^{j} b_{i,n},$$

approach a limit $b^{(1)}_i$; so that

$$\lim_{j=\infty} b^{(1)}_{i,j} = b^{(1)}.$$

Suppose further that the condition of finitude is satisfied in the form

$$|b^{(1)}_{i,j}| < C, \quad \text{for all values of } i, j;$$

it follows that the two series

$$\sum_{(i)} b^{(1)}_{i,j} x^i, \qquad \sum_{(i)} b^{(1)}_i x^i$$

are absolutely convergent. The same is true of the series

$$\sum a_{i,j} x^i,$$

since $$b_{i,j} = \Delta_{(j)} [j b^{(1)}_{i,j-1}], \qquad a_{i,j} = \Delta_{(j)} [b_{i,j-1}];$$

so that $$|b_{i,j}| < 2C(j+1), \qquad |a_{i,j}| < 4C(j+1).$$

Now write

$$X_j = \sum_{l=0}^{j} \sum_{(i)} a_{i,l} x^i = \sum_{(i)} b_{i,j} x^i$$

and $$X^{(1)}_j = \frac{1}{j+1} (X_0 + X_1 + X_2 + \ldots + X_j).$$

Then plainly

(20) $$X_j^{(1)} = \sum_{(i)} b_{i,j}^{(1)} x^i.$$

But, by the process adopted in proving the last equation of § 4, it follows that*

$$\lim_{j=\infty} \sum_{(i)} b_{i,j}^{(1)} x^i = \sum_{(i)} b_i^{(1)} x^i,$$

and so, from (20), we find

(21) $$\lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

Now it has been proved that

$$| a_{i,j} | < 4C(j+1),$$

and consequently $\Sigma a_{i,j} x^i y^j$ is absolutely convergent, its terms being less numerically than those in the expansion of $4C(1-x)^{-1}(1-y)^{-2}$. Thus

$$f(x, y) = \sum_{(j)} y^j \sum_{(i)} a_{i,j} x^i.$$

Frobenius's theorem can be applied to this series: and, in virtue of equation (21), it follows that

$$\lim_{(y)} f(x, y) = \lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

If now either the series $\sum_{(i)} b_i^{(1)}$ *converges to a sum* l, *or if the arithmetic mean process applied to* $b_i^{(1)}$ *gives a definite limit* l, *then*

$$\lim_{(x)(y)} f(x, y) = \lim_{(x)} \sum_{(i)} b_i^{(1)} x^i = l,$$

a result which follows at once from Abel's (or Frobenius's) theorem.

Obviously a similar method can be used to find the limit

$$\lim_{(y)(x)} f(x, y),$$

the necessary modifications being made in the hypotheses.

As an illustration, take the series given by

$$a_{i,j} = (-1)^{i+j},$$

---

* In § 4, the condition of finitude was stated in a slightly different form; but a glance at the proof will show that $| b_{i,j}^{(1)} | < C$ is sufficient for the truth of the conclusion.

which has the scheme

$$\begin{array}{ccccc} +1, & -1, & +1, & -1, & \ldots, \\ -1, & +1, & -1, & +1, & \ldots, \\ +1, & -1, & +1, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots . \end{array}$$

In this case $b_{i,j} = 0$, if j is odd; and $b_{i,j} = (-1)^i$, if j is even.

Hence $b_i^{(1)} = \lim\limits_{j=\infty} b_{i,j}^{(1)} = \frac{1}{2}(-1)^i$, and $|b_{i,j}^{(1)}| < 1$ for all values of i, j. Thus

$$\lim_{(y)} f(x, y) = \sum_{(i)} \tfrac{1}{2}(-1)^i x^i.$$

The series $\sum\limits_{(i)} \frac{1}{2}(-1)^i$ does not converge, but the arithmetic mean process leads to the limit $\frac{1}{4}$; so that

$$\lim_{(x)(y)} f(x, y) = \tfrac{1}{4},$$

which may be immediately verified, since $f(x, y) = (1+x)^{-1}(1+y)^{-1}$. In this case, as a matter of fact, the theorem of § 8 can be applied; for $s_{i,j} = 1$, if both i and j are even, while $s_{i,j} = 0$ in every other case. Thus

$$\lim_{(i,j)} s_{i,j}^{(1)} = \tfrac{1}{4},$$

and so

$$\lim_{(x,y)} f(x, y) = \tfrac{1}{4}.$$

It is clear that the method used in this paragraph is capable of immediate extension to any case in which a *finite* number* of arithmetic means must be taken in order to obtain a limit from each column of the scheme. A corresponding change must be made in the condition of finitude. Then, if the limits so found from the columns either form a convergent series with the sum l, or lead to a limit l after a finite number of arithmetic means, the equation

$$\lim_{(x)(y)} f(x, y) = l$$

is true.

A simple example which we do not pause to work out in detail is given by

$$a_{i,j} = (-)^{i+j}(i+1)^p (j+1)^q,$$

* This number may vary with i, so long as it has a finite maximum. This is clear, in consequence of a theorem proved in § 7, according to which, if a limit is obtained from an arithmetic mean of any order, the *same* limit will belong to all the subsequent arithmetic means.

or, more generally,

$$a_{i,j} = (i+1)^p (j+1)^q \exp\{(i\theta + j\phi)\sqrt{(-1)}\}.$$

11. *Extension of Hölder's Theorems to Double Series: Double Limit.*

Continuing the notation of equation (12), let us write

$$s_{m,n} = \sum_{i,j=0}^{m,n} a_{i,j},$$

$$s^{(1)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j},$$

(22) $$s^{(2)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(1)}_{i,j},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots,$$

$$s^{(k)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(k-1)}_{i,j}.$$

Suppose that the condition of finitude

$$|s^{(k)}_{i,j}| < C$$

is verified for all values of i, j; then, by a process analogous to that used in (14) and (15), we deduce

$$|a_{i,j}| < 4^{k+1}(i+1)^k(j+1)^k C,$$

(23) $$|s_{i,j}| < 4^k (i+1)^k (j+1)^k C,$$

$$|s^{(k-r)}_{i,j}| < 4^r (i+1)^r (j+1)^r C \quad (r = 0, 1, 2, \ldots, k-1).$$

From (23) it is clear that each of the series

$$\Sigma a_{i,j} x^i y^j, \quad \Sigma s_{i,j} x^i y^j, \quad \Sigma s^{(r)}_{i,j} x^i y^j \quad (r = 1, 2, \ldots, k)$$

is absolutely convergent; since their terms are numerically less than the corresponding terms in $4^{k+1}(k!)^2 C(1-x)^{-(k+1)}(1-y)^{-(k+1)}$.

We prove next the following preliminary lemma:—

Assuming the truth of the equation

(24) $$\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s^{(r)}_{i,j} x^i y^j = l,$$

where ϕ is a polynomial of the form

$$\phi(i,j) = \frac{i^p}{p!}\frac{j^q}{q!} + \text{terms of lower degree},$$

then also

$$(25)\qquad \lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s_{i,j}^{(r-1)} x^i y^j = l,$$

provided that (24) *is valid for all integers* p, q.

To prove the lemma, we use the identity

$$s_{i,j}^{(r-1)} = \Delta_{(i,j)} [ij\, s_{i-1,\,j-1}^{(r)}],$$

which gives

$$\begin{aligned}(26)\qquad \sum_{(i,j)} \phi(i,j) s_{i,j}^{(r-1)} x^i y^j = (1-x)(1-y) &\sum_{(i,j)} (i+1)(j+1)\,\phi(i,j)\, s_{i,j}^{(r)} x^i y^j \\ -x(1-y) &\sum_{(i,j)} (i+1)(j+1)[\Delta_{(i)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j \\ -y(1-x) &\sum_{(i,j)} (i+1)(j+1)[\Delta_{(j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j \\ +xy &\sum_{(i,j)} (i+1)(j+1)[\Delta_{(i,j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j.\end{aligned}$$

But the polynomials appearing in these series are of the forms

$$(i+1)(j+1)\phi(i,j) = (p+1)(q+1)\frac{i^{p+1}}{(p+1)!}\,\frac{j^{q+1}}{(q+1)!} + \text{lower terms},$$

$$(i+1)(j+1)[\Delta_{(i)}\phi(i,j)] = p(q+1)\frac{i^p}{p!}\,\frac{j^{q+1}}{(q+1)!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(j)}\phi(i,j)] = (p+1)q\,\frac{i^{p+1}}{(p+1)!}\,\frac{j^q}{q!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(i,j)}\phi(i,j)] = pq\,\frac{i^p}{p!}\,\frac{j^q}{q!} + \cdots.$$

Thus, in virtue of (24), we find

$$\lim_{(x,y)} (1-x)^{p+2}(1-y)^{q+2} \sum_{(i,j)} (i+1)(j+1)\phi(i,j)\, s_{i,j}^{(r)} x^i y^j = (p+1)(q+1)\,l,$$

$$\lim_{(x,y)} x(1-x)^{p+1}(1-y)^{q+2} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(i)}\phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = p(q+1)\,l$$

$$\lim_{(x,y)} (1-x)^{p+2} y(1-y)^{q+1} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(j)}\phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = (p+1)q\,l,$$

$$\lim_{(x,y)} x(1-x)^{p+1} y(1-y)^{q+1} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(i,j)}\phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = pq\,l.$$

Combining the last four equations with equation (26), we see that

$$\begin{aligned}&\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s_{i,j}^{(r-1)} x^i y^j \\ &\qquad = [(p+1)(q+1)-p(q+1)-(p+1)q+pq]\,l = l,\end{aligned}$$

and this is equation (25). Thus the lemma is proved.

It is now clear that, *if the equation*

$$(27) \qquad \lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}^{(k)} x^i y^j = l$$

is true for all integers p, q and for any particular integer k, then also the equation

$$(28) \qquad \lim_{(x,\,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,\,j)} \phi(i,\,j)\, s_{i,j}\, x^i y^j = l$$

is true.

We shall now establish the truth of (27), on the hypothesis that

$$\lim_{(i,\,j)} s_{i,j}^{(k)} = l.$$

Let us write for brevity

$$\psi(i,\,j) = [(i+1)(i+2)\dots(i+p)(j+1)(j+2)\dots(j+q)]/p!\,q!,$$

so that

$$\lim_{(i,\,j)} (\phi/\psi) = 1.$$

An integer N can now be found, corresponding to any assigned positive number σ, such that

$$|(\phi/\psi)\, s_{i,j}^{(k)} - l| < \sigma, \quad \text{if} \quad i,\, j \geqslant N.$$

Further, a number g can be found such that

$$|\phi/\psi| < g, \quad \text{for all values of } i,\, j;$$

and so, using the condition of finitude,

$$|\phi\, s_{i,j}^{(k)}| < gC\psi, \quad \text{for all values of } i,\, j,$$

and

$$|l| \leqslant C;$$

so that

$$|\phi\, s_{i,j}^{(k)} - l\psi| < (g+1)\,C\psi.$$

Now

$$\sum_{(i,\,j)} (\phi s_k - l\psi)\, x^i y^j = \sum_{i,\,j=0}^{N-1} + \sum_{i=0}^{N-1}\sum_{j=N}^{\infty} + \sum_{j=0}^{N-1}\sum_{i=N}^{\infty} + \sum_{i,\,j=N}^{\infty}$$

and

$$\left|\sum_{i,\,j=0}^{N-1}\right| < (g+1)\,C \sum_{i,\,j=0}^{N-1} \psi < (g+1)\,C\,\frac{(N+p)^{p+1}\,(N+q)^{q+1}}{(p+1)!\,(q+1)!},$$

$$\left|\sum_{i=0}^{N-1}\sum_{j=N}^{\infty}\right| < (g+1)\,C \sum_{i=0}^{N-1}\sum_{j=N}^{\infty} \psi y^j < (g+1)\,C\,\frac{(N+p)^{p+1}}{(p+1)!}\,(1-y)^{-(q+1)},$$

$$\left|\sum_{i=N}^{\infty}\sum_{j=0}^{N-1}\right| < (g+1)\,C \sum_{j=0}^{N-1}\sum_{i=N}^{\infty} \psi x^i < (g+1)\,C\,\frac{(N+q)^{q+1}}{(q+1)!}\,(1-x)^{-(p+1)},$$

$$\left|\sum_{i,\,j=N}^{\infty}\right| < \sigma \sum_{i,\,j=N}^{\infty} \psi x^i y^j < \sigma (1-x)^{-(p+1)}\,(1-y)^{-(q+1)}.$$

Hence we deduce

$$|(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s_{i,j}^{(k)}-l\psi)x^iy^j|$$
$$<\sigma+(q+1)C\left[\frac{(N+p)^{p+1}(N+q)^{q+1}}{(p+1)!\,(q+1)!}(1-x)^{p+1}(1-y)^{q+1}\right.$$
$$\left.+\frac{(N+p)^{p+1}}{(p+1)!}(1-x)^{p+1}+\frac{(N+q)^{q+1}}{(q+1)!}(1-y)^{q+1}\right],$$

and we can choose δ so that the right-hand side of this inequality is less than 2σ, provided that $1-x$, $1-y$ are each less than δ. Hence

$$\lim_{(x,y)}(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s_{i,j}^{(k)}-\psi l)x^iy^j=0.$$

But
$$(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}\psi l x^iy^j=l,$$

and equations (27), (28) follow at once.

If we now take in (28) the special values*

$$\phi(i,j)=1,\qquad p=0,\qquad q=0,$$

it will be seen that

$$\lim_{(x,y)}(1-x)(1-y)\sum_{(i,j)}s_{i,j}x^iy^j=l,$$

or, using equation (8),
$$\lim_{(x,y)}f(x,y)=l.$$

Thus the following theorem has been established:—

If, for all values of i, j, $|s_{i,j}^{(k)}|<C$, and if

$$\lim_{(i,j)}s_{i,j}^{(k)}=l,$$

then also
$$\lim_{(x,y)}f(x,y)=l.$$

This is the general extension of Hölder's theorem to double series; the method can be easily modified so as to include the possibility that $s_{i,j}^{(k)}$ may oscillate; the result is then

$$\varliminf_{(i,j)}s_{i,j}^{(k)}\leqslant\varliminf_{(x,y)}f(x,y)\leqslant\varlimsup_{(x,y)}f(x,y)\leqslant\varlimsup_{(i,j)}s_{i,j}^{(k)}.$$

12. *The General Theorem.*

We proceed now to the proof of the general theorem stated in § 2. It has been already pointed out that the argument of § 3 applies to the

* This appears to be the only case of practical importance, but the introduction of this specialization earlier does not materially simplify the work.

general case when it is the convergence of the multiple series proper

$$\sum_{(1,\,2,\,\ldots,\,n)} a$$

which is given. To prove the theorem in its most general form it is convenient to proceed by induction. We shall adopt the following contracted notation. We denote the *groups* of suffixes $(i_1, i_2, \ldots, i_p)$, $(i_{p+1}, \ldots, i_q)$, ..., $(i_{r+1}, \ldots, i_n)$ by (α), (β), ..., (μ); so that the series summed in the manner explained at the top of p. 162 will be written as

(29) $$\sum_{(\alpha)(\beta)\ldots(\mu)} a.$$

Further, by $\sum\limits_{\alpha=0}^{I} a$, we denote the sum in which i_1 ranges from 0 to I_1, i_2 from 0 to I_2, ..., i_p from 0 to I_p, and by $x^{(\alpha)}$ we denote $x_1^{i_1} x_2^{i_2} \ldots x_p^{i_p}$.

Let us then assume (i.) that the condition of finitude is satisfied, (ii.) that the series (29) is convergent, and (iii.) that the theorem holds in its most general form for any number of indices less than n. Let

(30) $$s_\alpha = \sum_{(\beta)\ldots(\mu)} a.$$

Then, since $$\sum_{\beta,\,\ldots,\,\mu=0}^{m} a = \Delta_{(\alpha)}\, s_{i_1-1,\,\ldots,\,i_p-1,\,m_{p+1},\,\ldots,\,m_n},$$

it follows, from the condition of finitude, that

(31) $$|\, s_\alpha \,| \leqslant 2^p C$$

and that

(32) $$\sum_{(\alpha)} s_\alpha x^{(\alpha)}$$

is absolutely convergent. And, since

$$|\, a_{i_1,\,i_2,\,\ldots,\,i_n} \,| = |\, \Delta_{(1,\,2,\,\ldots,\,n)}\, s_{i_1-1,\,\ldots,\,i_n-1} \,| < 2^n C,$$

the series

(33) $$\sum_{(\alpha)} a x^{(\alpha)}$$

is also absolutely convergent. We shall prove further that

(34) $$\sum_{(\beta)\ldots(\mu)} \sum_{(\alpha)} a x^{(\alpha)}$$

is convergent and equal to (32).

13.

Our first step will be to prove that

$$(35) \qquad \sum_{(\mu)} \sum_{(a)} a x^{(a)}$$

is convergent and equal to

$$(36) \qquad \sum_{(a)} x^{(a)} \sum_{(\mu)} a,$$

which is convergent for the same reasons as (32) and (33).

Let

$$\sum_{\mu=0}^{m} a = b_{a,\, m}$$

and

$$\sum_{\mu=0}^{\infty} a = \lim_{(m)} b_{a,\, m} = b_a \text{ *}$$

(m of course being a *group* of suffixes). We have to prove that

$$\lim_{(m)} \sum_{(a)} (b_{a,\, m} - b_a) x^{(a)} = 0.$$

Now

$$\left| \left(\sum_{a=0}^{\infty} - \sum_{a=0}^{I-1} \right) (b_{a,\, m} - b_a) x^a \right|$$
$$< 2^{r+1} C \frac{1-(1-x_1^{I_1})(1-x_2^{I_2}) \dots (1-x_p^{I_p})}{(1-x_1)(1-x_2) \dots (1-x_p)} < 2^{r+1} C \frac{x_1^{I_1} + \dots + x_p^{I_p}}{(1-x_1) \dots (1-x_p)},$$

since $\qquad | b_{a,\, m} - b_a | < 2^{r+1} C.$

We can choose I so that this is $< \sigma$. Then, I being fixed, we can choose M so that $| b_{a,\, m} - b_a | < \sigma / I_1 I_2 \dots I_p$ for all values of $(m) \geqslant M$, and all values of $(a) \leqslant I$; thus

$$\left| \sum_{(a)}^{I-1} (b_{a,\, m} - b_a) x^{(a)} \right| < \sigma \quad \text{and} \quad \left| \sum_{(a)} (b_{a,\, m} - b_a) x^{(a)} \right| < 2\sigma.$$

Hence (35) is convergent and equal to (36).

14.

This argument can now be repeated. Suppose that (λ) is the group of suffixes immediately preceding (μ). We have to show that

$$(37) \qquad \sum_{(\lambda)(\mu)} \sum_{(a)} a x^{(a)}$$

---

* The existence of this limit is, of course, implied in our data.

is convergent and equal to

(38) $$\sum_{(\alpha)} x^{(\alpha)} \sum_{(\lambda)(\mu)} a,$$

which is convergent for the same reasons as the series (30), (33), and (36). To prove this we have only to observe that (37) may (after § 13) be written in the form

$$\sum_{(\lambda)} \sum_{(\alpha)} x^{(\alpha)} \sum_{(\mu)} a$$

and that a repetition of the preceding argument with $\sum_{(\mu)} a$ in place of a proves that this is convergent and equal to (38).

By repeating this line of argument as often as may be necessary we conclude finally that (34) is convergent and equal to (32).

15.

We are now in a position to prove the theorem. For

$$\lim_{(\beta)\ldots(\mu)} f = \sum_{(\beta)\ldots(\mu)} \sum_{(\alpha)} a x^{(\alpha)}$$

(since the theorem holds for any number of indices less than n) and therefore is equal to $\sum_{(\alpha)} x^{(\alpha)} \sum_{(\beta)\ldots(\mu)} a$ (by §§ 13, 14). Hence, by a further application of the theorem for p indices,

$$\lim_{(\alpha)(\beta)\ldots(\mu)} f = \lim_{(\alpha)} \sum_{(\alpha)} x^{(\alpha)} \sum_{(\beta)\ldots(\mu)} a = \sum_{(\alpha)(\beta)\ldots(\mu)} a.$$

The theorem is therefore true for n indices if it is true for any number less than n; and therefore it is true generally.

16. *Multiplication of Series.*

It is well known that from Abel's theorem we can at once deduce that, *if the three series*

$$\sum a_i, \quad \sum b_i, \quad \sum c_i,$$

where $$c_i = \sum_{(k+l=i)} a_k b_l,$$

are convergent, the third series is the product of the other two. We have in fact only to make the first two series absolutely convergent by introducing a factor x^i in each term, to multiply the resulting power series, and to proceed to the limit.

By an exactly similar process we deduce from the theorem proved in § 15 that, *if the three series*

$$\sum a_{i_1, i_2, \ldots, i_n}, \quad \sum b_{i_1, i_2, \ldots, i_n}, \quad \textit{and} \quad \sum c_{i_1, i_2, \ldots, i_n},$$

where $$c_{i_1, i_2, \ldots, i_n} = \sum_{(k_1+l_1 = i_1, \ldots, k_n+l_n = i_n)} a_{k_1, \ldots, k_n} b_{l_1, \ldots, l_n},$$

satisfy the condition of finitude and are convergent when summed in the same way (*e.g.*, *in the way specified by* $\sum_{(1, 2, \ldots, p)(p+1, \ldots, q)\ldots(r+1, \ldots, n)}$), *then the third series is the product of the first two.*

Of course similar theorems can be proved for the product of any number of series.

17. *Mean Value Theorems for the General Series.*

It is easy to prove by the method of § 11 that, if $s^{(k)}_{i_1, \ldots, i_n}$ is the k-th arithmetic mean of $s_{i_1, \ldots, i_n}$, and $|s^{(k)}| < C$ for all suffixes, and

$$\lim_{(1, 2, \ldots, n)} s^{(k)} = s,$$

then

$$\lim_{(1, 2, \ldots, n)} f = s.$$

The form of the arithmetic mean theorem corresponding to the general theorem of §§ 11–15 is as follows:—

Let Σ' denote that a series is "summed" by taking any finite number of arithmetic means. *Suppose that*

$$\Sigma'_{(\alpha)} \Sigma'_{(\beta)} \ldots \Sigma'_{(\mu)} a$$

is determinate and equal to s, *and that a number* C *can be assigned such that the various quantities which we pass through before we arrive at* s *are all less than* C; *then*

$$\lim_{(\alpha)(\beta)\ldots(\mu)} f = s.$$

CORRECTIONS

p. 164, *line* 2 *up*. After 'absolutely convergent' add 'for $|x| < 1$, $|y| < 1$'.

p. 166, *line* 10 *up*. For 'series' read 'power series'.

p. 171, 1*st footnote, line* 2 *up*. At the end of the line, read $i_{n_2}, i_{n_4}, i_{n_6}, \ldots$.

—— 2*nd footnote, line* 2 *up*. After 'proved that if' add '$\sum a_n x^n$ converges for $0 < x < 1$ and'.

p. 173, *line* 3. The first sum should be $\sum\limits_{i,j=0}^{m,n}$.

p. 184, *line* 5 *up*. For s^k read $s^{(k)}_{i,j}$.

COMMENTS

The general result stated in § 2 (with the notation of § 1) is proved for double and repeated series in §§ 3–4 and for multiple series in §§ 12–15. The theorem shows that, in a multiple series, the grouping and order in which the coordinates of a rectangular partial sum tend to infinity, so as to obtain a limit, is imitated by the grouping and order in which the variables of the related multiple power series may tend along the principal radii so as to obtain an Abel limit.

The remark at the end of § 6 and the example constructed in the footnote (stated to be due to Hardy) show the beginning of Hardy's interest in the relations between the oscillation of the partial sum of an ordinary series and the oscillation of the sum of the related power series. In 1907, 5 and 1907, 6, he gives as further examples the 'gap' power series $\sum(-1)^m x^{a^m}$ ($a = 2, 3,\ldots$), whose sums oscillate as $x \to 1-$. In 1910, 3, he gives the series $\sum(n+1)^{-1-ai}$ (a real and $\neq 0$), and quotes from 1905, 6 (in Vol. IV), where other references are given, the result that

$$\sum(n+1)^{-1-ai}x^n \sim \Gamma(-ai)(\log 1/x)^{ai} \quad \text{as } x \to 1-.$$

These examples have the common feature that the partial sum s_n is bounded, and the mean $(s_0+\ldots+s_n)/(n+1)$ does not tend to a limit. Littlewood, in his paper of 1911 on the 'converse of Abel's theorem',† stated a theorem which, when combined with Frobenius's theorem, says that: *if* $s_n = O(1)$, *then* $f(x) \to s$ *if and only if* $s_n \to s$ $(C, 1)$. This is a corollary of Hardy and Littlewood's 'positive' Tauberian theorem, 1914, 4; see D.S., p. 155. A short deduction of Littlewood's statement from his $O(1/n)$ Tauberian theorem (loc. cit.) is given in D.S., p. 162; see also 1931, 8.

The method of summability used in the extension of Hölder's theorem to double series is summability (H, k, k). It has been proved by Adams‡ that summability (H, r, s) is equivalent to summability (C, r, s) for the class of series with bounded (C, r, s) (or (H, r, s)) transforms. Nothing was known about the relation between (H, k) and (C, k) summability before 1907 (see the Comments on 1907, 6).

In §§ 16–17 some further general results are indicated—in particular, an extension to multiple series of Abel's multiplication theorem. For an addendum see 1905, 3.

† *Proc. London Math. Soc.* (2), 9 (1911), 434–48.

‡ *Trans. American Math. Soc.* 34 (1932), 215–30.

NOTE IN ADDITION TO A FORMER PAPER ON CONDITIONALLY CONVERGENT MULTIPLE SERIES

By G. H. Hardy.

[Received March 23rd, 1904.—Read April 14th, 1904.]

In a paper which appeared recently in these *Proceedings** I proved the convergence of a general class of n-ple series, of which

$$\Sigma \frac{\begin{matrix}\cos\\ \sin\end{matrix}(i_1\theta_1+i_2\theta_2+\ldots+i_n\theta_n)}{(i_1a_1+i_2a_2+\ldots+i_na_n)^\rho}$$

is typical. Here $a_1, a_2, \ldots, a_n, \rho$ are all real and positive, and no one of $\theta_1, \ldots, \theta_n$ is a multiple of 2π. In that paper I was concerned entirely with *proper multiple series*; series of the type which, according to the notation developed by Prof. Bromwich and myself in the preceding paper, would be denoted by $\underset{(1,2,\ldots,n)}{\Sigma}$.

I wish in this note to point out that all these series are convergent also when summed according to the type $\underset{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)}{\Sigma}$ or $\underset{(\alpha)(\beta)\ldots(\mu)}{\Sigma}$. This follows at once from the following lemma, which is an obvious extension of a lemma proved by Pringsheim for double series.

Lemma.—*The quantity* $\overline{\lim_{(1,2,\ldots,n)}}\, s_{i_1, i_2, \ldots, i_n}$

is not increased, and the quantity

$$\underline{\lim_{(1,2,\ldots,n)}}\, s_{i_1, i_2, \ldots, i_n}$$

is not decreased, by replacing the single bracket $(1, 2, \ldots, n)$ *by any system of brackets* $(\alpha)(\beta)\ldots(\mu)$.

To prove this it is evidently enough to prove that

$$\overline{\lim_{(1,2,\ldots,n)}}\, s \geqslant \overline{\lim_{(1,2,\ldots,p)}}\ \overline{\lim_{(p+1,\ldots,n)}}\, s \qquad \text{or} \qquad \geqslant \overline{\lim_{(\alpha)}}\ \overline{\lim_{(\beta)}}\, s,$$

say. Denote the quantity on the left by L; then, however small be σ, we can determine I so that if $i > I$ then $s < L+\sigma$.

* "On the Convergence of Certain Multiple Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 124.

Making (β) tend to infinity, we deduce $\lim\limits_{(\beta)} s \leqslant L+\sigma$ for $(\alpha) > I$, and so $\lim\limits_{(\bar{\alpha})(\beta)} s \leqslant L$.

The lemma is therefore proved.

Now let a, u* be two systems of quantities satisfying the conditions of § 4 of my former paper. I proved there that $\sum\limits_{(1,\,\ldots,\,n)} au$ is convergent, and the same argument shows that $\sum\limits_{(\beta)} au$ is convergent. Now

$$\sum_{\overline{(1,\,2,\,\ldots,\,n)}} au = \sum_{\underline{(1,\,2,\,\ldots,\,n)}} au = S,$$

say. Hence, by the lemma, $\sum\limits_{(\bar{\alpha})(\beta)} au \leqslant S$ and also $\sum\limits_{(\underline{\alpha})(\beta)} au \geqslant S$. That is to say, $\sum\limits_{(\alpha)\,(\beta)} au$ is convergent and $= S$. Similarly, we can show (by repeating this argument a finite number of times) that, if we divide the indices into any number of groups $(\alpha)\,(\beta) \ldots (\mu)$, the resulting series is convergent. The most interesting special case of this theorem is that the series

$$\sum_1^\infty \sum_1^\infty \ldots \sum_1^\infty \frac{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (i_1\theta_1 + \ldots + i_n\theta_n)}{(i_1 a_1 + \ldots + i_n a_n)^\rho}$$

is convergent when the summations are carried out *successively*.

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* I write a for what was α in the former paper.

COMMENTS

This note is an addendum to both 1904, 1 and 1905, 2.

When all possible sums $\sum_{(\alpha)} u$ exist, where (α) denotes a subset (which may be the whole set) of the variables $i_1, \ldots, i_n$ and the (α)-sums are taken in the Pringsheim sense, the series

$$\sum_{i_1} \cdots \sum_{i_n} u_{i_1 \ldots i_n}$$

is said to be *regularly* (or *completely*) convergent.

Regular convergence for double series is defined in 1917, 3, and means that the three sums $\sum_{(m,n)} u_{mn}$, $\sum_{m} u_{mn}$, $\sum_{n} u_{mn}$ are convergent. The multiple form is given in Moore, § 1.20. Hardy shows here that, in the theorem of 1904, 1, § 4, the *convergence* of the final series may be replaced by *regular* convergence.

The general theorem stated in the Comments on 1904, 1 holds with *regular convergence* in the conclusion, in place of *convergence* or *bounded convergence*. Also, when condition (i)′ of the theorem holds, condition (ii)″ is equivalent to

$$\text{(ii)}''' \quad a_{i_1 \ldots i_n} \to 0 \text{ } \textit{regularly}.$$

For double series, this version was given by Hardy (sufficiency), in 1917, 3, and Kojima (necessity), with some redundant conditions. Hamilton gave all three versions of the theorem for double series. For references, see the Comments on 1917, 3.

Pringsheim† proved, for a real double sequence $u_{m,n}$, that (I):

$$\varliminf_{m,n} \leqslant \varliminf_{m} \varliminf_{n} \leqslant \varlimsup_{m} \varlimsup_{n} \leqslant \varlimsup_{m,n}.$$

In the general lemma here, it is to be understood that there are bars over or under the (α), (β),.... A corollary‡ of (I) is (II): *if* $\lim_{m,n}$ *exists, and* $\lim_{n}$ *exists for all m* (*or all large m*), *then* $\lim_{m,n} = \lim_{m} \lim_{n}$. The result required in the application is (in the notation of Bromwich and Hardy, 1905, 2, § 1): *if* $\sum_{(\alpha,\beta,\ldots,\gamma,\delta)}$, $\sum_{(\beta,\ldots,\gamma,\delta)}$,... $\sum_{(\gamma,\delta)}$, *and* $\sum_{(\delta)}$ *exist for all values of the remaining variables, then*

$$\sum_{(\alpha,\beta,\ldots,\gamma,\delta)} = \sum_{(\alpha)} \sum_{(\beta)} \cdots \sum_{(\gamma)} \sum_{(\delta)}.$$

A corollary is that: *if a multiple series is regularly convergent, then it converges* (*to the same sum*) *in every Bromwich–Hardy manner.*

In the last line of the first page, the phrase 'we can determine I so that if $i > I$' means 'we can determine $I = (I_1, \ldots, I_n)$ so that if $i_1 > I_1, \ldots, i_n > I_n$'.

† Pringsheim (1), *Math. Annalen* 53 (1900), 289–321.

‡ Stolz, *Math. Annalen* 24 (1884), 157–71; Pringsheim (1), and (2), *Sitz. d. K. Bayerischen Akad. d. Wiss.* 27 (1897), 101–52. Stolz had defined the 'Pringsheim' sum of a double series in 1884; see Pringsheim (3), *Encyk. Math. Wiss.* I, 1, A, 3, p. 98.

ON CERTAIN CONDITIONALLY CONVERGENT MULTIPLE SERIES CONNECTED WITH THE ELLIPTIC FUNCTIONS.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. In two earlier papers* I proved the convergence of a general class of multiple series of the form

$$\Sigma a_{n_1, n_2, \ldots, n_p} u_{n_1, n_2, \ldots, n_p},$$

of which the most interesting special cases were those obtained by taking

$$u_{n_1, n_2, \ldots, n_p} = \exp. \left(\sum_{\nu=1}^{p} \theta_\nu n_\nu \right),$$

and
$$a_{n_1, n_2, \ldots, n_p} = \phi \left(\sum_{\nu=1}^{p} a_\nu n_\nu \right),$$

$\phi(u)$ being a function of u which has zero for its limit for $u = \infty$ and continuous derivates $\phi'(u)$, $\phi^{(2)}(u)$, ..., $\phi^{(p)}(u)$ such that $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi'''(u) < 0$, ...; and $a_1, a_2, \ldots, a_p$ being real and positive, and $\theta_1, \theta_2, \ldots, \theta_p$ real and no multiples of 2π. The limits of summation for each variable are 1 and ∞. In particular, I proved that

$$(1) \quad \Sigma \frac{\exp. (i \Sigma \theta_\nu n_\nu)}{(u + \Sigma a_\nu n_\nu)^\beta}$$

is convergent if u is real and positive and $\beta > 0$, and the summation is effected according to any *type*†

$$(1, 2, \ldots, \lambda)(\lambda + 1, \ldots, \mu)\ldots(\tau + 1, \ldots, p).$$

In a note attached to the first paper above mentioned, I indicated that the restriction that β, u, a_ν should be real was unnecessary.

It is in fact possible to prove that if (i) the values of $u, a_1, \ldots, a$, real or complex, are such that we can assign a constant H, for which

$$| u + \Sigma a_\nu n_\nu | > H,$$

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* v. *Proc. Lond. Math. Soc.*, New Series, Vol. I., p. 124, and Vol. II., p. 190.

† For an explanation of this notation see a paper by Prof. Bromwich and myself, *Proc. Lond. Math. Soc.* (2), v. 2, p. 161.

1905, 14 Messenger of Mathematics, 34, 146–53.

and (ii) that value of the denominator of the general term is chosen, for which

$$(u + \Sigma\, a_\nu n_\nu)^\beta = \exp.\ \{\beta \log (u + \Sigma\, a_\nu n_\nu)\}$$

where $$-\pi \leq R\left[\frac{1}{i}\log(u + \Sigma\, a^\nu n_\nu)\right] < \pi,$$

then the series (1) is convergent.

I do not now propose to give a detailed proof of this, which must, from the nature of the case, be less concise than that which suffices in the case when β, u, a_ν are all real and positive. The nature of the arguments which must be employed will be made sufficiently clear by the discussion of a particular case which follows. This is a double series first studied by Kronecker and employed by him to obtain very beautiful results in the theory of elliptic functions.* Both Kronecker and Prof. Lerch, who has also considered the series, wrote about it before the appearance of Pringsheim's memoirs on the general theory of conditionally convergent series; and their discussions of its convergence are therefore much less simple than one based on Pringsheim's theory, and the generalisation of Abel's lemma proved in my paper quoted above. My object in this paper is to consider its convergence from the latter point of view. I have also summed the series by a method practically identical with Kronecker's, but so as obtain the sum directly in terms of the ordinary Weierstrassian elliptic functions.

§ 2. The series in question is

$$(2)\quad \Sigma\,\frac{e^{(2m\theta+2n\phi)\pi i}}{a + m\omega_2 + n\omega_1},$$

where the limits of summation are $-\infty$ and $+\infty$ for each variable, θ and ϕ are real, positive, and less than unity, ω_1 and ω_2 are any two complex quantities such that the imaginary part of $\tau = \omega_2/\omega_1$ is positive, and a is any quantity such that $a + m\omega_2 + n\omega_1$ does not vanish for any values of m and n. The series is obviously not absolutely convergent. I shall prove first that *the series* (2) *is a convergent double series in Pringsheim's sense.*

Although the series (2) is not precisely of the type (1), ω_2 and ω_1 being complex, and the limits different, the proof

* *vide* Kronecker, *Sitzungsberichte d. k. P. Ak.* (1890 and neighbouring years, *passim*, and especially 1890 (2), pp. 123 *et seq*); also Lerch, *Rozpravy Ceské Ak.*, 1895 and neighbouring years, *passim* (unfortunately written in Bohemian).

of its convergence is very similar to that which I gave of the convergence of (1). I shall therefore present the proof in as short a form as possible. Let

$$u_{m,n} = e^{(2m\theta+2n\phi)\pi i},$$

$$v_{m,n} = \frac{1}{a + m\omega_2 + n\omega_1}.$$

Then

$$(3)\quad \left(\sum_0^{M_1}\sum_0^{N_1} - \sum_0^{M}\sum_0^{N}\right) u_{m,n}v_{m,n}$$

$$= \left(\sum_0^{M}\sum_{N+1}^{N_1} + \sum_{M+1}^{M_1}\sum_0^{N} + \sum_{M+1}^{M_1}\sum_{N+1}^{N} \right) u_{m,n}v_{m,n}.$$

Now, by the extension of Abel's lemma proved in the paper referred to,

$$\sum_{M+1}^{M_1}\sum_{N+1}^{N_1} u_{m,n}v_{m,n} = \sum_{M+1}^{M_1-1}\sum_{N+1}^{N_1-1} \Delta_{m,n} \sum_{p=M+1}^{m}\sum_{q=N+1}^{n} u_{p,q}$$

$$+ \sum_{M+1}^{M_1-1} \Delta_m \sum_{M+1}^{m}\sum_{N+1}^{N_1} u_{p,q} + \sum_{N+1}^{N_1-1} \Delta_n \sum_{M+1}^{M_1}\sum_{N+1}^{n} u_{p,q} + \Delta \sum_{M+1}^{M_1}\sum_{N+1}^{N_1} u_{p,q},$$

where $\Delta = v_{M_1, N_1}$, $\Delta_m = v_{m, N_1} - v_{m+1, N_1}$,

$$\Delta_n = v_{M_1, n} - v_{M_1, n+1},$$

and $\Delta_{m,n} = v_{m,n} - v_{m+1,n} - v_{m,n+1} + v_{m+1,n+1}$.

The modulus of this is less than a constant multiple of

$$(4)\quad \sum_{M+1}^{M_1}\sum_{N+1}^{N_1} |\Delta_{m,n}| + \sum_{M+1}^{M_1-1} |\Delta_m| + \sum_{N+1}^{N_1-1} |\Delta_n| + |\Delta|,$$

since there is a constant K such that

$$\left|\sum_{m_1}^{m_2}\sum_{n_1}^{n_2} u_{p,q}\right| < K$$

for *all* values of m_1, n_1, m_2, n_2.

Now it is easy to see that

$$\Delta_{m,n} = \frac{\omega_1\omega_2\{a + (2m+1)\omega_2 + (2n+1)\omega_1\}}{(m, n)(m+1, n)(m, n+1)(m+1, n+1)},$$

where $(m, n) = a + m\omega_2 + n\omega_1$,

and therefore that, unless $m = n = 0$,

$$|\Delta_{m,n}| < \frac{K}{|m\omega_2 + n\omega_1|^3},$$

where K is independent of m, n, and a. But the right-hand side is the general term of a series known, by Eisenstein's theorem, to be absolutely convergent. It follows that given σ we can choose M_0, N_0, so that

$$\sum_{M+1}^{M_1} \sum_{N+1}^{N_1} |\Delta_{m,n}| < \sigma,$$

for all values of M, M_1, N, N_1, such that $M_1 > M \geqq M_0$, $N_1 > N \geqq N_0$.

Similar arguments may be applied to each of the other terms on the right of (4).

It follows that given σ we can choose M_0 and N_0, so that

$$\left| \sum_{M+1}^{M_1} \sum_{N+1}^{N_1} u_{m,n} v_{m,n} \right| < \sigma,$$

for all values of M, M_1, N, N_1, such that $M_1 > M \geqq M_0$, $N_1 > N \geqq N_0$, and moreover that M_0 and N_0 are independent of a.

A similar line of argument may be applied to the two other terms which occur on the right-hand side of (3). We conclude then that the double series

$$\sum_0^\infty \sum_0^\infty u_{m,n} v_{m,n}$$

is convergent. Exactly the same process proves that

$$\sum_0^\infty \sum_{-\infty}^{-1},\quad \sum_{-\infty}^{-1} \sum_0^\infty,\quad \sum_{-\infty}^{-1} \sum_{\infty}^{-1}$$

are convergent. Therefore (2) is convergent, when summed as a *double* series; that is to say

$$\sum_{-M'}^{M} \sum_{-N'}^{N} \frac{e^{(2m\theta+2n\phi)\pi i}}{a + m\omega_2 + n\omega_1}$$

tends to a finite limit when M, N, M', N' are made to tend simultaneously and independently to ∞; and moreover, that this convergence is *uniform* for all values of a which lie in any continuous domain which does not include any point, such that

$$a + m\omega_2 + n\omega_1 = 0,$$

for any values of m and n.

It follows by a theorem of Pringsheim's that *if* the series is convergent when summed as a repeated series, either by rows or by columns, its sum according to that method of summation is equal to its sum as a double series. Propositions substantially equivalent to these are proved by Kronecker with the aid of certain definite integrals. That the series is so convergent as a repeated series is easily proved, either by general considerations or by direct summation. The summation which follows is, as I stated above, practically the same as Kronecker's except in notation.

Summation of the series.

§ 3. If
$$w = -\frac{a+m\omega}{\omega_1},$$

$$\sum_{n=-\infty}^{\infty} \frac{e^{2n\phi\pi i}}{a+m\omega_2+n\omega_1} = -\frac{1}{\omega_1}\sum_{-\infty}^{\infty}\frac{e^{2n\phi\pi i}}{w-n} = -\frac{2\pi i}{\omega_1}\frac{e^{2w\phi\pi i}}{e^{2w\pi i}-1}.^*$$

$$= -\frac{2\pi i}{\omega_1}\frac{\exp.\left\{-2\pi i\phi\left(\frac{a+m\omega_2}{\omega_1}\right)\right\}}{\exp.\left\{-2\pi i\left(\frac{a+m\omega_2}{\omega_1}\right)\right\}-1}.$$

First suppose $m=\mu>0$. Then

$$\sum_{n=-\infty}^{\infty}\frac{e^{(2m\theta+2n\phi)\pi i}}{a+m\omega_2+n\omega_1} = -\frac{2\pi i}{\omega_1}\frac{q^{\mu}z}{1-q^{2\mu}z^2}t^{2\mu}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\},$$

where
$$q=e^{\pi i\tau},\quad z=e^{\frac{\pi i a}{\omega_1}},$$

$$t=\exp.\left[\frac{\pi i}{\omega_1}\{\omega_1\theta-\omega_2(\phi-\tfrac{1}{2})\}\right].$$

On the other hand, if m is negative, and $-m=\mu>0$,

$$\sum_{n=-\infty}^{\infty}\frac{e^{(2m\theta+2n\phi)\pi i}}{a+m\omega_2+n\omega_1} = \frac{2\pi i}{\omega_1}\frac{q^{\mu}z^{-1}}{1-q^{2\mu}z^{-2}}t^{-2\mu}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\}.$$

Thus the terms of (2) for which $m \gtrless 0$ contribute

$$-\frac{2\pi i}{\omega_1}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\}\sum_{\mu=1}^{\infty}\left[\frac{q^{\mu}t^{2\mu}z}{1-q^{2\mu}z^2}-\frac{q^{\mu}t^{-2\mu}z^{-1}}{1-q^{-2\mu}z^{-2}}\right].$$

* Kronecker, *Vorlesungen ueber Integrale*, p. 105.

Again, the terms for which $m = 0$ contribute

$$\sum_{-\infty}^{\infty} \frac{e^{2n\phi\pi i}}{a + n\omega_1} = -\frac{2\pi i}{\omega_1} \frac{\exp.\left(-\frac{2\pi i\phi a}{\omega_1}\right)}{\exp.\left(-\frac{2\pi i a}{\omega_1}\right) - 1}$$

$$= \frac{2\pi i}{\omega_1} \exp.\left\{(1 - 2\phi)\frac{\pi i a}{\omega_1}\right\} \frac{1}{z - z^{-1}}.$$

Thus in all we obtain

$$\frac{2\pi i}{\omega_1} \exp.\left\{(1 - 2\phi)\frac{\pi i a}{\omega_1}\right\}\left[\frac{1}{z - z^{-1}} - \sum_1^{\infty}\left(\frac{q^{\mu}t^{2\mu}z}{1 - q^{2\mu}z^2} - \frac{q^{\mu}t^{-2\mu}z^{-1}}{1 - q^{2\mu}z^{-2}}\right)\right].$$

But*

$$\frac{\sigma_3(u + v)}{\sigma_3(v)\,\sigma(u)} e^{-\frac{\eta_1 uv}{\omega_1}} = \frac{2\pi i}{\omega_1}\left[\frac{1}{z - z^{-1}} - \sum_1^{\infty}\left(\frac{q^{\mu}t^{2\mu}z}{1 - q^{2\mu}z^2} - \frac{q^{\mu}t^{-2\mu}z^{-1}}{1 - q^{2\mu}z^{-2}}\right)\right],$$

where† $\quad z = e^{\frac{i\pi u}{\omega_1}},\quad t = e^{\frac{i\pi v}{\omega_1}}.$

Thus the sum of our series is

$$(5) \quad \exp.\left\{(1 - 2\phi)\frac{\pi i a}{\omega_1}\right\} \frac{\sigma_3(v)\,\sigma(u)}{\sigma_3(u + v)} e^{-\frac{\eta_1 uv}{\omega_1}},$$

where $\quad u = a,\ v = \omega_1\theta - \omega_2(\phi - \tfrac{1}{2}).$

Now $\quad \sigma_3(v) = \dfrac{\sigma(u + \frac{1}{2}\omega_2)}{\sigma(\frac{1}{2}\omega_2)} e^{-\frac{1}{2}\eta_2 u},$

so that (5) reduces to

$$e^{a(\eta_2\phi - \eta_1\theta)} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)}.$$

Thus we arrive at the result

$$(6) \quad \sum \frac{e^{(2m\theta + 2n\phi)\pi i}}{a + m\omega_2 + n\omega_1} = e^{a(\eta_2\phi - \eta_1\theta)} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)},$$

the periods of the elliptic functions being ω_1, ω_2.

§ 4. From this result a variety of other interesting formulæ may be deduced. In the first place we observe that the series

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* Halphen, *Fonctions Elliptiques*, i., p. 422.

† I write ω_1, ω_2 for Halphen's $2\omega_1$, $2\omega_2$.

is uniformly convergent for values of a lying in any domain which does not include any of the points $-m\omega_2 - n\omega_1$. Now it is easy to see that if we remove the term for which $m = n = 0$, the resultant series is uniformly convergent for a region of values of a including $a = 0$. Hence

$$\Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{m\omega_n + n\omega_1} = \lim_{a=0} \left[-\frac{1}{a} + e^{(\eta_2\phi-\eta_1\theta)a} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)} \right]$$

$$= \zeta(\omega_1\theta - \omega_2\phi) - \eta_1\theta + \eta_2\phi,$$

the dash over the sign of summation implying that the pair of values $m = n = 0$ are excluded.

§ 5. Again it is easy to justify differentiating (6) any number of times with respect to a. In fact it is to be observed that the general theorems:

(i) A series is continuous for values of a parameter within a certain domain, if uniformly convergent throughout that domain;

(ii) The series may be integrated term by term along any path within the domain: and

(iii) The corresponding theorems for differentiation

are just as true for (proper) double series as for simple series; and the uniform convergence of the various derived series may be proved in this case just as in § 3.

Differentiate (6) once, subtract the term $-\frac{1}{a^2}$, and proceed to the limit for $a = 0$. We obtain

$$(8) \quad \Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{(m\omega_2 + n\omega_1)^2} = \tfrac{1}{2}\{p(u) - \zeta^2(u)\} + v\zeta(u) - \tfrac{1}{2}v^2,$$

where $\qquad u = \omega_1\theta - \omega_2\phi, \quad v = \eta_1\theta - \eta_2\phi.$

Again, we may differentiate (6) twice, obtaining

$$\Sigma \frac{e^{(2m\theta+2n\phi)\pi i}}{(a + m\omega_2 + n\omega_1)^3} = \tfrac{1}{2}\left(\frac{\partial}{\partial a}\right)^2 \left\{ e^{a(\eta_2\phi-\eta_1\theta)} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)} \right\}.$$

The series on the left is absolutely and uniformly convergent for all real values of θ and ϕ. Making $\theta = \phi = 0$, we obtain

$$\Sigma \frac{1}{(a + m\omega_2 + n\omega_1)^3} = \tfrac{1}{2}\left\{ \frac{\sigma'''(a)}{\sigma(a)} - \frac{3\sigma'(a)\,\sigma''(a)}{\sigma^2(a)} + \frac{2\sigma'^3(a)}{\sigma^3(a)} \right\}$$

$$= -\tfrac{1}{2}p'(a),$$

which agrees with the definition from which the theory of the Weierstrassian functions usually starts.

§ 6. The series

$$\Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{am^2 + 2bmn + cn^2} \quad (ac - b^2 > 0)$$

and other series considered by Kronecker and M. Lerch, may be treated in the same manner.

CORRECTIONS

p. 146, *line* 6. For $\sum_{\nu=1}^{p} \theta_\nu \eta_\nu$ read $i \sum_{\nu=1}^{p} \theta_\nu \eta_\nu$.

—— *line* 3 *up*. For a read a_p.

p. 147, *line* 4. For a^ν read a_ν.

—— *end of* § 1, 3–2 *lines up*. For 'so as' read 'so as to'.

p. 148, *line* 8. In the last sum, for $\sum_{N+1}^{N}$ read $\sum_{N+1}^{N_1}$.

p. 150, *line* 1 *of* § 3. For ω read ω_2.

—— *line* 2 *of* § 3. In the last denominator, for ω read w.

COMMENTS

The method of proof in § 2 anticipates that for Theorem 11 in 1917, 3, of which it is a corollary.

In the conditions for convergence of the series (1) with complex parameters, stated in § 1, the real part of β should be positive; see 1917, 3 (after Theorem 11). The example answers questions raised at the end of 1904, 1.

SOME THEOREMS CONNECTED WITH ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

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1. It will probably make the object of this paper more easily intelligible if, at the risk of repeating a certain number of well known facts, I preface it with a brief historical *résumé*.

In his famous memoir on the Binomial Series Abel proved that, *if a series Σa_n is convergent, the series $\Sigma a_n x^n$ is convergent for all positive values of x less than unity, and represents a function $f(x)$ which is continuous for all such values of x, unity included.**

An alternative proof of Abel's theorem was given later by Dirichlet.†

Stated in the language of the modern theory of functions, Abel's theorem runs: "If a power series in x converges to the sum s at a point P on its circle of convergence, and $f(x)$ is the function represented by the series within the circle, then $f(x)$ tends to the limit s when x tends to P along a radius vector from the origin."

This theorem has proved the starting point for a considerable number of later researches. Stolz was the first to prove that the result still holds if x tends to P along any path which lies entirely within the circle of convergence.‡ At a later date Pringsheim returned to the subject in a very instructive memoir,§ in which he shows that Abel's proof suffices to prove not only the continuity of $f(x)$, but also the *uniform convergence* of the series $\Sigma a_n x^n$ throughout the interval (0, 1). Of this the continuity of $f(x)$ for $x = 1$ is a corollary; but Abel had really proved more than mere continuity, and Pringsheim justly remarks that Dirichlet's proof is inferior to Abel's in that it obscures this fundamental point.

This is not the only direction in which Abel's theorem has been generalised. The property of the special function x^n, upon which Abel's

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* *Crelle*, Bd. I.; *Œuvres*, T. I., p. 223.

† *Liouville*, Sér. 2, T. VII.; *Werke*, Bd. II., p. 305.

‡ *Zeitschr. f. Math.*, Bd. XX., p. 370, and Bd. XXIX., p. 127. This statement is somewhat loose; see § 4.

§ *Münchener Sitzungsberichte*, 1897, p. 343.

proof was based, was simply that expressed by the inequality

$$x^n \geqslant x^{n+1} \quad (0 \leqslant x \leqslant 1),$$

and it was at once suggested that similar theorems must hold for more general classes of series of the type $\Sigma a_n f_n(x)$. And, in fact, Dirichlet and Dedekind* arrived at the following results, which for the sake of brevity I state on the hypothesis that the functions $f_n(x)$ are real functions of x defined for the interval $0 \leqslant x \leqslant 1$.

(*a*) If $$f_n(x) \geqslant f_{n+1}(x) \geqslant 0 \quad (0 \leqslant x \leqslant 1),$$
and Σa_n is convergent, then $\Sigma a_n f_n(x)$ is convergent and, if every f_n is continuous, the sum of the series is a continuous function of x.

(*b*) If Σa_n oscillates between finite limits of indetermination,

$$f_n(x) \geqslant f_{n+1}(x), \quad \text{and} \quad \lim f_n = 0,$$

then $\Sigma a_n f_n(x)$ is convergent; and, if every f_n is continuous, the sum of the series is a continuous function of x.

Dirichlet and Dedekind were concerned mainly with applications of these theorems to Dirichlet's series, and pass somewhat lightly over the general properties of series which are involved in them. Their exposition is also obscured to some extent by the fact that they do not utilize the notion of *uniform convergence.* I have therefore discussed the question further in § 2, and have stated a few theorems which summarize the conclusions which can be drawn from the discussion. I cannot claim any particular originality for these theorems, but, so far as I know, they have not, in the form in which I state them, been included in any published work. They would naturally suggest themselves to any one who undertook a careful analysis of the various theorems stated in this section, and Prof. Bromwich informs me that he has himself included Theorem I. *a* in a tract on the theory of series which will ultimately form one of the *Cambridge Tracts in Mathematics and Mathematical Physics.*

I have also included in §§ 3, 4 some applications of these theorems which do not appear to have been noticed hitherto, and in § 5 I have discussed a passage in Kronecker's *Vorlesungen über Integrale* which is concerned with the subject, but appears to contain serious errors.

There is yet another form of generalisation of Abel's theorem which has occupied the attention of mathematicians. It may happen that the series $\Sigma a_n x^n$ is divergent at a point on the circle of convergence, but is capable of "summation" by one or other of the methods furnished by

* *Vorlesungen über Zahlentheorie*, §§ 100 and 143–4.

the theory of divergent series, Cesàro's method of mean values, or Borel's method of exponential summation, or one of the various generalisations of either method. And it results from the combined researches of a number of writers that, *if* Σa_n *has the sum* s *when summed according to any of these methods, then* $f(x)$ *tends to the limit* s *when* x *tends to the point in question on the circle of convergence by any path subject to certain restrictions.* In the latter part of the paper I have occupied myself with series summable by Cesàro's method. The theorem for such series which corresponds to Abel's original theorem was first proved by Frobenius,* and states that, if

$$s_n = a_0 + a_1 + \dots + a_n,$$

and
$$\lim_{n=\infty} \frac{s_0+s_1+\dots+s_n}{n+1} = s,$$

then
$$\lim_{x=1} f(x) = s.$$

I have attempted to prove a general theorem which shall stand to this theorem in the same relation as Theorem I. to Abel's theorem. This theorem (Theorem II.) is the principal result of the paper: it will be found in § 6.

Finally, I have illustrated some of the most obvious applications of this general theorem, and I have indicated some further questions which are naturally suggested, but which I cannot profess to have completely solved.

I may remark that I was led to this investigation by considering various problems concerning the limits approached by the q-series of elliptic functions, when q tends to a point on the unit circle, and a number of my illustrations are furnished by q-series. But I have not in this paper attempted to treat any such particular class of problems systematically.

2. THEOREM I. *a.—If* $f_0(x)$, $f_1(x)$, $f_2(x)$, ... *is a series of real finite positive functions*† *such that*

(1) $$f_n(x) \geqslant f_{n+1}(x) \quad (0 \leqslant x \leqslant 1),$$

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* *Crelle*, Bd. LXXXIX., p. 262.

† A finite function (*fonction bornée*) is a function whose absolute value is, throughout the interval of variation of the independent variable, less than a constant K. It would obviously be enough to assert that $|f_0| < K$.

and Σa_n is any convergent series, then the series $\Sigma a_n f_n(x)$ is uniformly convergent throughout the interval (0, 1).

For

$$(1)\quad \sum_{\nu=m}^{n} a_\nu f_\nu = \sum_{\nu=m}^{n-1} (a_m+a_{m+1}+\ldots+a_\nu)(f_\nu-f_{\nu+1})+(a_m+a_{m+1}+\ldots+a_n)f_n.$$

Choose m_0 so that, for $\nu \geqslant m \geqslant m_0$,

$$|a_m+a_{m+1}+\ldots+a_\nu| < \epsilon.$$

Then
$$\left|\sum_{\nu=m}^{n} a_\nu f_\nu\right| < \epsilon f_m < \epsilon M,$$

where M is the maximum of $f_0(x)$ in the range (0, 1). The theorem is therefore proved.

COROLLARY.—*If the functions $f_n(x)$ are continuous, the series $\Sigma a_n f_n(x)$ represents a function of x continuous throughout the interval* $0 \leqslant x \leqslant 1$.

THEOREM I. *a* 1.—*If the restriction that f_n is real and positive is removed, and the condition* (1) *is replaced by the condition that*

$$(1a)\quad \sum_{m}^{n} |f_\nu(x)-f_{\nu+1}(x)| < K,$$

*where K is a constant, then the series $\Sigma a_n f_n$ is still uniformly convergent.**

We first observe that the existence of such a constant K involves that of a constant L, such that $|f_n(x)| < L$, for all values of x and n. For

$$|f_n(x)| \leqslant |f_0(x)| + \sum_{0}^{n-1} |f_\nu(x)-f_{\nu+1}(x)| < M+K.$$

Hence
$$\left|\sum_{m}^{n} a_\nu f_\nu\right| < \epsilon \left\{\sum_{m}^{n-1} |f_\nu-f_{\nu+1}| + |f_n|\right\} < \epsilon(M+2K),$$

and the result follows as before.

COROLLARY.—*If the functions f_n are continuous, the sum of the series is continuous.*

An obvious generalisation is—

THEOREM I. *a* 2. — *The conclusions of the preceding theorems and corollaries still hold if the terms of the series Σa_n are functions of x, provided the series is uniformly convergent, and (in the corollaries) the functions a_n are continuous.*

* We may suppose either that f_n is a complex function of a real variable, or a function of a complex variable; in the latter case the interval (0, 1) must be replaced by a region.

These theorems all arise from the Theorem (*a*) of Dirichlet-Dedekind. It is with this rather than with Theorem (*b*) that I am concerned in this paper; but the latter also raises interesting questions.

THEOREM I. *b*.—*If the functions $f_n(x)$ satisfy, in addition to the conditions of I., the condition $\lim_{n=\infty} f_n(x) = 0$, and if Σa_n oscillates between finite limits of indetermination,* then the series $\Sigma a_n f_n$ is uniformly convergent.*

In the first place there is a number K such that

$$|a_m+a_{m+1}+\ldots+a_\nu| < K$$

for all values of m and ν. In the second place $f_n(x)$ is a function of x which never increases as n increases, and whose limit zero is a continuous function of x. The convergence of $f_n(x)$ to its limit is therefore *uniform*,† and we can choose m_0 so that, for $m \geqslant m_0$, and for all values of x,

$$|f_m(x)| < \epsilon.$$

The theorem now follows immediately from (1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma\, a_n f_n(x)$ is a continuous function of x.*

THEOREM I. *b* 1.—*If the restriction that the functions $f_n(x)$ are real and positive is removed, and the conditions to which they are subject are replaced by the condition that the series $\Sigma\,|f_n(x)-f_{n+1}(x)|$ is convergent, the series $\Sigma a_n f_n$ is convergent.*

THEOREM I. *b* 2.—*If in addition the functions f_n are continuous and either of the equivalent conditions* (i.) *that the series $\Sigma|f_n-f_{n+1}|$ is uniformly convergent, or* (ii.) *that its sum represents a continuous function of x, is satisfied, the series $\Sigma a_n f_n$ will be uniformly convergent and continuous.*

THEOREM I. *b* 3.—*The preceding conclusions are not affected if the a_n's are functions of x, provided a constant K exists such that*

$$|a_0+a_1+\ldots+a_n| < K$$

for all values of n and x, and (if the continuity of the series is asserted) the functions a_n are continuous.

These theorems follow at once by trifling modifications of the preceding arguments. It will be seen that the series of theorems I. *b*, *b* 1, *b* 2, *b* 3 runs almost, though not exactly, parallel to the series I. *a*, *a* 1, *a* 2.

* *I.e.*, $|a_0+a_1+\ldots+a_n| < K$.

† Dini, *Grundlagen*, pp. 148, 149. The corollary is substantially Dedekind's theorem: his proof is less simple, owing to the fact that he does not employ the notion of uniform convergence.

3. Of the preceding theorems those of which the applications are most interesting are I. a and its extension I. a 1.

Since $f_n \geqslant f_{n+1}$, f_n tends to a limit for $n = \infty$ for all values of x; but in general it will not tend *uniformly* to this limit, and the limit will not be a continuous function of x. In the most important applications such a non-uniformity or discontinuity occurs at one or other end of the interval (0, 1), and the interest of the theorem lies in its application to establish the continuity of the series $\Sigma a_n f_n$ at this end. Thus

(i.) If
$$f_n(x) = x^n, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 \leqslant x < 1), \qquad \lim f_n = 1 \ (x = 1),$$
and we obtain Pringsheim's form of Abel's theorem.

(ii.) If
$$f_n(x) = n^{-x}, \quad f_n \geqslant f_{n+1},$$
$$\lim f_n = 0 \ (0 < x \leqslant 1), \qquad \lim f_n = 1 \ (x = 0),$$
and we deduce that the Dirichlet's series
$$\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x}, \dots$$
is uniformly convergent throughout (0, 1), and so continuous for $x = 0$, which is one of the Dirichlet-Dedekind theorems.

(iii.) If (denoting the independent variable now by q) we take
$$f_n(q) = \frac{q^n}{1+q^n},$$
so that
$$f_n - f_{n+1} = \frac{q^n(1-q)}{(1+q^n)(1+q^{n+1})} \geqslant 0,$$
and
$$\lim_{n=\infty} f_n = 0 \ (q < 1), \quad = \frac{1}{2} \ (q = 1),$$
and we deduce that, if Σa_n is convergent,
$$\lim_{q=1} \Sigma \frac{a_n q^n}{1+q^n} = \tfrac{1}{2} \Sigma a_n,$$
numerous applications of this result [and the similar results for $\Sigma a_n q^n/(1+q^{2n})$, ...] may be made in the theory of elliptic functions. For instance, from
$$\log k = \log 4\sqrt{q} + 4\Sigma \frac{(-q)^n}{n(1+q^n)} \, *$$
we deduce
$$\lim_{q=1} \log k = 2\log 2 - 4(\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \dots) = 0,$$
as may be verified independently.

(iv.) Let us next consider the series
$$\Sigma \frac{n a_n q^n (1-q)}{1-q^n} = \Sigma \frac{n a_n q^n}{1+q+q^2+\dots+q^{n-1}}.$$
Here
$$f_n(q) = \frac{nq^n}{1+q+q^2+\dots+q^{n-1}},$$
$$f_n(q) - f_{n+1}(q) = \frac{(1-q)^2 q^n}{(1-q^n)(1-q^{n+1})} (n-1-q-\dots-q^{n-1}) \geqslant 0.$$

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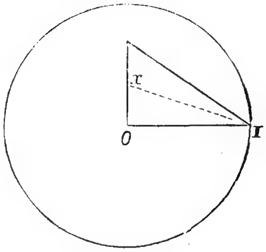
* Jacobi, *Fundamenta Nova*, p. 103.

We deduce that
$$\lim_{q=1}(1-q)\sum\frac{na_nq^n}{1-q^n}=\sum a_n,$$
provided only the latter series is convergent. This result has been proved (by a special method depending upon integrals) by Franel.* Similar results may, of course, be proved for such series as
$$\sum\frac{2na_nq^n}{1-q^{2n}},\quad \sum\frac{(2n+1)\,a_nq^{2n+1}}{1-q^{4n+2}},\quad \ldots.$$
For instance, from
$$-\log k'=8\sum_0^\infty\frac{q^{2n+1}}{(2n+1)(1-q^{4n+2})}\dagger$$
we deduce
$$-\log_{q=1} k'\sim\frac{\pi^2}{2\,(1-q)},$$
and from
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)}=\operatorname{cosec}\frac{u\pi}{2\omega}+4\sum\frac{q^{2n+1}}{1-q^{2n+1}}\sin\left\{(2n+1)\frac{u\pi}{2\omega}\right\}\ddagger$$
we deduce
$$\frac{2\omega}{\pi}\sqrt{(\wp u-e_3)}\sim\frac{4}{1-q}\sum\frac{\sin\left\{(2n+1)\dfrac{u\pi}{2\omega}\right\}}{2n+1}=\pm\frac{\pi}{1-q},$$
according to the value of u. In the last equation we must suppose that ω is constant and that ω' varies in such a way that q tends to 1 along the real axis.

In an interesting note recently published in the *Messenger of Mathematics*,§ Prof. Bromwich establishes the asymptotic equality
$$f(\theta)=\sum_1^\infty\frac{(-)^{n-1}}{\sinh n\theta}\sim\frac{\log 2}{\theta}$$
for $\theta=0$. This result follows immediately from what precedes if we write q for e^θ. I shall refer later on to Prof. Bromwich's further results.

4. I shall now consider some examples of the use of Theorem I. α 1.

(i.) Suppose that $f_n(x)=x^n$, and that the region of variation of x is a triangle formed by joining 0 and 1 to any point inside the unit circle.



It is easily verified that a constant K (depending only on the triangle) can be found such that for all points within or on the boundary of the triangle
$$\frac{|1-x|}{1-|x|}<K.$$

Hence, if $|x|=r$,
$$\sum_m^n|f_\nu(x)-f_{\nu+1}(x)|=\sum_m^n r^\nu|1-x|<K\sum_m^n r^\nu(1-r)<K,$$

* *Math. Annalen*, Bd. LII.

† *Fundamenta Nova, l.c.*

‡ Halphen, *Fonctions Elliptiques*, t. I., p. 431.

§ "Some Contributions to the Theory of Two Electrified Spheres," *Messenger*, Vol. XXXV., p. 1.

and the conditions of the theorem are satisfied. We thus obtain Pringsheim's generalisation of Abel's theorem.*

(ii.) The theorem may be applied to q-series such as those previously considered when q moves (let us say) along a radius vector to a *rational point* on the unit circle, *i.e.*, a point $e^{\pi ib/a}$, where a and b are integers. Take, *e.g.*, the series for $\log k$ considered above,† and suppose that $q = re^{\pi ib/a}$, where b is even and a odd, and that r tends to unity along the radius vector $(0, 1)$. Then none of the terms of the series become infinite in the limit; also

$$\sum_1^\infty \frac{(-q)^n}{n(1+q^n)} = \sum_{s=1}^a \sum_{m=0}^\infty \frac{(-q)^{ma+s}}{(ma+s)(1+q^{ma+s})} = \sum_{s=1}^a (-)^s r^s e^{s\pi ib/a} F_s(r^a),$$

where
$$F_s(\rho) = \sum_0^\infty \frac{(-\rho)^m}{(ma+s)(1+\rho^{m+s/a}e^{s\pi ib/a})}.$$

This last series satisfies the criteria of I. a 1 for uniform convergence throughout the interval $(0, 1)$ of values of ρ. For, if $a_m = (-)^m/(ma+s)$, Σa_m is convergent. Also, if

$$f_m(\rho) = \frac{\rho^m}{1+\rho^{m+s/a}e^{s\pi ib/a}},$$

$$f_m(\rho) - f_{m+1}(\rho) = \frac{\rho^m(1-\rho)}{(1+A\rho^{m+s/a})(1+A\rho^{m+1+s/a})},$$

where $A = e^{s\pi ib/a}$. Now

$$|1+A\rho^{m+s/a}| = \sqrt{\{1+\rho^{2(m+s/a)}+2\rho^{m+s/a}\cos(s\pi b/a)\}}.$$

If $\cos(s\pi b/a) > 0$, this is greater than unity; if $\cos(s\pi b/a) < 0$, it has a minimum when $\rho^{m+s/a} = -\cos(s\pi b/a)$, this minimum being $|\sin(s\pi b/a)|$. And in any case

$$|f_m(\rho) - f_{m+1}(\rho)| < K\rho^m(1-\rho),$$

from which it follows at once that the conditions of I. a 1 are satisfied.

Hence the original series for $\log k$ converges uniformly when $q = re^{\pi ib/a}$, $0 \leqslant r \leqslant 1$. For $r = 1$ it assumes the form

$$2\log 2 + \frac{\pi ib}{2a} + 2\sum_1^\infty \frac{(-)^n}{n}\left(1 + i\tan\frac{n\pi b}{a}\right) = \frac{\pi ib}{a} + 2i\sum_1^\infty \frac{1}{n}\tan\frac{n\pi b}{a},$$

and this is therefore the value to which $\log k$ tends as r approaches unity. The series on the right may be summed in finite terms.‡

5. In a passage in his *Vorlesungen über Integrale*, which has doubtless puzzled many readers besides myself, Kronecker apparently essays to prove a theorem designed to be a generalisation of Abel's theorem somewhat on the lines of Theorem I. a, except that there is no mention of uniform convergence. The whole passage is obscure: but the suggested

* *Münchener Sitzungsberichte*, *l.c.*

† § 3, iii.

‡ See H. J. S. Smith, "On some Discontinuous Series considered by Riemann" (*Messenger*, Vol. XI., pp. 1–11; *Collected Math. Papers*, Vol. II., p. 312); Dedekind's Note in Riemann's *Werke*, pp. 427–447; G. H. Hardy, "Note on the Limiting Values of the Elliptic Modular Functions," *Quarterly Journal*, Vol. XXXIV., pp. 76–86.

theorem seems to be as follows:—* "If

(i.) Σa_n is a convergent series,

(ii.) the functions $f_n(x)$ are positive and continuous throughout (a, A),

(iii.) $f_n(x) \geqslant f_{n+1}(x)$,

(iv.) $\lim\limits_{x=A} f_m(x) = \lim\limits_{x=A} f_n(x)$, for all values of m and n,

then $\Sigma a_n f_n(x)$ will be convergent and continuous for $x = A$."

My criticisms on the passage are in brief (i.) that the conditions are redundant, the fourth of them being quite unnecessary and having nothing to do with the essence of the matter; and (ii.) that the proof is altogether unsound. The unsoundness of the proof appears to have arisen from a mistaken idea of the importance of condition (iv.). Kronecker argues as follows. Starting from Abel's partial summation lemma, the origin of all these theorems, viz.,

$$c_0 f_0 + \sum_1^n (c_\nu - c_{\nu-1}) f_{\nu-1} = \sum_1^n c_\nu (f_{\nu-1} - f_\nu) + c_n f_n,$$

and putting $$c_\nu = -(a_\nu + a_{\nu+1} + \ldots),$$

he deduces

$$-f_0 \sum_0^\infty a_\nu + \sum_1^n a_{\nu-1} f_{\nu-1} = -\sum_1^n (f_{\nu-1} - f_\nu) \Sigma a_\kappa - f_n \sum_n^\infty a_\kappa$$

$$= -(f_0 - f_n) M_n - f_n \sum_n^\infty a_\kappa,$$

where M_n lies between the least and greatest of the values of

$$\sum_\nu^\infty a_\kappa \quad (\nu = 1, 2, \ldots, n).$$

Making n tend to infinity, and observing that $\sum_1^\infty a_{\nu-1} f_{\nu-1}$ is convergent, we obtain

$$-f_0 \sum_0^\infty a_\nu + \sum_1^\infty a_{\nu-1} f_{\nu-1} = -(f_0 - \lim_{n=\infty} f_n) M,$$

where M lies between the least and greatest of all the values of $\sum_\nu^\infty a_\kappa$.

He then makes x tend to A, and (unless his meaning has been entirely obscured by misprints), argues that, because

$$\lim_{x=A} f_0 = \lim_{x=A} f_n$$

* I have altered Kronecker's notation so as to agree with my own (Kronecker, *l.c.*, pp. 88, 89).

for all values of n, therefore

$$\lim_{x=A}(f_0-\lim_{n=\infty} f_n) = 0;$$

and therefore
$$\lim_{x=A}\sum_1^\infty a_{\nu-1} f_{\nu-1} = \lim_{x=A} f_0 \times \sum_0^\infty a_\nu.$$

But it is obvious that all that he is justified in asserting is that

$$\lim_{x=A} f_0 = \lim_{n=\infty}(\lim_{x=A} f_n),$$

and *not*
$$\lim_{x=A} f_0 = \lim_{x=A}(\lim_{n=\infty} f_n),$$

the two repeated limits only being equal in exceptional circumstances. And, in fact, in the very simplest case, when $f_n(x) = x^n$ and $A = 1$,

$$\lim_{n=\infty}\lim_{x=1} x^n = 1, \qquad \lim_{x=1}\lim_{n=\infty} x^n = 0;$$

so that his argument does not even suffice to prove Abel's theorem itself. And a careful examination of the passage will, I think, lead any reader to the conclusion that the flaw in it is fundamental and not to be repaired by any alterations merely of detail.

6. I shall now consider the case in which the series Σa_n is divergent but summable by Cesàro's method of mean values. I use the following notation and terminology. We shall say that Σa_n is *summable* if

$$\frac{s_0+s_1+\ldots+s_n}{n+1},$$

where
$$s_n = a_0+a_1+\ldots+a_n,$$

tends to a finite limit for $n = \infty$; and, if the terms a_n are functions of a variable x, and the convergence of this mean value to its limit is uniform throughout a certain interval or region, we shall say that Σa_n is *uniformly summable*. It is evident that the sum of a uniformly summable series of continuous terms is a continuous function of x.

THEOREM 2.—*If the functions f_n are finite, real, and positive, and f_n-f_{n+1} and $f_n-2f_{n+1}+f_{n+2}$, their first and second differences, are positive for $0 \leqslant x \leqslant 1$ and for all values of n, and if the series Σa_n is summable, then the series $\Sigma a_n f_n$ is uniformly summable throughout* (0, 1).

COROLLARY.—*If the functions f_n are continuous, the sum of the series $\Sigma a_n f_n$ is a continuous function of x.*

The proof of this theorem presents somewhat greater difficulties than those of the simpler theorems of § 2. We shall find it a necessary preliminary to establish a series of lemmas.

LEMMA 1.—*If s_n tends uniformly to a limit s, the series Σa_n is uniformly summable and has the sum s.*

If we omit " uniformly," this is a well known theorem* asserting the consistency of the new definition with the old. The insertion of " uniformly " in no way affects the proof.

LEMMA 2.—*If*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$
we can determine a series of positive quantities $\epsilon_1, \epsilon_2, \ldots$, whose limit is zero, such that
$$\left| \frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1} \right| < \epsilon_p$$
for all values of r.

For we may write $s_0+s_1+\ldots+s_n = (n+1)\,\eta_n$,

where $\lim \eta_n = 0$. And then
$$s_p+s_{p+1}+\ldots+s_{p+r} = (p+r+1)\,\eta_{p+r} - p\eta_{p-1},$$
from which the lemma follows; for we can choose p so that, for $\nu \geqslant p-1$, $|\eta_\nu| < \epsilon$, however small be ϵ, and then
$$\left| \frac{s_p+s_{p+1}+\ldots+s_{p+r}}{p+r+1} \right| < 2\epsilon$$
for all values of r. In particular, as is well known,
$$\lim s_p/(p+1) = 0.$$

LEMMA 3.—*If f_n is finite, real, and positive and $f_n \geqslant f_{n+1}$ for all values of n and x, and*
$$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$
then
$$\lim \frac{s_0 f_0+s_1 f_1+\ldots+s_n f_n}{n+1} = 0$$
uniformly for all values of x.

For
$$s_0 f_0+\ldots+s_n f_n = \sum_{\nu=0}^{n-1} (s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1}) + (s_0+\ldots+s_n) f_n$$
$$= \left(\sum_{\nu=0}^{r-1} + \sum_{r}^{n-1}\right)(s_0+\ldots+s_\nu)(f_\nu-f_{\nu+1}) + (s_0+\ldots+s_n) f_n$$
$$= (f_0-f_r)\,M_{0,\,r-1} + f_r\,M_{r,\,n},$$

* See, *e.g.*, Bromwich and Hardy, *Proceedings*, Vol. II., p. 172.

where $M_{0,r-1}$ lies between the least and greatest of

$$s_0,\ s_0+s_1,\ \ldots,\ s_0+s_1+\ldots+s_{r-1},$$

and $M_{r,n}$ between the least and greatest of

$$s_0+s_1+\ldots+s_r,\ \ldots,\ s_0+s_1+\ldots+s_n.$$

Let ϵ be an assigned positive small quantity. We can choose r so that for $\nu \geqslant r$

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{\nu+1}\right| < \epsilon,$$

and, *a fortiori*,

$$\left|\frac{s_0+s_1+\ldots+s_\nu}{n+1}\right| < \epsilon$$

for $n \geqslant \nu \geqslant r$; and therefore we can choose r so that

$$\left|\frac{M_{r,n}}{n+1}\right| < \epsilon$$

for all values of $n \geqslant r$. But when r is fixed we can obviously choose n so that

$$\left|\frac{M_{0,r-1}}{n+1}\right| < \epsilon.$$

When r and n are thus chosen

$$\left|\frac{s_0f_0+s_1f_1+\ldots+s_nf_n}{n+1}\right| < 2M\epsilon,$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 4.—*If the conditions of* 3 *are satisfied except that*

$$\lim_{n=\infty}\frac{s_0+s_1+\ldots+s_n}{n+1} = s\ (\neq 0),$$

then

$$\lim_{n=\infty}\frac{s_0f_0+s_1f_1+\ldots+s_nf_n}{n+1} = s\lim_{n=\infty} f_n\,;$$

but the convergence to this limit will in general not be uniform.

For let $s_0 = s+t_0$, $s_1 = s+t_1$, ... Then

$$\lim_{n=\infty}\frac{t_0+t_1+\ldots+t_n}{n+1} = 0\,;$$

and therefore

$$\frac{t_0f_0+t_1f_1+\ldots+t_nf_n}{n+1}$$

converges *uniformly* to zero. Also

$$\lim_{n=\infty} \frac{s(f_0+f_1+\ldots+f_n)}{n+1} = s \lim_{n=\infty} f_n;$$

but the convergence to this limit will not in general be uniform unless f_n converges to its limit uniformly, which will not generally be the case.

LEMMA 5.—*If* $$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = 0,$$

and the f_n's satisfy the further condition

$$f_n - f_{n+1} \geqslant f_{n+1} - f_{n+2}$$

for all values of n and x in question, then the series

$$\sum_0^\infty s_n (f_n - f_{n+1})$$

is uniformly convergent.

In the first place

$$f_0 - f_n = (f_0 - f_1) + \ldots + (f_{n-1} - f_n) \geqslant n(f_{n-1} - f_n).$$

Hence a constant K can be assigned so that for all values of x and n

$$f_{n-1} - f_n < K/n.$$

Now $s_p(f_p - f_{p+1}) + s_{p+1}(f_{p+1} - f_{p+2}) + \ldots + s_{q-1}(f_{q-1} - f_q)$

$$\begin{aligned} &= s_p(f_p - 2f_{p+1} + f_{p+2}) + (s_p + s_{p+1})(f_{p+1} - 2f_{p+2} + f_{p+3}) \\ &\quad + \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\quad + (s_p + s_{p+1} + \ldots + s_{q-2})(f_{q-2} - 2f_{q-1} + f_q) \\ &\quad + (s_p + s_{p+1} + \ldots + s_{q-1})(f_{q-1} - f_q), \end{aligned}$$

the modulus of which is less than

$$\begin{aligned} &\epsilon_p \{(p+1)(f_p - 2f_{p+1} + f_{p+2}) + (p+2)(f_{p+1} - 2f_{p+2} + f_{p+3}) + \ldots \\ &\qquad \ldots + (q-1)(f_{q-2} - 2f_{q-1} + f_q) + q(f_{q-1} - f_q)\} \\ &= \epsilon_p \{p(f_p - f_{p+1}) + f_p - f_q\} < \epsilon_p \{K + 2M\}, \end{aligned}$$

where M is the maximum of $f_0(x)$. The lemma is therefore proved.

LEMMA 6.—*If the f_n's satisfy the conditions of* 5, *but*

$$\lim_{n=\infty} \frac{s_0+s_1+\ldots+s_n}{n+1} = s\,(\neq 0),$$

the series

$$\sum_0^\infty s_n(f_n-f_{n+1})$$

is convergent (but, in general, not uniformly convergent).

Let

$$s_n = s+t_n\,;$$

then, by 3, the series $\Sigma t_n(f_n-f_{n+1})$ is uniformly convergent. On the other hand, the series $\Sigma s(f_n-f_{n+1})$ is convergent, but not uniformly convergent, unless f_n tends to its limit uniformly.

7. *Proof of Theorem* 2.—Let s be the sum of the divergent series Σa_n, and let

$$a_0' = a_0-s,\ a_1' = a_1,\ a_2' = a_2,\ \ldots,\ s_n' = a_0'+a_1'+\ldots+a_n' = s_n-s\,;$$

then $\Sigma a_n'$ is summable, and its sum is zero; *i.e.*,

$$\lim \frac{s_0'+s_1'+\ldots+s_n'}{n+1} = 0.$$

By Lemma 3,

$$\frac{s_0'f_0+s_1'f_1+\ldots+s_n'f_n}{n+1}$$

tends uniformly to 0 for $n=\infty$; and, by Lemma 5, the series

$$\Sigma s_n'(f_n-f_{n+1})$$

is uniformly convergent. Hence, if

$$S_n' = \sum_0^n s_\nu(f_\nu-f_{\nu+1}),$$

S_n' tends uniformly to a limit for $n=\infty$, and so, by Lemma 1,

$$\frac{S_0'+S_1'+\ldots+S_n'}{n+1}$$

does the same.

Now $\quad a_\nu'f_\nu = (s_\nu-s_{\nu-1}')f_\nu = s_\nu'f_\nu-s_{\nu-1}'f_{\nu-1}+s_{\nu-1}'(f_{\nu-1}-f_\nu).$

Hence, if $\sigma_n = a_0f_0+a_1f_1+\ldots+a_nf_n$, $\sigma_n' = a_0'f_0+a_1'f_1+\ldots+a_n'f_n$,

$$\sigma_n' = s_n'f_n+\sum_1^n s_{\nu-1}'(f_{\nu-1}-f_\nu) = s_n'f_n+S_{n-1}',$$

and $$\frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1} = \frac{s_0'f_0+\ldots+s_nf_n}{n+1}+\left(\frac{n}{n+1}\right)\frac{S_0'+S_1'+\ldots+S_{n-1}}{n},$$

and therefore tends uniformly to a limit for $n = \infty$. But

$$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1} = sf_0 + \frac{\sigma_0'+\sigma_1'+\ldots+\sigma_n'}{n+1},$$

and therefore also tends uniformly to a limit for $n = \infty$. Hence *the series $\Sigma a_n f_n$ is uniformly summable, and, if the functions f_n are continuous, its sum is a continuous function of n.* The theorem is therefore proved.

8. In order to show more precisely the relations of the preceding lemmas and theorem I take a very simple example.

Let $$a_0 = 1, \quad a_1 = -2, \quad a_2 = 2, \quad a_3 = -2, \quad \ldots,$$

so that $$s_{2n} = 1, \quad s_{2n+1} = -1,$$

and $$\lim \frac{s_0+s_1+\ldots+s_n}{n+1} = 0;$$

and suppose $f_n(x) = x^n$. Then

(i.) $$s_n f_n = (-)^n x^n,$$

$$\frac{s_0 f_0 + s_1 f_1 + \ldots + s_n f_n}{n+1} = \frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

which converges *uniformly* to 0 for $n = \infty$ (*Lemma* 3).

(ii.) Again $\sum_1^n s_{\nu-1}(f_{\nu-1}-f_\nu) = \sum_1^n (-)^{\nu-1} x^{\nu-1}(1-x) = (1-x)\{1+(-)^{n-1}x^n\}/(1+x)$,

which tends *uniformly* to $(1-x)/(1+x)$ for $n = \infty$ (*Lemma* 5). For, although x^n does *not* tend uniformly to its limit,

$$x^n - x^{n+1} - (x^{n+1} - x^{n+2}) = x^n(1-x)^2 \geqslant 0,$$

and $$1-x^{n+1} = (1-x)+(x-x^2)+\ldots+(x^n-x^{n+1}) \geqslant (n+1)(x^n-x^{n+1}),$$

so that $$x^n(1-x) < \frac{1}{n+1},$$

and therefore does tend uniformly to zero.

(iii.) Finally, $\sigma_n = 1-2x+2x^2-\ldots+(-)^n 2x^n = \dfrac{1-x}{1+x} + 2(-)^n \dfrac{x^{n+1}}{1+x}$,

and $$\frac{\sigma_0+\sigma_1+\ldots+\sigma_n}{n+1} = \frac{1-x}{1+x} + \frac{2\{x+(-)^n x^{n+2}\}}{(n+1)(1+x)^2},$$

which tends *uniformly* to $(1-x)/(1+x)$ for $n = \infty$ (*Theorem* 2).

If the conditions were altered by changing a_0 into $1+\alpha$ $(\alpha \neq 0)$, we should have

$$s_n f_n = \{\alpha + (-1)^n\} x^n,$$

and $$\frac{s_0 f_0 + s_1 f_1 + \ldots + s_n f_n}{n+1} = \phi + \frac{1+(-)^n x^{n+1}}{(n+1)(1+x)},$$

where $$\phi = \frac{\alpha}{n+1}\,\frac{1-x^{n+1}}{1-x} \quad (x < 1),$$

$$\phi = \alpha \quad (x = 1),$$

and the convergence of ϕ to its limit is not uniform (*Lemma* 4). Similarly $\Sigma s_{\nu-1}(f_{\nu-1}-f_\nu)$ is increased by the addition of the non-uniformly convergent series $\Sigma \alpha(x^{n-1}-x^n)$ (*Lemma* 6); but it is easily verified that the uniformity of convergence which is prescribed by Theorem 2 is not affected, the two non-uniformities (so to say) cancelling one another.

9. *Applications of Theorem* 2.—(i.) If $f_n(x) = x^n$,

$$f_n - 2f_{n+1} + f_{n+2} = x^n(1-x)^2 \geqslant 0$$

for $0 \leqslant x \leqslant 1$ and all values of n. Hence, if Σa_n is summable, $\Sigma a_n x^n$ is uniformly summable for $0 \leqslant x \leqslant 1$; and its sum is a continuous function of x for $x = 1$, which is Frobenius's theorem cited in § 1.

(ii.) If $f_n(x) = n^{-x}$ ($n \geqslant 1$, $x \geqslant 0$), it is easy to see that the first and second differences of f_n are positive (or zero). Hence we obtain the theorem that, if $\sum_1^\infty a_n$ is summable, $\sum_1^\infty a_n n^{-x}$ is uniformly summable for all positive values of x, including zero, and its sum is a continuous function of x for $x = 0$. That is to say

$$\lim_{x=0}\left(\frac{a_1}{1^x}+\frac{a_2}{2^x}+\frac{a_3}{3^x}+\dots\right) = \lim_{n=\infty}\frac{s_0+s_1+\dots+s_n}{n+1},$$

if the latter limit exists. For example,

$$\lim_{x=0}\left(\frac{1}{1^x}-\frac{1}{2^x}+\frac{1}{3^x}-\dots\right) = \tfrac{1}{2}.$$

(iii.) If
$$f_n(q) = \frac{q^n}{1+q^n} \quad (0 \leqslant q \leqslant 1),$$

$$f_n - 2f_{n+1} + f_{n+2} = \frac{q^n(1-q)^2(1-q^{n+1})}{(1+q^n)(1+q^{n+1})(1+q^{n+2})} \geqslant 0.$$

Hence, if Σa_n is summable, $\Sigma a_n q^n/(1+q^n)$ is uniformly summable for $0 \leqslant q \leqslant 1$, and represents a continuous function of q, in particular for $q = 1$.

For instance, from the formula

$$\frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^3} - \frac{4q^6}{1+q^5} + \dots*$$

we deduce that
$$\lim_{q=1}\frac{2k'K}{\pi} = 1 - 4\left(\tfrac{1}{2}-\tfrac{1}{2}+\dots\right) = 1 - 4 \cdot \tfrac{1}{4} = 0.\dagger$$

(iv.) Consider the series
$$\frac{q}{1-q^2} - \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} - \dots,$$

whose sum is easily found‡ to be
$$\frac{K}{2\pi^2}(E - k'^2 K).$$

We may write this in the form
$$\frac{q}{1-q^2}\,\Sigma a_n f_n(q),$$

where
$$a_n = (-1)^n \quad \text{and} \quad f_n(q) = \frac{(n+1)q^n}{1+q^2+\dots+q^{2n}},$$

and it is easy to verify that the first and second differences of f_n are positive. Hence $\Sigma a_n f_n$ is uniformly summable. For $q = 1$ it takes the form

$$1 - 1 + 1 - \dots = \tfrac{1}{2}.$$

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* *Fundamenta Nova*, § 40, (6).

† Strictly speaking, the divergent series should be written

$$\tfrac{1}{2} + 0 - \tfrac{1}{2} - 0 + \tfrac{1}{2} + 0 - \tfrac{1}{2} - \dots.$$

‡ *E.g.*, by making $x = \frac{1}{2}\pi$ in formula (1) of § 41 of the *Fundamenta Nova*.

We deduce that
$$K(E-k'^2K) \sim \frac{\pi^2}{2(1-q)}$$
for $q = 1$.

10. It would be easy to multiply instances of interesting applications of Theorem 2. Those which I have given are fair examples of some of the simplest types which naturally occur, and the length of this paper forbids that I should attempt to treat them in a more systematic manner. I shall conclude by indicating briefly certain actual or possible further generalisations.

In the first place we may at once enunciate

THEOREM 2 a 1.—*The conclusions of Theorem 2 (and the lemmas preliminary to it) are still valid if the functions $f_n(x)$ are not restricted to be real and positive, and the condition that the first and second differences of the functions are not negative is replaced by the conditions*

$$\sum_m^n |f_\nu - f_{\nu+1}| < K, \qquad \sum_m^n (\nu+1)\,|f_\nu - 2f_{\nu+1} + f_{\nu+2}| < K,$$

for all values of m, n, and x.

The course of the proof is unaffected save for slight modifications in the case of Lemmas 3 and 5.

Consider, for example, the series

$$\vartheta_4(v, q) = 1 + 2\sum_1^\infty (-)^n q^{n^2} \cos 2n\pi v.$$

Taking $a_n = 2(-)^n \cos 2n\pi v$ $(n > 0)$ and $f_n = q^{n^2}$, we may verify without difficulty that the conditions of the theorem are satisfied. Since the series

$$1 - 2\cos 2\pi v + 2\cos 4\pi v - \dots$$

has the sum zero when summed by Cesàro's method, we deduce that

$$\lim_{q=1} \vartheta_4(v, q) = 0.*$$

THEOREM 2 a 2.—*The preceding conclusions are not affected if the terms of the series Σa_n are functions of x, provided the series be uniformly summable.*

A much more interesting and more difficult question is that of the extension of Theorem II. to cases in which the summation of Σa_n requires

* See Borel, *Leçons sur les Séries divergentes*, p. 7; L. Fejér, *Math. Annalen*, Bd. LVIII., p. 66; Hardy, "Note on Divergent Fourier Series," *Messenger*, Vol. XXXIII., p. 144. I refer later to Herr Fejér's investigations.

one of the extended forms of the mean value process, *e.g.*, when, if

$$s_n^{(1)} = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

$s_n^{(1)}$ oscillates for $n=\infty$, but

$$s_n^{(2)} = \frac{s_0^{(1)}+s_1^{(1)}+\ldots+s_n^{(1)}}{n+1}$$

has a limit.

The following more general theorem is naturally suggested, and I have no doubt that it is true. We define "summable" to mean "summable by k repetitions of the mean value process." Then,

If the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero) for all values of x and n in question, and the series Σa_n is summable, then the series $\Sigma a_n f_n(x)$ is uniformly summable, and therefore its sum is a continuous function of x

—with corollaries and generalisations in every way analogous to those of Theorems I. *a* and II. Such a theorem would be related to Hölder's extensions of Frobenius's theorem as is II. to Frobenius's and I. *a* to Abel's theorem. But I have not up to the present succeeded in overcoming the algebraical difficulties attendant upon a complete and rigorous proof.

In the most interesting cases Theorem II. is generally sufficient. But the latter theorem does not cover such cases as those in which Σa_n is a series like $1-2+3-4+\ldots$ or $1^2-2^2+3^2-4^2+\ldots$.

An example in which a result more general than that of II. is needed may be found in the theory of two electrified spheres. In the paper already referred to, Prof. Bromwich, seeking a rigorous proof of Lord Kelvin's theorem that the force acting between two spheres in contact and at potential V is $\frac{1}{6}V^2(\log 2-\frac{1}{4})$, requires to show that, for small values of θ,

$$f(\theta) = \Sigma \frac{(-)^{n-1}}{\sinh n\theta} = \frac{\log 2}{\theta} - \tfrac{1}{24}\theta + \ldots.$$

The first approximation was established in § 3 (iv.). To obtain the second we must prove that

$$\lim_{\theta=0} \frac{1}{\theta} \Sigma (-)^{n-1} \left(\frac{1}{n\theta} - \frac{1}{\sinh n\theta}\right) = \tfrac{1}{24}.$$

The limiting form of the series is $\quad \frac{1}{6}(1-2+3-4+\ldots)$,

which is summable by *two* repetitions of the mean value process, and has the sum $\frac{1}{24}$. Here we could take $a_n = (-)^{n-1}n$ and $f_n(\theta) = \frac{1}{n\theta}\left(\frac{1}{n\theta} - \frac{1}{\sinh n\theta}\right)$, and so obtain the result desired.

Although I have not succeeded in proving the suggested general theorem, I have, starting from a theorem of Herr Fejér's, succeeded in proving a number of theorems of a more special character which do enable us to deal effectively with cases such as these: *e.g.*, to assign the limit of

$$\frac{q}{1+q} - \frac{2q^2}{1+q^2} + \frac{3q^3}{1+q^3} - \ldots$$

for $q = 1$. I confine myself at present to stating one of these theorems. Herr Fejér's theorem (modified so as to correspond to Theorem 2*) runs as follows:—*If*

(i.) Σa_n *is summable (to the sum s),*

(ii.) *the functions* $f_n(x)$ *and their first and second differences are positive (or zero),*

(iii.) $\Sigma n f_n(x)$ *is convergent for* $x > 0$,

(iv.) $\lim_{x=0} f_n(x) = 1$ *for all values of n,*

then $\Sigma a_n f_n(x)$ *is absolutely convergent for* $x > 0$, *and its limit for* $x = 0$ *is s.*

The more general theorem is that the same conclusion holds when k repetitions of the mean value process are necessary in order to sum the series Σa_n, and

(ii.)′ the first, second, ..., $(k+1)$-th differences of the functions $f_n(x)$ are positive (or zero),

(iii.)′ $\Sigma n^k f_n(x)$ is absolutely convergent.

The proof is not difficult. The other theorems relate to cases in which condition (ii.) or (ii.)′ is not satisfied. I have included proofs of these theorems in a paper which will be published in the *Mathematische Annalen.*

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* The conditions actually stated by Herr Fejér differ from the above in the restriction of $f_n(x)$ to be of the form $\phi(nx)$, and the substitution for (ii.) and (iii.) of the conditions

$$|\phi(t)| < \frac{K}{t^{2+\rho}}, \qquad |\phi''(t)| < \frac{K}{t^{2+\rho}},$$

where $\rho > 0$. The proof of the theorem as I state it may be made a good deal simpler than Herr Fejér's proof.

CORRECTIONS

p. 249, *line* 16, *and elsewhere.* For Theorem II read Theorem 2.

p. 250, *line* 4. For (1) read (1)′.

p. 251, *lines* 4–5. For 'conditions of I' read 'conditions of I.a'.

p. 252, *line* 6 *up*. After $2\log 2$ read $-4(\frac{1}{2}$.

p. 260, *line* 11. For $a_1 = a_1$ read $a_1' = a_1$.

—— *lines* 4 *and* 8 *up*. For s_ν read s_ν'.

—— *last line*. For s_n and S_{n-1} read s_n' and S_{n-1}'.

COMMENTS

In his revised edition of Dirichlet's *Vorlesungen über Zahlentheorie*,† Dedekind included two results more general than (*a*) and (*b*) of § 1.

(A) *If* $\sum a_n$ *converges, then*

(i) $\sum a_n f_n$ *converges when* $\sum |\Delta f_n| < \infty$;

(ii) $\sum a_n f_n(x)$ *is continuous when* $f_n(x)$ *is continuous and* $\sum |\Delta f_n(x)| < K$.

(B) *If* $\sum a_n$ *has bounded partial sums, then*

(i) $\sum a_n f_n$ *converges when* $\sum |\Delta f_n| < \infty$ *and* $f_n \to 0$;

(ii) $\sum a_n f_n(x)$ *is continuous when* $f_n(x)$ *and* $\sum |\Delta f_n(x)|$ *are continuous and* $f_n(x) \to 0$ *as* $n \to \infty$.

Cahen‡ proved:

(C) *If* $\sum a_n$ *has bounded partial sums, then* $\sum a_n f_n(x)$ *converges uniformly when* $\sum |\Delta f_n(x)|$ *converges uniformly and* $f_n(x) \to 0$ *as* $n \to \infty$.

Hardy's Corollary to I.a 1 is Dedekind's (A) (ii); I.b 1 is (B) (i); version (ii) of I.b 2 (with 'uniformly convergent' omitted) is (B) (ii); version (i) of I.b 2 (with 'continuous' omitted) is Cahen's (C).

In I.a 1 it is to be understood that $|f_0(x)| < K$ (as in I.a); this is not needed in I.b. In I.b 1 and I.b 2 it is to be understood that $f_n \to 0$ as $n \to \infty$ (as in the statement of I.b).

In I.b it is necessary that $f_n(x) \to 0$ *uniformly*. But if this property is not assumed, and is deduced by means of Dini's theorem, it must be assumed that $f_n(x)$ is *continuous*.

Arguments of Hadamard§ (for fixed x) and Hahn|| (for variable x) show that, in Dedekind's and Cahen's theorems, *the conditions on* f_n *are necessary and sufficient for the series* $\sum a_n f_n$ *to have the relevant property for all series* $\sum a_n$ *of the class considered.*

If (A) (ii) is restated with continuity at a single point (as established by Dedekind's proof), it may be recognized as the first general formulation of the conditions for a series-to-function method of summability to be convergence-preserving. These conditions were shown to be necessary and sufficient by Hahn (loc. cit.); for other references see Agnew†† and D.S., p. 91.

Theorems 2 and 2.a 1 are an important introduction to 1908, 1. In particular, they give correct versions,‡‡ in the case $k = 1$, of Theorems A and B of 1908, 1, which meet the criticisms made by Bohr in the general case; see 1910, 1, § 2. The unpublished paper mentioned at the end is 1907, 6.

† 2nd edn. (1871) Suppl. IX, § 143, pp. 376–9; see the Comments on 1904, 1.

‡ *Ann. de l'École norm. sup.* (3), 11 (1894), 75–164 (79).

§ *Acta Math.* 27 (1903), 177–84.

|| *Monatsh. für Math. u. Phys.* 32 (1922), 3–88.

†† *American Math. Monthly* 53 (1946), 251–9.

‡‡ It is to be understood that $|f_0(x)| < K$.

ON CERTAIN OSCILLATING SERIES.

By G. H. HARDY, Trinity College, Cambridge.

§ 1. LET

$$f(x) = \Sigma a_n x^n$$

be a power series whose radius of convergence is unity. Abel's theorem states that, if Σa_n is convergent, $f(x)$ tends to a limit, equal to the sum of the series, as x approaches the point $x = 1$ along the real axis; and this theorem has led to a whole series of more general theorems, differing widely in their details, but resembling one another in their general aim. They assume that the characteristics of the sequence (a_n) are known, and their aim is to state in terms of these characteristics conditions sufficient to ensure that $f(x)$ shall tend to a limit when x, in one way or another, approaches the point $x = 1$.

A number of these theorems, for example, are of the following nature. It is supposed that the series Σa_n, although divergent, is 'summable' by one or other of the various methods which have been given for the summation of divergent series, such as Césaro's method of mean values, or Borel's exponential method. It is then proved that $f(x)$ tends to the limit s, s being the 'sum' of the series.

In this paper I propose to start from one of these methods of summation, indeed the simplest—Césaro's original method of mean values. The fundamental theorem dealing with this method of summation was first proved by Frobenius. A well known generalisation of Frobenius' theorem was given by Hölder. But I shall be concerned not with Hölder's theorems but with some generalisations of a somewhat different kind indicated more recently by Mr. L. Fejèr. These generalisations are of particular interest because, when we attempt to discover how far they may be pushed, we find that after a certain point further generalisation is impossible and the results of such generalisation demonstrably false. We are in fact led to consider sequences (a_n) summable (by one method or another) to the sum s, but such that the associated function $f(x)$, instead of tending to the limit s as x approaches 1, oscillates between finite or infinite limits of indetermination. The results are thus in a sense negative, whereas the results of all the theorems to which I have referred are positive.

1907, 5 Quarterly Journal of Mathematics, 38, 269–88.

§ 2. A series Σa_n may be said to be *summable* by Césaro's method of mean values if

$$s_n = a_0 + a_1 + \ldots + a_n,$$

and
$$s_n^{(1)} = \frac{s_0 + s_1 + \ldots + s_n}{n+1}$$

tends to a finite limit s for $n = \infty$. The simplest example of such a summable series is Leibniz' series

$$1 - 1 + 1 - 1 + \ldots,$$

whose sum is $\frac{1}{2}$. Since

$$a_n = s_n - s_{n-1} = (n+1)\, s_n^{(1)} - 2ns_{n-1}^{(1)} + (n-1)\, s_{n-2}^{(1)}$$

it is easy to see that

$$\lim a_n/n = 0.$$

The class of series which are summable in this sense is therefore a comparatively restricted one.

Frobenius, generalising Abel's well known theorem, proved that if Σa_n *is summable to the sum* s, $\Sigma a_n x^n$ *is convergent for* $|x| < 1$, *and its sum* $F(x)$ *tends to the limit* s *when* x *tends to* 1 *along the real axis.* As was stated above, a generalisation of this theorem was given by Hölder: and other more general theorems have been proved which adopt more general hypotheses as to the way in which x approaches its limiting value. With these theorems we shall not be directly concerned at present.

§ 3. In a recent paper in the *Mathematische Annalen* Mr. L. Fejér has proved a theorem which shows, as a particular case, that a theorem similar to that of Frobenius holds for series of the type

$$\Sigma a_n x^{n^2} \text{ or } \Sigma a_n x^{n^k},$$

where k is an integer greater than 2: *e.g.* that *if* Σa_n *is summable to sum* s

$$\lim_{x=1} \Sigma a_n x^{n^2} = s.$$

This, it should be observed, is not obviously a mere corollary of Frobenius' theorem. We may of course regard $\Sigma a_n x^{n^2}$ as an ordinary power series lacking a number of terms: but what Frobenius' theorem asserts is that if

$$(1) \quad a_0 + a_1 + 0 + 0 + a_2 + 0 + 0 + 0 + 0 + a_3 + \ldots$$

is summable, then $\lim \Sigma a_n x^{n^2} = s$ and not that this is so if $a_0 + a_1 + a_2 + a_3 + \ldots$ is summable; and the two hypotheses are *prima facie* quite distinct. It is *a priori* quite probable that there are summable series Σa_n such that (1) is not summable, or summable series (1) such that Σa_n is not summable.

On the other hand it seems *a priori* about equally likely that the summability of one series involves that of the other and the equality of the two sums. If this should prove to be the case Herr Fejèr's theorem will prove to be a corollary of that of Frobenius.

It seems worth while, before we proceed further, to settle this point. The attempt to do so leads to a general theorem of some interest in the abstract theory of divergent series.

§4. THEOREM. *If* $b_0 + b_1 + b_2 + \ldots,$

$$c_0 + c_1 + c_2 + \ldots$$

are two divergent series of positive terms, the equation

$$\lim_n \frac{b_0 s_0 + b_1 s_1 + \ldots + b_n s_n}{b_0 + b_1 + \ldots + b_n} = s$$

involves the equation

$$\lim_n \frac{c_0 s_0 + c_1 s_1 + \ldots + c_n s_n}{c_0 + c_1 + \ldots + c_n} = s,$$

if either (a) $\frac{c_n}{b_n} > \frac{c_{n+1}}{b_{n+1}}$ *for all values of* n, *or (b)* $\frac{c_n}{b_n} < \frac{c_{n+1}}{b_{n+1}}$ *for all values of* n, *and*

$$\frac{b_0 + b_1 + \ldots + b_n}{b_n} < K \frac{c_0 + c_1 + \ldots + c_n}{c_n}$$

for all values of n.*

Before proceeding to the proof of this I note that it is sufficient to postulate the divergence of Σc_ν only.

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* I am indebted to Prof. Bromwich, to whom I had communicated this theorem, for the following information.

Césaro gave the theorem for the case in which $\frac{c_\nu}{b_\nu} < \frac{c_{\nu+1}}{b_{\nu+1}}$ (Case (*a*) above) in the *Bulletin des Sc. Math*, Sér. 2, t. 13, p. 51, and refers to it elsewhere, particularly with reference to the case in which $b_n = 1$. The second part of the theorem, which is what is wanted for our present purposes, appears to be new.

Prof. Bromwich has constructed a simpler proof, which I do not give here, as it depends upon a generalisation of 'Abel's lemma' which would have to be established first. I may refer the reader to §§ 153–155 of Prof. Bromwich's forthcoming book on the Theory of Infinite Series (Macmillan).

For (*a*) if $\frac{c_\nu}{b_\nu} > \frac{c_{\nu+1}}{b_{\nu+1}}$ or $\frac{b_{\nu+1}}{b_\nu} > \frac{c_{\nu+1}}{c_\nu}$ it is obvious that the series Σb_ν diverges more rapidly than the series Σc_ν. If (*b*) $\frac{c_\nu}{b_\nu} < \frac{c_{\nu+1}}{b_{\nu+1}}$ or $\frac{b_{\nu+1}}{b_\nu} < \frac{c_{\nu+1}}{c_\nu}$ the series of *b*'s is *less* divergent: but it must diverge if the conditions of the theorem are satisfied. For if Σc_ν is divergent, so (as is well known) is

$$\Sigma \frac{c_\nu}{c_0 + c_1 + \ldots + c_\nu},$$

and so therefore is

$$\Sigma \frac{b_\nu}{b_0 + b_1 + \ldots + b_\nu};$$

and this last series would converge if Σb_ν converged

I proceed now to prove the theorem.

Let
$$B_\nu = b_0 + b_1 + \ldots + b_\nu,$$
$$C_\nu = c_0 + c_1 + \ldots + c_\nu,$$
$$\beta_\nu = \frac{1}{B_\nu}(b_0 s_0 + \ldots + b_\nu s_\nu),$$
$$\gamma_\nu = \frac{1}{C_\nu}(c_0 s_0 + \ldots + c_\nu s_\nu).$$

Then
$$b_\nu s_\nu = B_\nu \beta_\nu - B_{\nu-1}\beta_{\nu-1},$$

and so
$$\gamma_n = \frac{1}{C_n}\sum_{\nu=0}^{n} \frac{c_\nu}{b_\nu}(B_\nu \beta_\nu - B_{\nu-1}\beta_{\nu-1}).$$

Now
$$\beta_\nu = s + \epsilon_\nu \quad (\lim \epsilon_\nu = 0),$$

and so
$$\gamma_n - s = \frac{1}{C_n}\sum_{\nu=0}^{n} \frac{c_\nu}{b_\nu}(B_\nu \epsilon_\nu - B_{\nu-1}\epsilon_{\nu-1}) = \frac{1}{C_n}\Big(\sum_{\nu=0}^{N-1} + \sum_{\nu=N}^{n}\Big) = R_1 + R_2, \text{ say.}$$

First, let us suppose $\frac{c_\nu}{b_\nu} > \frac{c_{\nu+1}}{b_{\nu+1}}$. Then

$$R_2 = \frac{1}{C_n}\sum_{\nu=N}^{n} \frac{c_\nu}{b_\nu}(B_\nu \epsilon_\nu - B_{\nu-1}\epsilon_{\nu-1}) = \frac{1}{C_n}\Big\{-\frac{c_N}{b_N}B_{N-1}\epsilon_{N-1}$$
$$+ \sum_{\nu=N}^{n-1}\Big(\frac{c_\nu}{b_\nu} - \frac{c_{\nu+1}}{b_{\nu+1}}\Big)B_\nu \epsilon_\nu + \frac{c_n}{b_n}B_n \epsilon_n\Big\}.$$

Choose N so that $|\epsilon_\nu| < \epsilon$ for $\nu \geqq N-1$.

Then
$$\left|\sum_N^{n-1}\left(\frac{c_\nu}{b_\nu}-\frac{c_{\nu+1}}{b_{\nu+1}}\right)B_\nu\epsilon_\nu+\frac{c_n}{b_n}B_n\epsilon_n\right|$$
$$<\epsilon\left\{\sum_N^{n-1}\left(\frac{c_\nu}{b_\nu}-\frac{c_{\nu+1}}{b_{\nu+1}}\right)B_\nu+\frac{c_n}{b_n}B_n\right\}$$
$$=\epsilon\left\{\frac{c_N}{b_N}B_N+\sum_{N+1}^{n}\frac{c_\nu}{b_\nu}(B_\nu-B_{\nu+1})\right\}$$
$$=\epsilon\left(\frac{c_N}{b_N}B_N+c_{N+1}+c_{N+2}+\ldots+c_n\right)$$
$$<\epsilon\left(\frac{c_N}{b_N}B_N+C_n\right),$$

and hence
$$|R_2|<\frac{\epsilon}{C_n}\frac{c_N}{b_N}(B_{N-1}+B_N)+\epsilon.$$

When N is fixed we can certainly determine n_0 so that

$$(1) \quad |R_2|<2\epsilon \ (n \geqq n_0).$$

But we can also choose n_1 so that

$$|R_1|=\frac{1}{C_n}\left|\sum_{\nu=0}^{N-1}\frac{c_\nu}{b_\nu}(B_\nu\epsilon_\nu-B_{\nu-1}\epsilon_{\nu-1})\right|<\epsilon,$$

for $n \geqq n_1$: and so if n is not less than the greater of n_0 and n_1

$$|\gamma_n-s|<3\epsilon,$$

or
$$\lim_{n=\infty}\gamma_n=s.$$

The theorem is therefore established under condition (a).

Secondly, let us suppose the conditions (b) satisfied. Then if N is chosen as before

$$\left|-\frac{c_N}{b_N}B_{N-1}\epsilon_{N-1}+\sum_{\nu=N}^{n-1}\left(\frac{c_\nu}{b_\nu}-\frac{c_{\nu+1}}{b_{\nu+1}}\right)B_\nu\epsilon_\nu\right|$$
$$<\epsilon\left\{\frac{c_N}{b_N}B_{N-1}+\sum_N^{n-1}\left(\frac{c_{\nu+1}}{b_{\nu+1}}-\frac{c_\nu}{b_\nu}\right)B_\nu\right\}$$
$$=\epsilon\left\{\frac{c_nB_{n-1}}{b_n}-c_N-c_{N+1}-\ldots-c_{n-1}\right\}$$
$$<\epsilon\frac{c_nB_{n-1}}{b_n}<\epsilon\frac{c_nB_n}{b_n},$$

and so

$$|R_2| < \frac{2\epsilon c_n B_n}{b_n C_n} < 2K\epsilon,$$

by the second of conditions (b). The proof of the theorem is then completed as before.

§5. The most interesting applications of this theorem are obtained by supposing either the b's or the c's all to be equal to 1.

(1) Let $b_\nu = 1$. Then we deduce that *if* Σa_n *is summable to sum s by Césaro's method, then*

$$\lim_{n=\infty} \frac{c_0 s_0 + c_1 s_1 + \ldots + c_n s_n}{c_0 + c_1 + \ldots + c_n} = s,$$

if either (a) Σc_n *is divergent and* $c_{n+1} < c_n$ *or* (b) $c_{n+1} > c_n$ *and*

$$\frac{c_0 + c_1 + \ldots + c_n}{c_n} > Kn.$$

As examples, suppose

(i) $$a_n = (-1)^n, \quad s_{2m} = 1, \quad s_{2m+1} = 0,$$

and condition (a) satisfied. We deduce that if Σc_n is any divergent series of decreasing positive terms

$$\lim_{m=\infty} \frac{c_0 + c_2 + \ldots + c_{2m}}{c_0 + c_1 + c_2 + \ldots + c_{2m}} = \tfrac{1}{2},$$

or $$\lim_{m=\infty} \frac{c_1 + c_3 + \ldots + c_{2m\pm 1}}{c_0 + c_2 + \ldots + c_{2m}} = 1,$$

a result which is in any case obvious.

(ii) Suppose

$$c_n = 2n + 1,$$
$$c_0 + c_1 + \ldots + c_n = (n+1)^2,$$

and conditions (b) are satisfied.

Thus if Σa_n is summable

$$\lim_{n=\infty} \frac{s_0 + 3s_1 + 5s_2 + \ldots + (2n+1)\, s_n}{(n+1)^2} = s.$$

(iii) Suppose

$$c_n = 2^n,$$
$$c_0 + c_1 + \ldots + c_n = 2^{n+1} - 1,$$

and conditions (b) are *not* satisfied. It is easy to see that in this case the result of the theorem is not true. In fact if we

start from the series $1-1+1-\ldots$ as in example (i)

$$\frac{c_0s_0+\ldots+c_ns_n}{c_0+\ldots+c_n}=\frac{1+2^2+2^4+\ldots+2^n}{1+2+2^2+\ldots+2^n}$$
$$=\frac{2^{n+2}-1}{3\,(2^{n+1}-1)}\quad (n \text{ even}),$$

but
$$=\frac{1+2^2+\ldots+2^{n+1}}{1+2+2^2+\ldots+2^n}=\tfrac{1}{3}\quad (n \text{ odd}),$$

and the limits of these two expressions for $n=\infty$ are $\frac{2}{3}$, $\frac{1}{3}$ respectively.

(2) Taking the c's all equal to unity we find that *if*

$$\lim_{n=\infty}\frac{b_0s_0+b_1s_1+\ldots+b_ns_n}{b_0+b_1+\ldots+b_n}=s,$$

Σb_n being a divergent series such that either (a) $b_{n+1}>b_n$ or (b) $b_{n+1}<b_n$ and

$$\frac{b_0+b_1+\ldots+b_n}{b_n}<Kn,$$

then the series Σa_n is summable. In particular if

$$\lim_{n=\infty}\frac{s_0+3s_1+5s_2+\ldots+(2n+1)\,s_n}{(n+1)^2}=s$$

the series is summable, which is the converse of the result proved under (ii).

§ 6. Now if σ_n denote the sum of $n+1$ terms of the series

(1) $\quad a_0+a_1+0+0+a_2+0+0+0+0+a_3+\ldots,$

$$\sigma_0=s_0,\quad \sigma_1=s_1=\sigma_2=\sigma_3,$$
$$\sigma_4=s_2=\sigma_5=\ldots\ldots\ldots=\sigma_8,$$
$$\sigma_9=s_3=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence

$$\frac{\sigma_0+\sigma_1+\ldots+\sigma_{\nu^2-q}}{\nu^2}=\frac{s_0+3s_1+5s_2+\ldots+(2\nu-1)\,s_{\nu-1}}{\nu^2},$$

and, if Σa_n has the sum s, the limit of this for $\nu=\infty$ is s, and conversely, by the results of § 3. Also if $0\leqq q<2\nu+1$,

$$\frac{\sigma_0+\sigma_1+\ldots+\sigma_{\nu^2+q}}{\nu^2+q+1}=\frac{\nu^2}{\nu^2+q+1}\left(\frac{\sigma_0+\sigma_1+\ldots+\sigma_{\nu^2-1}}{\nu^2}\right)$$
$$+\frac{\sigma_{\nu^2}+\sigma_{\nu^2+1}+\ldots+\sigma_{\nu^2+q}}{\nu^2+q+1}$$
$$=\frac{\nu^2}{\nu^2+q+1}\left(\frac{\sigma_0+\sigma_1+\ldots+\sigma_{\nu^2-1}}{\nu^2}\right)+\frac{(q+1)\,s_\nu}{\nu^2+q+1},$$

T 2

And, since $\lim \frac{s_\nu}{\nu} = 0$, the limit of this for $\nu = \infty$ is s. Hence the summability of (1) to sum s involves that of Σa_n, and conversely.

We can therefore deduce the equation

$$\lim_{x=1} \Sigma a_n x^{n^2} = s$$

directly from Frobenius' theorem. The same is true for $\Sigma a_n x^{n^p}$. On the other hand, it follows from § 3 that this line of proof would fail for

$$\Sigma a_n x^{2^n},$$

and generally for $\Sigma a_n x^{a^n}$, where a is any integer > 1.

§ 7. Let us consider more generally the function

$$F(x) = \Sigma a_n x^{\psi(n)},$$

where Σa_n is a summable series, and $\psi(n)$ is a function of n which is integral for all integral values of n, and tends to ∞ with n steadily and according to some simple and regular law. The most obvious forms of $\psi(n)$ are

$$n,\ n^2,\ n^3,\ \ldots,\ 2^n,\ 3^n,\ 4^n,\ \ldots,\ n!,\ (n!)^2,\ \ldots,$$

$$2^{n^2},\ 3^{n^2},\ \ldots,\ 2^{n^3},\ \ldots,\ 2^{2^n},\ \ldots.$$

We have seen that for $\psi(n) = n,\ n^2,\ n^3,\ \ldots$

$$\lim_{x=1} F(x) = s,$$

while our proof fails for $\psi(n) = 2^n$. It is suggested by this and by other considerations* that this result may no longer be true when the increase of $\psi(n)$ is rapid, and that the form of $\psi(n)$ (among those written above) whose increase is least rapid, and for which $F(n)$ does not tend to s for $x = 1$, may be 2^n. I propose to show that this conjecture is correct—incidentally I shall be led to some formulæ connected with the function

$$F_a(x) = x - x^a + x^{a^2} - \ldots,$$

* It can be shown directly that if the increase of $\psi(n)$ is sufficiently rapid the function

$$x^{\psi(0)} - x^{\psi(1)} + x^{\psi(2)} - \ldots$$

oscillates between finite limits of indeterminations for $x = 1$. See Bromwich and Hardy, *Proc. L.M.S.*, Ser. 2, Vol. 2, p. 171, and the last paragraph of this paper.

where a is an integer $\geqq 2$, which seem to me to be of some interest.

§ 8. In the first place I snall give a very simple proof, suggested to me by Mr. J. H. Maclagan Wedderburn, that $F_a(x)$ oscillates between finite limits of indetermination for $x=1$.

It is clear that $F_a(x)$ satisfies the equation

$$(1) \qquad F_a(x)+F_a(x^a)=x.$$

Also we can determine a function of the real variable x, which satisfies this equation and is regular at $x=1$. Such a function is (as is easily verified) the function

$$\Phi_a(x)=\sum_0^\infty \frac{(-)^n\{\log(1/x)\}^n}{n!\,(1+a^n)}.$$

The function

$$\theta_a(x)=F_a(x)-\Phi_a(x)$$

satisfies the equation

$$\theta_a(x)+\theta_a(x^a)=0.$$

Moreover it is obvious that $\theta_a(x)$ *cannot be identically zero.**

Now $\qquad \theta_a(x)=-\theta_a(x^a)=\theta_a(x^{a^2})=\theta_a(x^{a^{2\mu}})$

for all positive integral values of μ: or if

$$\theta_a(x)=\Theta_a\{\log\log(1/x)\},$$

$$\Theta_a\{\log\log(1/x)\}=\Theta_a\{\log\log(1/x)+2\mu\log a\}.$$

Hence $\theta_a(x)$ *is a periodic function of* $\log\log(1/x)$, *with a period* $2\log a$.

But as x tends to unity $\log\log(1/x)$ tends to $-\infty$. *Hence* $\theta_a(x)$ *oscillates for* $x=1$.

Thus $f_a(x)$ is the sum of two functions, one of which tends for $x=1$ to the limit $\frac{1}{2}$, while the second oscillates; and therefore $f_a(x)$ itself oscillates.

§ 9. There is another method of an even more elementary character by which we may attempt to establish this result. It does not indeed succeed completely, but is interesting in that it leads to *numerical* results.

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* $\Phi_a(a)$ is an integral function of $\log(1/x)$, with an essential singularity for $x=0$; $F_a(x)$ is regular at $x=0$, and has the unit circle as a line of essential singularities.

Since

$$F_a(x) + F_a(x^a) = x,$$

it is evident that *if* $F_a(x)$ *tends to a limit for* $x = 1$ *that limit must be* $\frac{1}{2}$. Hence if we can prove that for an infinite series of values of x, whose limit is 1,

$$F_a(x) > \tfrac{1}{2} + \delta \quad (\delta > 0).$$

it will follow that $F_a(x)$ oscillates for $x = 1$. Now

$$F_a(x) = \sum_{\nu=0}^{\infty} (x^{a^{2\nu}} - x^{a^{2\nu+1}}) = \sum_{\nu=0}^{\infty} u_\nu(x),$$

say. All the terms $u_\nu(x)$ are positive. Also $u_\nu(x)$, considered as a function of x, is a maximum when

$$a^{2\nu} x^{a^{2\nu}} - a^{2\nu+1} x^{a^{2\nu+1}} = 0,$$

$$x^{a^{2\nu}} = (1/a)^{1/(a-1)},$$

$$u_\nu(x) = (1/a)^{1/(a-1)} - (1/a)^{a/(a-1)} = (1/a)^{1/(a-1)} (1 - 1/a).$$

If this is greater than $\frac{1}{2}$ the question is immediately settled. Now, if $a = 5$,

$$(\tfrac{1}{5})^{\frac{1}{4}} (1 - \tfrac{1}{5}) > \tfrac{4}{5} \times \cdot 66 > \cdot 52.$$

Hence the series oscillates when $a = 5$, and it is easy to see that the same conclusion holds for any larger value of a. The limits of oscillation for $a = 5$ extend at any rate beyond ·48 and ·52. For $a = 6$ they extend at any rate beyond ·42 and ·58. When a is large

$$(1/a)^{1/(a-1)} = 1 - \frac{\log a}{a-1} + \left[\left(\frac{\log a}{a}\right)^2\right]$$

(the square bracket denoting the *order* of the omitted terms). Hence, when a is large the range of oscillation is very nearly the whole interval (0, 1).

When $a = 2, 3, 4$,

$$(1/a)^{1/(a-1)} (1 - 1/a) < \tfrac{1}{2},$$

and this proof fails. In the cases of $a = 3, 4$ it may (as Mr. Wedderburn has shown) be reestablished by taking account of the values of the terms adjacent to $u_\nu(x)$, when x has the value which makes $u_\nu(x)$ a maximum. And, of course, for $x = 5, 6, \ldots$ the limits of the range of oscillation given above may be extended by taking account of these terms. When

$n=2$ it does not seem possible (at any rate without laborious numerical calculations) to complete the result in this way.

§ 10. The actual form of the function $\theta_a(x)$ of § 8 may be found as follows. Take the contour integral

$$\int \Gamma(-u)\frac{y^u}{1+a^u}du$$

round the rectangle

$$-\kappa+iH,\ -\kappa-iH,\ N+\tfrac{1}{2}-iH,\ N+\tfrac{1}{2}+iH,$$

where M, N are large positive integers and

$$H=\frac{(2M+\frac{1}{2})\pi}{\log a},$$

and κ is positive.

First make M tend to ∞. If $u=\xi+i\eta$, and $|\eta|$ tends to ∞, while $|\xi|$ remains between fixed limits, $|\Gamma(u)|$ tends to zero like

$$e^{-\frac{1}{2}\pi|\eta|}.$$

Moreover, if $y=\rho e^{i\phi}$, and y^u has its principal value,

$$|y^u|=e^{\xi\log\rho-\eta\phi}$$

and so, if

$$-\tfrac{1}{2}\pi+\epsilon<\phi<\tfrac{1}{2}\pi-\epsilon,$$

$$|\Gamma(-u)y^u|<e^{-\epsilon|\eta|}.$$

Finally, if $u=\xi+iH$,

$$a^u=ia^\xi,$$

and so $|1+a^u|>1$, as ξ varies from $-\kappa$ to $N+\frac{1}{2}$; and similarly if $u=\xi-iH$. Hence, when M tends to infinity, the contributions of the sides of the rectangle parallel to the real axis tend to zero; and hence

$$\left(-\int_{-\kappa-i\infty}^{\kappa+i\infty}+\int_{N+\frac{1}{2}-i\infty}^{N+\frac{1}{2}+i\infty}\right)\Gamma(-u)\frac{y^u}{1+a^u}du=2\pi i\Sigma R,$$

where R denotes a residue of the subject of integration at a pole within the strip bounded by the two lines of integration.

Again,

$$\Gamma(-N-\tfrac{1}{2}-i\eta)=-\frac{\pi}{\sin\{(N+\frac{1}{2}+i\eta)\pi\}\Gamma(N+\frac{3}{2}+i\eta)}$$

$$=\frac{(-)^{N-1}\pi}{\Gamma(N+\frac{3}{2}+i\eta)\cosh\eta\pi},$$

from which it follows immediately that the limit of the rectilinear integral along the distant side of the contour vanishes for $N=\infty$.

Hence in the limit

$$(1)\qquad -\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\frac{y^u}{1+a^u}\,du=2\pi i\Sigma R,$$

the summation being now extended to all poles whose real part $>-\kappa$.

There are two series of poles. The poles $u=n$, where n is a positive integer, contribute

$$-\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\frac{y^n}{1+a^n}.$$

The poles of $1/(1+a^u)$ are given by

$$u=\frac{(2k+1)\pi i}{\log a}$$

where k is any integer; and these poles contribute

$$-\frac{1}{\log a}\sum_{-\infty}^{\infty}\Gamma\left\{-\frac{(2k+1)\pi i}{\log a}\right\}y^{\{(2k+1)\pi i\}/\log a},$$

where

$$y^{\{(2k+1)\pi i\}/\log a}=\exp\left\{\frac{(2k+1)\pi i\log y}{\log a}\right\},$$

$\log y$ having its principal value.

Hence

$$(2)\quad -\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\frac{y^u}{1+a^u}\,du=2\pi i\left\{-\sum_{0}^{\infty}\frac{(-)^n}{n!}\frac{y^n}{1+a^n}\right.$$
$$\left.-\frac{1}{\log a}\sum_{-\infty}^{\infty}\Gamma\left\{-\frac{(2k+1)\pi i}{\log a}\right\}y^{\{(2k+1)\pi i\}/\log a}\right\}.$$

Now

$$-\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\frac{y^u}{1+a^u}\,du=-Y_0+Y_1-Y_2+\ldots+(-)^n Y_{n-1}+R_n,$$

where $$Y_\nu=\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\,y^u a^{\nu u}du,$$

and $$R_n=(-)^{n+1}\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\,y^u\frac{a^{nu}}{1+a^u}\,du.$$

Since $$|a^{nu}| = a^{-n\kappa},$$
it is easy to see that $\lim_{n=\infty} R_n = 0$.

The integral Y_ν may be evaluated by the method used earlier in this section. We find that

$$Y_\nu = 2\pi i \sum_{m=0}^{\infty} \frac{(-)^m}{m!} (ya^\nu)^m$$
$$= 2\pi i e^{-ya^\nu}.$$

Hence finally, *if* $-\pi/2 + \epsilon <$ am $y < \pi/2 - \epsilon$, *we have proved the relation*

$$(3) \qquad e^{-y} - e^{-ya} + e^{-ya^2} + \ldots = \sum_{0}^{\infty} \frac{(-)^n}{n!} \frac{y^n}{1+a^n}$$
$$+ \frac{1}{\log a} \sum_{-\infty}^{\infty} \Gamma\left\{-\frac{(2k+1)\pi i}{\log a}\right\} y^{\{(2k+1)\pi i\}/\log a}.$$

§ 11. In this equation we put
$$e^{-y} = x, \; y = \log(1/x).$$

Then $R(y) > 0$, if $|x| < 1$. The positive half of the y-plane corresponds to the interior of the unit circle in the x-plane, taken however infinitely many times over.

We suppose
$$x = re^{i\theta}, \; \log(1/x) = \log(1/r) - i\theta,$$
where $$-\pi < \theta < \pi,$$
and make the logarithm one-valued by a cut along the negative real axis. Within the region thus defined

$$(4) \qquad F_a(x) = x - x^a + x^{a^2} - \ldots = \sum_{0}^{\infty} \frac{(-)^n}{n!} \frac{\{\log(1/x)\}^n}{1+a^n}$$
$$+ \frac{1}{\log a} \sum_{-\infty}^{\infty} \Gamma\left\{-\frac{(2k+1)\pi i}{\log a}\right\} \{\log(1/x)\}^{\{(2k+1)\pi i\}/\log a},$$

where $$\{\log(1/x)\}^n = \exp(n \log\log 1/x)$$
and $$\{\log(1/x)\}^{\{(2k+1)\pi i\}/\log a} = \exp\left\{\frac{(2k+1)\pi i}{\log a} \log\log(1/x)\right\},$$
and $$\log\log(1/x) = \log\{\log(1/r) - i\theta\}$$
$$= \log\sqrt{[\{\log(1/r)\}^2 + \theta^2]} - i \tan^{-1}[\theta/\{\log(1/r)\}]$$
where $$-\tfrac{1}{2}\pi < \tan^{-1}[\theta/\{\log(1/r)\}] < \tfrac{1}{2}\pi.$$

The series $F_a(x)$ has the unit circle as a *coupure*. The series $\Sigma \frac{(-)^n}{n!} \frac{\{\log(1/x)\}^n}{1+a^n}$ is on the other hand an integral function of $\log(1/x)$. It is interesting to notice that these two functions may be directly transformed into one another (of course by a purely formal process) as follows. In fact

$$\sum_{(n)}\sum_{(m)} (-)^{m+n} \frac{y^m a^{mn}}{m!} = \sum_{(n)} (-)^n e^{-ya^n},$$

$$\sum_{(m)}\sum_{(n)} (-)^{m+n} \frac{y^m a^{mn}}{m!} = \sum_{(n)} (-)^m \frac{y^m}{m!(1+a^m)}.$$

The *double* series

$$\sum_{(m,n)}\sum \frac{(-)^{m+n} y^m a^m}{m!}$$

is of course divergent.

§ 12. It is interesting to notice the corresponding formulæ connected with the function

$$\psi_a(x) = x + x^a + x^{a^2} + \ldots,$$

which evidently tends to $+\infty$ as x tends to unity. The functional equation is in this case

$$\psi_a(x) - \psi_a(x^a) = x.$$

Now the function

$$\chi_a(x) = \sum_1^\infty \frac{(-)^{n-1}\{\log(1/x)\}^n}{n!(a^n - 1)} - \frac{\log\log(1/x)}{\log a}$$

is also a solution of the equation, and

$$\theta_a(x) = \psi_a(x) - \chi_a(x)$$

satisfies $\theta_a(x) = \theta_a(x^a)$. We deduce, as before, that $\theta_a(x)$ is a periodic function of $\log\log(1/x)$, in this case with a period $\log a$.

Hence

$$x + x^a + x^{a^2} + \ldots = -\frac{\log\log(1/x)}{\log a} + \lambda(x),$$

where $\lambda(x)$ is a function of x which oscillates between finite limits of indetermination as x tends to unity.

That

$$x + x^a + x^{a^2} + \ldots \sim -\frac{\log\log(1/x)}{\log a} \sim \frac{\log\{1/(1-x)\}}{\log a},$$

for $x=1$, is easily deduced from known theorems. For let $a^n \leq \nu \leq a^{n+1}$. The sum of the first ν coefficients of $x+x^a+x^{a^2}+\ldots$ is $\geq n+1$ and $\leq n+2$. That of the first ν coefficients of $\log\{1/(1-x)\}$ is

$$1+\tfrac{1}{2}+\ldots+1/\nu=\log\nu+\gamma+\epsilon_\nu,$$

where $\lim\epsilon_\nu=0$. The ratio of these two sums lies between

$$\frac{(n+1)}{\log(a^{n+1})+\gamma+\epsilon_1},\quad \frac{(n+2)}{\log(a^n)+\gamma+\epsilon_2},$$

where ϵ_1 and ϵ_2 are small when n is large. Hence the limit of the ratio is $1/\log a$; and therefore

$$x+x^a+x^{a^2}+\ldots\sim\frac{\log\{1/(1-x)\}}{\log a}.$$

§ 13. The explicit formula corresponding to that of § 11 may be deduced from the integral

$$\int\Gamma(-u)\frac{y^u}{a^u-1}\,du.$$

The argument is exactly the same, except that the residue for $u=0$ requires especial calculation. We find that

$$-\int_{-\kappa-i\infty}^{-\kappa+i\infty}\Gamma(-u)\frac{y^u}{a^u-1}\,du=2\pi i\left\{\sum_1^\infty\frac{(-)^{n-1}y^n}{n!\,(a^n-1)}\right.$$

$$-\frac{1}{\log a}(\log y-\tfrac{1}{2}\log a+\gamma)$$

$$\left.-\frac{1}{\log a}\Sigma'\Gamma\left(-\frac{2k\pi i}{\log a}\right)y^{2k\pi i/\log a}\right\},$$

the dash denoting summation with respect to all values of k but $k=0$.

Hence we deduce

$$x+x^a+x^{a^2}+\ldots=\sum_1^\infty\frac{(-)^{n-1}\{\log(1/x)\}^n}{n!\,(a^n-1)}-\frac{1}{\log a}\log\log(1/x)$$

$$+\tfrac{1}{2}-\frac{\gamma}{\log a}-\frac{1}{\log a}\Sigma'\Gamma\left(-\frac{2k\pi i}{\log a}\right)\{\log(1/x)\}^{2k\pi i/\log a}.$$

§ 14. The considerations of §§ 6—10 are capable of generalisation.*

* The substance of this section was pointed out to me by Mr. Wedderburn.

Let us consider the difference equation

(1) $$f(x)+f(ax)=g(x).$$

A solution is given by

(2) $$f(x)=\sum_{\nu=0}^{\infty}(-)^{\nu} g(a^{\nu}x),$$

if this series is convergent.

Let us suppose this condition satisfied at any rate for real positive values of x. And let us suppose that

$$g(x)=\sum_0^{\infty}\alpha_n x^n$$

is a function of x expansible in a Taylor's series.

It is evident that

(3) $$F(x)=\sum_0^{\infty}\frac{\alpha_n x^n}{1+a^n}$$

is also a solution of (1). In the particular case hitherto considered we had

$$g(x)=e^{-x}, \quad f(x)=\Sigma(-)^{\nu}e^{-a^{\nu}x},$$
$$F(x)=\Sigma\frac{(-)^n x^n}{n!(1+a^n)},$$

The functions $f(x)$, $F(x)$ may be formally transformed into one another as follows:

$$f(x)=\Sigma(-)^{\nu}g(a^{\nu}x)=\sum_{(\nu)}\sum_{(\mu)}(-)^{\nu}\alpha_{\mu}x^{\mu}a^{\mu\nu}$$
$$=\sum_{(\mu)}\sum_{(\nu)}(-)^{\nu}\alpha_{\mu}x^{\mu}a^{\mu\nu}$$
$$=\sum_{(\mu)}\frac{\alpha_{\mu}x^{\mu}}{1+a^{\mu}}=F(x);$$

but, as the particular case shows, they will not be really the same function in general, the double series being divergent and the two repeated series not equivalent.

§ 15. If, *e.g.*, $g(x)=\dfrac{1}{1+x}$, our two solutions are

$$\frac{1}{1+x}-\frac{1}{1+ax}+\frac{1}{1+a^2x}-\dots,$$

and

$$\tfrac{1}{2} - \frac{x}{1+a} + \frac{x^2}{1+a^2} - \dots .$$

The two functions are obviously not identical: the first has poles for

$$x = -1, \ -1/a, \ -1/a^2, \ \dots,$$

and therefore has the origin for an essential singularity: the second is regular at the origin.

Arguing as in § 6 we arrive at the conclusion that *the series*

$$\frac{1}{1+x} - \frac{1}{1+ax} + \frac{1}{1+a^2x} - \dots$$

oscillates as x tends to zero along the positive real axis. It is equal to the sum of a function which tends to $\tfrac{1}{2}$ for $x=0$, and a periodic function of $\log x$.

The explicit expression of this function may be obtained much as in § 8. We use the contour integral

$$\int \frac{\pi}{\sin \pi u} \frac{x^u}{1+a^u} du,$$

and we find

$$\frac{1}{1+x} - \frac{1}{1+ax} + \frac{1}{1+a^2x} - \dots = \tfrac{1}{2} - \frac{x}{1+a} + \frac{x^2}{1+a^2} - \dots$$

$$- \frac{2\pi}{\log a} \sum_1^\infty \frac{\sin[\{(2k+1)\pi \log x\}/\log a]}{\sinh[\{(2k+1)\pi^2\}/\log a]}.$$

The series on the left furnishes another simple example of a series which does not tend for a limiting value of x to a limit equal to the sum of the divergent series $1-1+1\dots$, which is its limiting form.

We can obtain more general results concerning the series

$$x - \lambda e^{i\alpha} x^a + \lambda^2 e^{2i\alpha} x^{a^2} - \dots,$$

$$\frac{1}{1+x} - \frac{\lambda e^{i\alpha}}{1+ax} + \frac{\lambda^2 e^{2i\alpha}}{1+a^2x} - \dots,$$

where λ and α are real, by considering the contour integrals

$$\int \Gamma(-u) \frac{y^u}{1+\lambda e^{i\alpha} a^u} du, \quad \int \frac{\pi}{\sin \pi u} \frac{x^u}{1+\lambda e^{i\alpha} a^u} du.$$

The behaviour of these series as x tends to 1 (or 0) may also be discussed by the elementary methods of §§ 8, 12, 14.

§ 16. The series $\sum_0^\infty \frac{(-)^n y^n}{n!\,(1+a^n)}$, which occurs in § 8, may be transformed as follows:—

$$\sum_0^\infty \frac{(-)^n y^n}{n!\,(1+a^n)} = \tfrac{1}{2} + \sum_{n=1}^\infty \frac{(-)^n y^n}{n!} \sum_{m=1}^\infty (-)^{m-1} a^{-mn}$$

$$= \tfrac{1}{2} + \sum_{m=1}^\infty (-)^{m-1} \sum_{n=1}^\infty \frac{(-ya^{-m})^n}{n!}$$

$$= \tfrac{1}{2} + \sum_{m=1}^\infty (-)^{m-1} (e^{-ya^{-m}} - 1)$$

$$= \tfrac{1}{2} - (1 - e^{-y/a}) + (1 - e^{-y/a^2}) - \ldots.$$

This transformation is legitimate, since the double series

$$\sum_1^\infty \sum_1^\infty \frac{y^n a^{-mn}}{n!} = \sum_{m=1}^\infty (e^{ya^{-m}} - 1)$$

is convergent.

This transformation may also be deduced from the contour integral of § 10 by supposing $\kappa < 0$ (which cuts out the zeroes of $1 + a^u$) and expanding $1/(1+a^u)$ in descending, instead of ascending, powers of a^u.

Similarly

$$\sum_1^\infty \frac{(-)^{n-1} y^n}{n!\,(a^n - 1)} = (1 - e^{-y/a}) + (1 - e^{-y/a^2}) + (1 + e^{-y/a^3}) + \ldots.$$

§ 17. I return now to the series

$$\Sigma\,(-)^n x^{\psi(n)}$$

considered in § 7. The case next in interest is that in which $\psi(n) = n!$. Let then

$$F(x) = x - x^{1!} + x^{2!} - x^{3!} + \ldots$$

$$= \sum_{\nu=0}^\infty (x^{2\nu!} - x^{2\nu+1!}) = \sum_{\nu=0}^\infty u_\nu,$$

say. The maximum of u_ν is given by

$$x^{2\nu!} = (2\nu + 1)^{-1/2\nu}.$$

Then $x^{2\nu!} = e^{-\log(2\nu+1)/2\nu}$ is nearly equal to unity, while

$$x^{2\nu+1!} = (2\nu + 1)^{-(2\nu+1)/2\nu}$$

is small and of order $1/\nu$, $x^{2\nu+2}$ is small of order $\nu^{-\nu}$, and so on.

We deduce that

$$u_\nu + u_{\nu+1} + \ldots = 1 - [1/\nu].$$

On the other hand,

$$x^{2\nu-1!} = \exp\left\{-\frac{1}{(2\nu+1)\,2\nu}\log(2\nu+1)\right\}$$

$$= 1 - \left[\frac{\log\nu}{\nu^4}\right],$$

$$x^{2\nu-2!} = 1 - \left[\frac{\log\nu}{\nu^3}\right],$$

and so on. Hence we easily deduce that

$$u_0 + u_1 + \ldots + u_{\nu-1} = \left[\frac{\log\nu}{\nu^2}\right].$$

Thus, for $x = (2\nu+1)^{-1/(2\nu.2\nu!)}$,

$$f(x) = 1 - [1/\nu].$$

It is equally easy to show that, for $x = (2\nu+2)^{-1/\{(2\nu+1)(2\nu+1)!\}}$,

$$f(x) = [1/\nu].$$

Hence $f(x)$ oscillates for $x = 1$, the limits of oscillation being precisely 0 *and* 1.

The same conclusion obviously holds for all the forms of $\psi(n)$, included in our table of § 5, whose increase is greater than that of $n!$

Thus our general view of the behaviour of series of the type

$$x^{\psi(0)} - x^{\psi(1)} + x^{\psi(2)} - \ldots$$

for $x = 1$, $\psi(n)$ being an increasing function of n of a regular and simple type, such as

$$n,\ n^2,\ n^3,\ \ldots,\ 2^n,\ 3^n,\ \ldots,\ n!,\ (n!)^2, \ldots$$

is this:—

If the increase of $\psi(n)$ is not too rapid, the series tends for $x = 1$ to the limit $\frac{1}{2}$, the sum of the divergent series

$$1 - 1 + 1 - \ldots.$$

When the increase of $\psi(n)$ passes a certain limit indicated by the vertical bar the series oscillates for $x = 1$. The swing of the oscillations is greater the greater the increase of $\psi(n)$. If the increase of $\psi(n)$ is greater than or equal to that of $n!$

the swing covers the whole interval (0, 1). Beyond these limits it obviously cannot go.

I may remark in conclusion that it was shown that in a footnote to a paper, by Prof. Bromwich and myself, which appeared in 1904 in the *Proc. Lond. Math. Soc.*,* that functions of this kind could be constructed directly. It is easy to show by the method there outlined that

$$x^{p_0} - x^{p_1} + x^{p_2} - \dots$$

oscillates for $x = 1$, if

(1) $$p_{\nu+1} > p_\nu^k \ (k > 1).$$

This condition, however, postulates a very rapid increase for the indices. It is satisfied, *e.g.* by the series

$$x^2 - x^{2^2} + x^{2^{2^2}} - x^{2^{2^3}} + x^{2^{2^4}} - \dots.$$

It is interesting to see how far one may limit the increase of the p_ν's without impairing the validity of the conclusion. The most that I have been able to do in this direction is to replace the condition (1) by a condition of the type

(1′) $$p_{\nu+1} > K\nu p_\nu \log p_\nu.$$

This is satisfied if

$$p_\nu = (\nu!)^3.$$

But the results proved in this paper show that the series do begin to oscillate for $x = 1$ for values of p_ν considerably less than this.

I have in the preceding pages (except for the general theorem of § 4) considered only particularly simple functions of quite special forms. It would be easy to generalise the results: but the points of real interest seem to me to be quite adequately illustrated by these special cases.

* *Series* 2, *Vol.* II, *p.* 171.

CORRECTIONS

p. 271, *footnote, line* 3. For $<$ read $>$.

p. 273, *line* 4. For $B_{\nu+1}$ read $B_{\nu-1}$.

p. 275, *line* 6 *up*. For σ_{ν^2-q} read σ_{ν^2-1}.

—— *line* 5 *up*. For 'has the sum s' read 'is summable to sum s'.

—— *line* 4 *up*. For § 3 read § 5.

p. 277, *lines* 7 *and* 5 *up*. For $f_a(x)$ read $F_a(x)$.

—— *footnote, line* 1. For $\Phi_a(a)$ read $\Phi_a(x)$.

p. 278, *line* 2 *up*. For $x = 5, 6,...$ read $a = 5, 6,...$.

p. 279, *line* 1. For $n = 2$ read $a = 2$.

p. 287, *line* 3 *up*. The 'vertical bar' should be between $n^3,...$ and 2^n, in *line* 9 *up*.

p. 288, *line* 3. For 'shown that in' read 'shown in'.

COMMENTS

The paper of Fejér, referred to in §§ 1 and 3, is in *Math. Annalen*, 58 (1904), 51–69. Fejér gave sufficient conditions for a series $\sum a_n \phi(nt)$ to converge and its sum tend to s, whenever $\sum a_n$ is summable $(C, 1)$ to s. The present paper is concerned with the cases $\phi(nt) = e^{-nt}$ or $e^{-n^2t^2}$.

Hardy extends Fejér's theorem in 1907, 6. A further extension was made by Bromwich, and the final necessary and sufficient conditions were obtained by Kojima and Hurwitz; see the Comments on 1907, 6.

In D.S., pp. 58–9, Hardy proves the theorem of § 4 by applying Toeplitz's theorem of 1911 to the transformation

$$\gamma_n = \frac{1}{C_n}\sum_{1}^{n-1} B_r\,\Delta(c_r/b_r)\beta_r + \frac{B_n c_n}{C_n b_n}\beta_n,$$

where $\Delta u_n = u_n - u_{n+1}$. The analysis shows that, if $b_n > 0$, $c_n > 0$, $\sum c_n = \infty$ and c_n/b_n is monotonic, then $\gamma_n \to s$ whenever $\beta_n \to s$ *if and only if* $(H): B_n/b_n < HC_n/c_n$; when c_n/b_n decreases condition (H) is always satisfied. Compare Garabedian and Randels,† who gave necessary and sufficient conditions in the case where c_n/b_n is not assumed to be monotonic. In the footnote to the theorem, Hardy attributes case (a) to Cesàro.‡ In 1910, 3 (§ 8, footnote) he attributes case (b) also to Cesàro.§ Hardy, however, proves rather more than Cesàro.

† *Duke Math. J.* 4 (1938), 529–33.

‡ Cesàro (1), *Bull. des sci. math.* (2), 13 (1889), 51–4.

§ Cesàro (2), *Atti d. Reale Accad. d. Lincei Rend.* (4), 4 (1888), 452–7; see also Cesàro (3), ibid., 133–8, where he gave an earlier proof of case (a), deducing it from the theorem in Cesàro (2).

Cesàro (in papers (1) and (2)) proceeded from the formula||

$$\gamma_n = \beta_n - \frac{W_n}{C_n}\Big(\beta_n - \frac{1}{W_n}\sum_1^n w_r\beta_r\Big),$$

where $w_r = B_r\,\Delta(c_r/b_r)$. He observed (in paper (2)) that if $b_n > 0$, $c_n > 0$, $\sum c_n = \infty$, c_n/b_n is monotonic and (i) $\sum w_r$ diverges, (ii) $W_n/C_n = O(1)$, then $\beta_n \to s$ implies $\gamma_n \to s$.

Condition (ii) may be written:

$$B_n/b_{n+1} < KC_n/c_{n+1}.$$

This *implies* (H) if c_n/b_n increases, but there are sequences for which (H) holds while (ii) does not, e.g.

$$b_n = 2^n, \qquad c_n = 2^{n^2};$$

thus (ii) is not a necessary condition. When c_n/b_n decreases (ii) always holds.

Cesàro proved (in papers (1) and (3)) that (i) holds whenever c_n/b_n decreases to the limit zero. But condition (i) may be omitted. For if $\sum w_n$ converges, then $W_n/C_n = o(1)$ and the conclusion still holds.

Cesàro gave as an example (in paper (2)) the result that the mean with weights $1^k, 2^k,\ldots$, where $k > -1$, is equivalent to the arithmetic mean.

The example in § 5, showing that the mean with weights 1, 3, 5,... is equivalent to the arithmetic mean, anticipates the result that the Riesz means $(R, n^2, 1)$ and $(R, n, 1)$ are equivalent; see 1916, 5.

Maclagan Wedderburn's proof†† that $\sum(-1)^m x^{a^m}$ oscillates as $x \to 1$ (§ 8) holds for all real $a > 1$, if non-integral indices are admitted; see D.S., p. 77. The remarks about oscillating gap series, in § 17, arise out of a construction in 1905, 2 (§ 6, footnote), by Bromwich and Hardy. The culminating result is the 'high indices' theorem, proved by Hardy and Littlewood in 1926, 5, which may be expressed in the form: *if* $p_0 > 0$, $p_{n+1}/p_n \geqslant \theta > 1$ *for* $n \geqslant 0$, *and the series* $\sum a_n x^{p_n}$ *converges for* $0 < x < 1$, *then its sum oscillates as* $x \to 1$, *if and only if* $\sum a_n$ *is divergent.*

|| The notation is that of Cesàro (1).

†† Also given in 1907, 6.

Some theorems concerning infinite series.

By

G. H. HARDY of Cambridge, England.

1. In the course of his most interesting memoir „Untersuchungen über Fouriersche Reihen"*) Mr. L. Fejér proves the following theorem:

If Σa_n is a divergent series summable by Cesàro's method of mean values — i. e., if

$$\lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1},$$

where $s_n = a_0 + a_1 + \cdots + a_n$, is a determinate and finite quantity s; and if $\varphi(t)$ is a continuous function of t such that $\varphi(0) = 1$ and

$$|\varphi(t)| < \frac{M}{t^{2+\varrho}}, \qquad \left|\frac{d^2\varphi}{dt^2}\right| < \frac{M}{t^{2+\varrho}},$$

where M and ϱ are positive constants, for all values of $t \geqq 1$: then the series

$$\Sigma a_n \varphi(nt)$$

is convergent for all positive values of t, and if $F(t)$ denote its sum,

$$\lim_{t=0} F(t) = s.$$

The following pages are the result of an attempt to generalise this theorem in two directions, (1) by considering the case in which the series Σa_n is summable only by repeated applications of the mean value method of summation, (2) by replacing $\varphi(nt)$ by a function of the more general form $\varphi_n(t)$. I have thus been led to a number of results which seem to me to be interesting, and which I have found useful in applications, more especially in the theory of elliptic functions.

2. I shall suppose throughout that, for all positive integral values of n, $\varphi_n(t)$ is a function of t continuous throughout a certain interval which we may suppose to be (0, 1). And I shall use the notation

$$\Delta\varphi_n = \varphi_n - \varphi_{n+1}, \qquad \Delta^2\varphi_n = \varphi_n - 2\varphi_{n+1} + \varphi_{n+2},$$
$$\Delta^3\varphi_n = \varphi_n - 3\varphi_{n+1} + 3\varphi_{n+2} - \varphi_{n+3}, \cdots$$

*) Math. Annalen, Bd. 58, pp. 51—69.

1907, 6 Mathematische Annalen, **64, 77–94.**

for the successive differences of the functions φ_n. Further, I shall say that Σa_ν is '*summable* (k)', if the first of the quantities

$$\begin{aligned} s_n &= a_0 + a_1 + \cdots + a_n, \\ s_n^{(1)} &= \frac{1}{n+1}(s_0 + s_1 + \cdots + s_n), \\ s_n^{(2)} &= \frac{1}{n+1}(s_0^{(1)} + s_1^{(1)} + \cdots + s_n^{(1)}) \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

which tends to a determinate limit for $n = \infty$, is $s_n^{(k)}$. The series, when convergent, is summable (0): this simple case is included in all the results which follow, by taking $k = 0$.

The first theorem which I shall prove relates to the case in which $\varphi_n, \Delta\varphi_n, \Delta^2\varphi_n, \cdots, \Delta^{k+1}\varphi_n \geqq 0$ for all values of n and t in question. The simplest example is given by

$$\varphi_n(t) = e^{-nt} = x^n \qquad (x = e^{-t})$$

in which case we obtain the well known results of Frobenius and Hölder for power series.

3. Theorem I. *If* (I) Σa_n *is summable* (k) *to a sum* s,

(II) $$\varphi_n \geqq 0,\ \Delta\varphi_n \geqq 0,\ \cdots,\ \Delta^{k+1}\varphi_n \geqq 0$$

for all values of n and t in question,

(III) $$\Sigma n^k \varphi_n$$

is convergent for $t > 0$,

(IV) $$\lim_{t=0} \varphi_n(t) = 1$$

for every n; then the series

$$F(t) = \Sigma a_n \varphi_n(t)$$

is convergent for $t > 0$ *and*

$$\lim_{t=0} F(t) = s.$$

It will be most convenient to prove this first in the case when $k = 1$. Since*)

$$a_n = (n+1)s_n^{(1)} - 2ns_{n-1}^{(1)} + (n-1)s_{n-2}^{(1)}$$

and $\lim s_n^{(1)} = s$, it is clear that $\lim \frac{a_n}{n} = 0$. Also

$$\begin{aligned} F(t) = \sum_{n=0}^{\infty} a_n \varphi_n(t) &= \sum_{n=0}^{\infty} \{(n+1)s_n^{(1)} - 2ns_{n-1}^{(1)} + (n-1)s_{n-2}^{(1)}\}\, \varphi_n(t) \\ &= \sum_{n=0}^{\infty} (n+1)s_n^{(1)}\, \Delta^2 \varphi_n, \end{aligned}$$

*) In this equation $s_{-1}^{(1)}$, $s_{-2}^{(1)}$ must be interpreted as being equal to zero.

the rearrangement of the series being legitimate in virtue of the condition (III). We now put

$$s_n^{(1)} = s + \varepsilon_n \quad (\lim \varepsilon_n = 0)$$

and deduce

$$F(t) = s\varphi_0(t) + \sum_{n=0}^{\infty} (n+1)\varepsilon_n \Delta^2 \varphi_n$$

$$= s\varphi_0(t) + \sum_0^{N-1} + \sum_N^{\infty} = s\varphi_0(t) + R_1 + R_2,$$

say.

Choose N so that $|\varepsilon_n| < \varepsilon (n \geqq N)$. Then since $\Delta^2\varphi_n \geqq 0$

$$|R_2| < \varepsilon \sum_N^{\infty} (n+1)\Delta^2\varphi_n = \varepsilon\{(N+1)\varphi_N - N\varphi_{N+1}\}.$$

But

$$(N+1)(\varphi_N - \varphi_{N+1}) \leqq \sum_0^N (\varphi_n - \varphi_{n+1}) < \varphi_0 < M,$$

and $\varphi_{N+1} < \varphi_0 < M$, where M is the maximum of φ_0.

Hence

$$|R_2| < 2M\varepsilon.$$

But when N is once fixed

$$\lim_{t=0} s\varphi_0 = s, \qquad \lim_{t=0} R_1 = 0.$$

Hence $\lim\limits_{t=0} F(t) = S$, and the theorem is proved in the case of $k = 1$.

4. In order to prove the theorem in the general case we shall require two Lemmas.

Lemma A. *If* $\varphi_n \geqq 0$, $\Delta\varphi_n \geqq 0, \cdots, \Delta^{k+1}\varphi_n \geqq 0$, *then*

(I) $\Sigma n^\lambda \Delta^{\lambda+1}\varphi_n$ *is convergent for* $\lambda = 0, 1, \cdots, k$,

(II) $n^{\lambda+1}\Delta^{\lambda+1}\varphi_n < K$, *where* K *is a constant, for* $\lambda = 0, 1, \cdots, k-1$ *and for all values of* n *and* t.

(III) $\lim n^{\lambda+1}\Delta^{\lambda+1}\varphi_n = 0$ *for* $\lambda = 0, 1, \cdots, k-1$ *and any particular value of* t.

We have already proved the truth of (I) and (II) for $\lambda = 0$. To prove (III) we observe that since $\Sigma\Delta\varphi_n$ is convergent we can choose N so that for $N' > N$

$$(N' - N)\Delta\varphi_{N'} \leqq \sum_{N+1}^{N'} \Delta\varphi_n < \varepsilon.$$

Taking $N' = 2N$ we see that $N\Delta\varphi_{2N} < \varepsilon$, and so $\lim\limits_{n=\infty} N\Delta\varphi_N = 0$.

Now assume that the conclusions of the lemma have been established for $\lambda = 0, 1, \cdots, \mu - 1$ $(\mu < k)$. Then

$$\sum_0^N n^\mu \Delta^{\mu+1} \varphi_n = \sum_1^N \{n^\mu - (n-1)^\mu\} \Delta^\mu \varphi_n - N^\mu \Delta^\mu \varphi_{N+1}.$$

The last term has the limit zero for $N = \infty$. Also, since

$$n^\mu - (n-1)^\mu < K n^{\mu-1} \text{*)}$$

the series $\Sigma\{n^\mu - (n-1)^\mu\}\Delta^\mu \varphi_n$ is convergent. Hence $\Sigma n^\mu \Delta^{\mu+1}\varphi_n$ is convergent: moreover

$$\Sigma n^\mu \Delta^{\mu+1}\varphi_n < K \Sigma n^{\mu-1} \Delta^\mu \varphi_n.$$

Again

$$\Delta^{\mu+1}\varphi_N \sum_0^N n^\mu \leqq \sum_0^N n^\mu \Delta^{\mu+1} \varphi_n < K$$

and therefore

$$N^{\mu+1} \Delta^{\mu+1} \varphi_N < K.$$

Finally, when t has any fixed value we can choose N so that for $N' > N$

$$\sum_{N+1}^{N'} n^\mu \Delta^{\mu+1} \varphi_n < \varepsilon.$$

Taking $N' = 2N$ we deduce

$$\Delta^{\mu+1}\varphi_{2N} \sum_{N+1}^{2N} n^\mu < \varepsilon$$

and therefore

$$\lim_{N=\infty} N^{\mu+1} \Delta^{\mu+1} \varphi_N = 0,$$

for any particular value of t.

Hence if the conclusions of the lemma are valid for $\lambda = 0, 1, \cdots, \mu - 1$ they are valid for $\lambda = \mu$. The lemma is therefore proved.**) I will only observe that, as the convergence to a limit postulated by (III) is generally not uniform for all values of t, (II) is not a mere corollary of (III).

5. Lemma B. *If $p < q \leqq k + 1$, and $f_p(n)$ is a polynomial of degree p in n, the series $\Sigma f_p(n) \Delta^q \varphi_n$ is convergent, and*

$$\sum_N^\infty f_p(n) \Delta^q \varphi_n = \sum_{\nu=0}^p (-1)^\nu \Delta^\nu f_p(N) \Delta^{q-\nu-1} \varphi_{N+\nu}.$$

---

*) I use K in the manner introduced by Borel, viz., to denote a quantity not necessarily the same in all inequalities, but always lying between certain fixed limits, e. g., $\frac{1}{1000}$ and 1000.

**) All three conclusions of the lemma hold up to $\lambda = k - 1$. The first may be extended as above to the case $\lambda = k$.

The convergence of the series follows immediately from Lemma A. Also

$$\sum_{N}^{N'} f_p(n)\Delta^q\varphi_n = f_p(N)\Delta^{q-1}\varphi_N - f_p(N')\Delta^{q-1}\varphi_{N'+1}$$
$$+\sum_{N}^{N'-1}\{f_p(n+1)-f_p(n)\}\,\Delta^{q-1}\varphi_{n+1},$$

and therefore, using the results of Lemma A, we have, in the limit for $N'=\infty$

$$\sum_{N}^{\infty} f_p(n)\Delta^q\varphi_n = f_p(N)\Delta^{q-1}\varphi_N - \sum_{N}^{\infty}\Delta f_p(n)\Delta^{q-1}\varphi_{n+1}$$

Repeating this transformation p times, and observing that $\Delta^{p+1}f_p(n)\equiv 0$, we obtain the result of the lemma.

6. We may now proceed with the proof of Theorem I. If $\lim s_n^{(k)} = s$ it is easy to see that

$$\lim\frac{s_n^{(k-1)}}{n}=0,\quad \lim\frac{s_n^{(k-2)}}{n^2}=0,\;\cdots,\;\lim\frac{s_n}{n^k}=0,\quad \lim\frac{a_n}{n^k}=0.$$

Hence the series

$$\Sigma a_n\varphi_n,\;\Sigma s_n\varphi_n,\;\Sigma n s_n^{(1)}\varphi_n,\;\cdots,\;\Sigma n^k s_n^{(k)}\varphi_n$$

are absolutely convergent for $t>0$. Also

$$\sum_{0}^{\infty} a_n\varphi_n(t) = \sum_{0}^{\infty}(n+1)s_n^{(1)}\,\Delta^2\varphi_n$$
$$=\sum_{0}^{\infty}\{(n+1)^2 s_n^{(2)} - n(n+1)s_{n-1}^{(2)}\}\,\Delta^2\varphi_n{}^{*)}$$
$$=\sum_{0}^{\infty}\{(n+1)^2 s_n^{(2)} - n^2 s_{n-1}^{(2)}\}\,\Delta^2\varphi_n - \sum_{1}^{\infty} n s_{n-1}^{(2)}\,\Delta^2\varphi_n$$
$$=\sum_{0}^{\infty}(n+1)^2 s_n^{(2)}\Delta^3\varphi_n - \sum_{1}^{\infty} n\, s_{n-1}^{(2)}\Delta^2\varphi_n.$$

By a repetition of this transformation, applied to each of the above series, we find

$$\sum_{0}^{\infty} a_n\varphi_n(t) = \sum_{0}^{\infty}(n+1)^3 s_n^{(3)}\,\Delta^4\varphi_n - \sum_{1}^{\infty}(3n^2+n)s_{n-1}^{(3)}\,\Delta^3\varphi_n$$
$$+\sum_{2}^{\infty}(n-1)s_{n-2}^{(3)}\,\Delta^2\varphi_n;$$

*) Here again $s_{-1}^{(2)}$ must be interpreted as being zero.

and by repeated application of the transformation we find that

$$(1) \qquad F(t) = \sum_0^\infty a_n \varphi_n(t) = \sum_{\mu=0}^{k-1} (-1)^\mu \sum_\mu^\infty f_{k-\mu}^{(k)}(n) s_{n-\mu}^{(k)} \Delta^{k+1-\mu} \varphi_n,$$

where $f_{k-\mu}^{(k)}(n)$ is a polynomial in n of degree $k-\mu$, whose leading coefficient is positive; and in particular

$$f_k^{(k)}(n) = (n+1)^k.$$

The only point which requires proof is that the first coefficient in $f_{k-\mu}^{(k)}(n)$ is positive. Let us suppose that this has been proved for all values of k up to a certain value. When we transform

$$\sum_\mu^\infty f_{k-\mu}^{(k)}(n) s_{n-\mu}^{(k)} \Delta^{k+1-\mu} \varphi_n$$

as above we obtain

$$\sum_\mu^\infty \left\{ (n-\mu+1) f_{k-\mu}^{(k)}(n) s_{n-\mu}^{(k+1)} - (n-\mu) f_{k-\mu}^{(k)}(n) s_{n-\mu-1}^{(k+1)} \right\} \Delta^{k+1-\mu} \varphi_n$$

$$= \sum_\mu^\infty \left\{ (n-\mu+1) f_{k-\mu}^{(k)}(n) s_{n-\mu}^{(k+1)} - (n-\mu) f_{k-\mu}^{(k)}(n-1) s_{n-\mu-1}^{(k+1)} \right\} \Delta^{k+1-\mu} \varphi_n$$

$$- \sum_{\mu+1}^\infty (n-\mu) \left\{ f_{k-\mu}^{(k)}(n) - f_{k-\mu}^{(k)}(n-1) \right\} s_{n-\mu-1}^{(k+1)} \Delta^{k+1-\mu} \varphi_n$$

$$= \sum_\mu^\infty (n-\mu+1) f_{k-\mu}^{(k)}(n) s_{n-\mu}^{(k+1)} \Delta^{k+2-\mu} \varphi_n$$

$$- \sum_{\mu+1}^\infty (n-\mu) \left\{ f_{k-\mu}^{(k)}(n) - f_{k-\mu}^{(k)}(n-1) \right\} s_{n-\mu-1}^{(k+1)} \Delta^{k+1-\mu} \varphi_n.$$

Similarly

$$\sum_{\mu+1}^\infty f_{k-\mu-1}^{(k)}(n) s_{n-\mu-1}^{(k)} \Delta^{k-\mu} \varphi_n = \sum_{\mu+1}^\infty (n-\mu) f_{k-\mu-1}^{(k)}(n) s_{n-\mu-1}^{(k+1)} \Delta^{k+1-\mu} \varphi_n$$

$$- \sum_{\mu+2}^\infty (n-\mu-1) \left\{ f_{k-\mu-1}^{(k)}(n) - f_{k-\mu-1}^{(k)}(n-1) \right\} s_{n-\mu-2}^{(k+1)} \Delta^{k-\mu} \varphi_n.$$

Combining the two terms in these expressions which contain

$$s_{n-\mu-1}^{(k+1)} \Delta^{k+1-\mu} \varphi_n$$

we see that

$$f_{k-\mu-1}^{(k+1)}(n) = (n-\mu) \left\{ f_{k-\mu}^{(k)}(n) - f_{k-\mu}^{(k)}(n-1) + f_{k-\mu-1}^{(k)}(n) \right\},$$

an expression in which the leading coefficient is obviously positive. It is therefore established by induction that for all values of k and μ the first coefficient of $f_{k-\mu}^{(k)}(n)$ is positive. Now when k is once fixed we have only to deal with a limited number of fonctions of this kind, and *a value of n can therefore be assigned from and after which all the functions are positive.*

7. We return now to the equation (1) and suppose that $s_n^{(k)}$ tends for $n = \infty$ to a finite limit s, so that

$$s_n^{(k)} = s + \varepsilon_n \qquad (\lim \varepsilon_n = 0).$$

If we substitute s for $s_n^{(k)}$ throughout (1) we obtain what would have been the value of $F(t)$ if we had

$$a_0 = s, \ a_n = 0 \qquad (n > 0).$$

Hence

$$(2) \qquad F(t) = s\varphi_0(t) + \sum_{\mu=0}^{k-1}(-1)^\mu \sum_{\mu}^{\infty} f_{k-\mu}^{(k)}(n)\varepsilon_{n-\mu}\Delta^{k+1-\mu}\varphi_n$$

$$= s\varphi_0(t) + \sum_{\mu=0}^{k-1}(-1)^\mu\left(\sum_{n=\mu}^{N-1} + \sum_{N}^{\infty}\right)$$

$$= s\varphi_0(t) + R_1 + R_2,$$

say. We suppose N so chosen that for $n \geqq N$ (I) all of $f_{k-\mu}^{(k)}(n)$ are positive, (II) $|\varepsilon_{n-\mu}| < \varepsilon$. Then, since the differences $\Delta^{k+1-\mu}\varphi_n$ are all positive (or zero)

$$|R_2| < \varepsilon \sum_{\mu=0}^{k-1} \sum_{N}^{\infty} f_{k-\mu}^{(k)}(n)\,\Delta^{k+1-\mu}\varphi_n.$$

We have now only to prove that for $\mu = 0, 1, \cdots, k-1$, for all values of N, and for $0 \leqq t < 1$

$$\sum_{N}^{\infty} f_{k-\mu}^{(k)}(n)\,\Delta^{k+1-\mu}\varphi_n < K.$$

For then

$$|R_2| < \varepsilon K$$

and as $\lim\limits_{t=0} s\varphi_0(t) = s$, and, when N is once fixed, $\lim R_1 = 0$, we shall have proved that

$$\lim_{t=0} F(t) = s.$$

8. But by Lemma B

$$\sum_{N.}^{\infty} f_{k-\mu}^{(k)}(n)\,\Delta^{k-\mu+1}\varphi_n = \sum_{\nu=0}^{k-\mu}(-1)^\nu \Delta^\nu f_{k-\mu}^{(k)}(N)\Delta^{k-\mu-\nu}\varphi_{N+\nu}.$$

Also

$$\Delta^{\nu} f_{k-\mu}^{(k)}(N) < K N^{k-\mu-\nu}$$

and by Lemma A

$$N^{k-\mu-\nu} \Delta^{k-\mu-\nu} \varphi_{N+\nu} < K$$

for $k - \mu - \nu = 0, 1, 2, \cdots, k$. Hence

$$\Delta^{\nu} f_{k-\mu}^{(k)}(N) \Delta^{k-\mu-\nu} \varphi_{N+\nu} < K$$

for

$$\nu = 0, 1, \cdots, k-\mu; \quad \mu = 0, 1, \cdots, k-1.$$

Hence

$$\sum_{N}^{\infty} f_{k-\mu}^{(k)}(n) \Delta^{k-\mu+1} \varphi_n < K$$

and the theorem follows as was indicated in the preceding paragraph.

9. *Applications: limitations of the theorem in practice.* The preceding theorem is simple, and interesting on account of the apparently natural character of the conditions which restrict its application. In practice, however, these conditions prove unduly narrow, and exclude a number of very interesting cases for which the result is true: and in order to deal with these cases we shall find it necessary to prove some supplementary theorems.

1). (I) If $\varphi_n(t) = e^{-nt}$ (or $= x^n$, if $x = e^{-t}$)

$$\Delta^{\lambda} \varphi_n = x^n (1-x)^{\lambda} \geqq 0.$$

and $\Sigma n^{\lambda} \varphi_n$ is convergent for $t > 0$ $(x < 1)$ for any value of λ. We thus obtain the known results of Frobenius*) and Holder**).

(II) In general, if $\varphi_n(t) = \varphi(nt)$, and for $u > 0$

$$\varphi(u) > 0, \ \varphi'(u) < 0, \ \varphi''(u) > 0, \cdots$$

all the differences of the functions φ_n are positive. We can apply this to the functions

$$\varphi(u) = \frac{1}{(1+u)^s} \quad (s > 0), \qquad \varphi(u) = e^{-u^{\alpha}} \quad (0 < \alpha < 1).$$

In the first case it is obvious that the conditions are satisfied; in the second case we observe that

$$\varphi'(u) = -\frac{\alpha}{u^{1-\alpha}} e^{-u^{\alpha}},$$

$$\varphi''(u) = \left\{ \frac{\alpha(1-\alpha)}{u^{2-\alpha}} + \frac{\alpha^2}{u^{2(1-\alpha)}} \right\} e^{-u^{\alpha}},$$

and generally $\varphi^{(k)}(u)$ is of the form

$$(-1)^k e^{-u^{\alpha}} \sum \frac{A_{\nu}}{u^{a_{\nu}}} \quad (a_{\nu} > 0, \ A_{\nu} > 0).$$

*) Crelles J., Bd. 89, p. 262.

**) Math. Annalen, Bd. 20, p. 535.

The theorem may therefore be applied if $\varphi(u)$ has either of these two forms; but in the first case we must suppose also that $s > k + 1$, so as to ensure the convergence of $\Sigma n^k \varphi_n$. Hence *if* Σa_n *is summable* (k) *the series*

$$a_0 + \frac{a_1}{(1+t)^s} + \frac{a_2}{(1+2t)^s} + \cdots (s > k+1)$$

and

$$a_0 + a_1 e^{-t^\alpha} + a_2 e^{-(2t)^\alpha} + \cdots (0 < \alpha < 1)$$

are convergent for $t > 0$; *and the limit of each for* $t = 0$ *is the sum of the series* Σa_n.

2). (I) In the case of *Dirichlet's Series* the theorem fails to give us any information. This, however, is not due to the special conditions which we have chosen, but is inherent in the nature of the case. Suppose, for example

$$\varphi_n(t) = (n+1)^{-t}.$$

Then all the differences are positive, so that the characteristic condition of the theorem is satisfied. The theorem however fails because $\Sigma a_n \varphi_n(t)$ is not necessarily convergent for $t > 0$. Take e. g., the case of $k = 1$. The series

$$\sqrt{1} - \sqrt{2} + \sqrt{3} - \cdots$$

is summable (1), but the series

$$\Sigma(-)^{n-1}(n+1)^{\frac{1}{2}-t}$$

is obviously divergent for $t \leqq \frac{1}{2}$. And in fact Dirichlet's Series must be dealt with by theorems of a more general character, which permit the series to be divergent but summable throughout the whole range of values of t and not merely at one end. Such a theorem I have proved, in the case of $k = 1$, in a recent paper*) in the *Proc. Lond. Math. Soc.*, where I have shown that '*if* φ_n, $\Delta\varphi_n$, $\Delta^2\varphi_n \geqq 0$ *and* Σa_n *is summable* (1), *then* $\Sigma a_n \varphi_n$ *is uniformly summable* (1), *and its sum is, therefore, a continuous function of* t.' I do not doubt that the obvious generalisation of this theorem to the case in which $k > 1$ and $k + 1$ sets of differences are positive, is true: but I have not up to the present been able to prove it to be so in all cases.

(II) There are however cases in which the result stated by the theorem is in fact true, although the conditions are violated owing to the fact that the differences are not all positive.

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*) Proc. Lond. Math. Soc., Series 2, Vol. III, pp. 247—265.

If

$$\varphi_n(t) = \frac{e^{-nt}}{1+e^{-nt}} = \frac{q^n}{1+q^n},$$

(where $q = e^{-t}$)

$$\Delta\varphi_n = \frac{q^n(1-q)}{(1+q^n)(1+q^{n+1})} \geqq 0,$$

$$\Delta^2\varphi_n = \frac{q^n(1-q)^2(1-q^{n+1})}{(1+q^n)(1+q^{n+1})(1+q^{n+2})} \geqq 0,$$

but the *third* differences are not essentially positive. Thus the theorem shows that

$$\lim_{q=1}\left(\frac{2q}{1+q} - \frac{2q^2}{1+q^2} + \frac{2q^3}{1+q^3} - \cdots\right) = 1 - 1 + 1 - \cdots = \frac{1}{2},$$

but not that

$$\lim_{q=1}\left(\frac{2q}{1+q} - \frac{4q^2}{1+q^2} + \frac{6q^3}{1+q^3} - \cdots\right) = 1 - 2 + 3 - \cdots = \frac{1}{4},$$

as the last divergent series is only summable (2). Similar remarks apply to the series

$$\sum \frac{a_n q^n}{1+q^{2n}},\quad \sum \frac{a_n q^{2n+1}}{1+q^{2n+1}},\quad \sum \frac{a_n q^{2n+1}}{1+q^{4n+2}}, \ldots$$

which occur in the theory of elliptic functions. On the other hand if $\varphi_n(t) = \log(1+e^{-nt}) = \log(1+q^n)$ the third differences also are essentially positive, but not the fourth; and the theorem applies to the cases $k = 0, 1, 2$ only. Again if

$$\varphi_n(t) = e^{-n^2t^2} = q^{n^2} \qquad (q = e^{-t^2})$$

the second differences are not all positive: so that in this case the theorem does not give us as much information as Mr. Fejér's theorem.

10. The preceding examples make it clear that it is worth while to return to the proof of Theorem I and to see whether, when the condition concerning the differences is not satisfied, other sufficient conditions cannot be found to take its place. The particular case considered by Mr. Fejér, in which $\varphi_n(t) = \varphi(nt)$, is of such especial importance that it deserves special treatment.

The following theorem is a very easy generalisation of Mr. Fejér's theorem:

Theorem II. *If* (I) Σa_n *is summable* (k),

(II) $\Sigma n^k\varphi(nt)$ *is absolutely convergent for* $t > 0$,

(III) $$\frac{d^{k+1-\mu}\varphi(u)}{du^{k+1-\mu}} < \frac{K}{u^{k+1-\mu+\varrho}} \qquad (\varrho > 0)$$

for $u \geqq 1$ *and* $\mu = 0, 1, \cdots, k-1$,

(IV) $\varphi(0) = 1$,

then $F(t) = \Sigma a_n \varphi(nt)$ *is absolutely convergent for* $t > 0$ *and*

$$\lim_{t=0} F(t) = s.$$

The proof proceeds in its early stages as before, until we find that it depends upon showing that

$$\sum_N^\infty \left| f_{k-\mu}^{(k)}(n)\, \Delta^{k+1-\mu} \varphi_n \right| < K.$$

Now for every n and t

$$\Delta^{k+1-\mu} \varphi(nt) = (-1)^{k+1-\mu} t^{k+1-\mu} \varphi^{(k+1-\mu)} \xi,$$

where $nt \leqq \xi \leqq (n+k+1-\mu)^t$. Choose N' so that

$$(N'-1)t < 1 \leqq N't.$$

Then for $n \geqq N'$

$$\left| f_{k-\mu}^{(k)}(n) \Delta^{k+1-\mu} \varphi_n \right| < K n^{k-\mu} t^{k+1-\mu} \left| \varphi^{(k+1-\mu)}(\xi) \right| < K t^{-\varrho} n^{-1-\varrho},$$

and so

$$(1) \qquad \sum_{N'}^\infty \left| f_{k-\mu}^{(k)}(n)\, \Delta^{k+1-\mu} \varphi_n \right| < K t^{-\varrho} . \sum_{N'}^\infty \frac{1}{n^{1+\varrho}} < K N'^{\varrho} \int_{N'}^\infty \frac{d\eta}{\eta^{1+\varrho}} < K.$$

On the other hand

$$\sum_N^{N'-1} \left| f_{k-\mu}^{(k)}(n) \Delta^{k+1-\mu} \varphi_n \right| < K t^{k+1-\mu} M \sum_N^{N'-1} f_{k-\mu}^{(k)'}(n)$$

(where M is the maximum of $\varphi^{(k+1-\mu)}(\xi)$ for $0 \leqq \xi \leqq 1$)

$$(2) \qquad < K t^{k+1-\mu} \sum_N^{N'-1} n^{k-\mu} < K (N't)^{k+1-\mu} < K.$$

From (1) and (2) the desired conclusion follows.

Theorem 2 may at once be applied to the cases mentioned in 9. (2), in which $\varphi_n(t)$ is of the form $\varphi(nt)$ and Theorem 1 fails, such as

$$\varphi_n(t) = e^{-(nt)^2}, \quad (k > 0); \quad = \frac{e^{-nt}}{1 + e^{-nt}}, \quad (k > 1);$$
$$= \log(1 + e^{-nt}) \quad (k > 2);$$

but it fails when $\varphi_n(t)$ is not of this form, as when

$$\varphi_n(t) = \frac{n e^{-nt}(1 - e^{-t})}{1 - e^{-nt}}, \quad n^{-nt}, \cdots.$$

Thus, for example, neither Theorem 1 nor Theorem 2 enables us to assign the limit of

$$(1-q)\left\{\frac{q}{1-q}-\frac{2^k q^2}{1-q^2}+\frac{3^k q^3}{1-q^3}-\cdots\right\} \quad (k \geqq 2)$$

for $q=1$. This and similar questions occur in the problem of determining the behaviour of the modular functions of q as q approaches a point on the unit circle, and also (as Professor Bromwich has shown*)) in that of determining the repulsion between two electrified spheres in contact.

Cases such as these present somewhat greater difficulties; but the following theorem will be found effective in dealing with a great many of them.

11. Theorem 3. *If*

(I) Σa_n *is summable* (k) *to a sum* s,

(II) $\varphi_n(t)$ *is real and continuous,*

(III) $\Sigma n^k \varphi_n(t)$ *is absolutely convergent for* $t>0$,

(IV) *for any value of* $t>0$ *the* λ^{th} *differences* $\Delta^\lambda \varphi_n (0 \leqq \lambda \leqq k+1)$ *can be divided into* r_λ *groups of successive terms, the terms of each group having the same sign, but the sign alternating from group to group, while* r_λ *depends (possibly) upon* t, *but remains less than a constant as* t *tends to zero,*

(V) $|n^\lambda \Delta^\lambda \varphi_n| < K$ *for* $\lambda = 0, 1, \cdots, k$ *and all values of* n *and* t,

(VI) $\varphi_n(0) = 1$ *for every* n;

then $F(t) = \Sigma a_n \varphi_n(t)$ *is absolutely convergent for* $t>0$, *and* $\lim_{t=0} F(t) = s$.

To show the meaning of condition (IV) let us suppose $\varphi_n(t) = e^{-n^2 t^2}$. All the first differences are positive, so that $r_1 = 1$. On the other hand

$$\Delta^2 \varphi_n = e^{-n^2 t^2}(1-2e^{-(2n+1)t^2}+e^{-(4n+4)t^2}) = e^{-n^2 t^2}(1-2ue^{-t^2}+u^2 e^{-4t^2})$$

where $u=e^{-2nt^2}$. This expression vanishes if

$$u = e^{3t^2}\left(1 \pm \sqrt{(1-e^{-2t^2})}\right),$$

or

$$2nt^2 = -3t^2 - \log\left\{1 \pm \sqrt{(1-e^{-2t^2})}\right\},$$

which gives one positive value of n, viz.,

$$n = \frac{1}{t\sqrt{2}} - \frac{3}{2} + \cdots.$$

Thus for $0 \leqq n \leqq \left[\frac{1}{t\sqrt{2}} - \frac{3}{2}\right]$ the second differences are negative, while for larger values of n they are positive. And $r_1 = 2$.

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*) Messenger of Mathematics, Vol. XXXV, p. 1.

In the general case we can, in virtue of condition (IV), form a certain finite number of groups of terms

$$\begin{array}{llll}
\Delta^{k+1-\mu}\varphi_0\,, & \Delta^{k+1-\mu}\varphi_1 & ,\cdots, & \Delta^{k+1-\mu}\varphi_{n_1-1}, \\
\Delta^{k+1-\mu}\varphi_{n_1}, & \Delta^{k+1-\mu}\varphi_{n_1+1}, & \cdots, & \Delta^{k+1-\mu}\varphi_{n_2-1}, \\
\Delta^{k+1-\mu}\varphi_{n_2}, & \cdots & \cdots & \cdots\,, \\
\cdots & \cdots & \cdots & \cdots \\
\Delta^{k+1-\mu}\varphi_{n_r}, & \cdots & \cdots & \cdots\,,
\end{array}$$

the last group extending to infinity, and the sign of every term in any one group being the same. The number of groups may depend upon the value of t, and the numbers $n_1, n_2, \cdots, n_n$ will depend upon t and in general tend to ∞ as t tends to zero. But the number of groups is less than a constant K.

Now

$$F(t) = \sum_{\mu=0}^{k-1} (-)^\mu \sum_{\mu}^{\infty} f_{k-\mu}^{(k)}(n)\, s_{n-\mu}^{(k)} \Delta^{k+1-\mu} \varphi_n = s\varphi_0(t) + \sum_{\mu=0}^{k-1} (-)^\mu (S_\mu + R_\mu),$$

say, where

$$S_\mu = \sum_{\mu}^{N-1} f_{k-\mu}^{(k)}(n)\, \varepsilon_{n-\mu} \Delta^{k+1-\mu} \varphi_n\,,$$

$$R_\mu = \sum_{N}^{\infty} f_{k-\mu}^{(k)}(n)\, \varepsilon_{n-\mu} \Delta^{k+1-\mu} \varphi_n\,.$$

We choose N so that for $n \geqq N$

$$\text{(I)}\quad f_{k-\mu}^{(k)}(n) > 0, \qquad \text{(II)}\quad |\varepsilon_{n-\mu}| < \varepsilon;$$

and everything depends on our being able to prove that

$$|R_\mu| < K\varepsilon.$$

Now

$$|R_\mu| < \left|\sum_{N}^{n_1-1}\right| + \left|\sum_{n_1}^{n_2-1}\right| + \cdots + \left|\sum_{n_{r-1}}^{n-1}\right| + \left|\sum_{n_r}^{\infty}\right|.\text{*)}$$

Also

$$\sum_{\varkappa}^{\lambda-1} f_{k-\mu}^{(k)}(n)\, \Delta^{k+1+\mu} \varphi_n = \Phi_\varkappa - \Phi_\lambda$$

where

$$\Phi_\varkappa = \sum_{\nu=0}^{k-\mu} (-)^\nu \Delta^\nu f_{k-\mu}^{(k)}(\varkappa)\, \Delta^{k-\mu-\nu} \varphi_{\varkappa+\nu}$$

and so

*) Should n_1 be $\leqq N$ (or $n_1, n_2, \cdots, n_s \leqq N$) a certain number of these terms may be omitted.

$$|R_\mu| < 2\varepsilon\left\{|\Phi_N| + \sum_{s=1}^{r}|\Phi_{n_s}|\right\}.$$

But from condition (V) it follows as in § 8 that

$$|\Phi_\varkappa| < K;$$

and hence

$$|R_\mu| < K\varepsilon.$$

The theorem is therefore proved.

12. Theorem 3 may be applied to such series as

$$\sum a_n q^{n^2}, \quad \sum a_n q^{n^k}. \cdots$$

but these series have already been disposed of more simply under Theorem 2. I shall therefore illustrate 3 by applying it to some cases in which $\varphi_n(t)$ is not of (and cannot be reduced to) the form $\varphi(nt)$.

(I) Suppose

$$\varphi_n(t) = n^{-nt} = e^{-nt\log n}.$$

Then

$$\Delta\varphi_n(t) = -\varphi'_{n+\theta}(t)$$

where $0 < \theta < 1$, the dash denoting differentiation with respect to n.

Now

$$|\varphi'_n(t)| = n^{-nt}t(1+\log n).$$

Keeping n fixed, let us choose t so that this is a maximum. We find $t = \frac{1}{n\log n}$, $n^{nt} = e$, and

$$|\varphi'_n(t)| = \frac{1+\log n}{en\log n}$$

from which it follows at once that $|n\Delta\varphi_n| < K$ for all values of n and t. Also

$$\Delta^2\varphi_n(t) = \left[\xi^{-\xi t}\left\{t^2(1+\log\xi)^2 - \frac{t}{\xi}\right\}\right] \quad (n < \xi < n+2),$$

and $t^2(1+\log\xi)^2 - \frac{t}{\xi}$ changes sign once only as ξ increases from 0 to ∞. Hence the conclusions of the theorem are satisfied for $k = 1$. Thus *if Σa_n is summable* (1), *to a sum s, then*

$$\lim_{t=0}\left(\frac{a_1}{1^t} + \frac{a_2}{2^{2t}} + \frac{a_3}{3^{3t}} + \cdots\right) = s.$$

It may be verified without difficulty that the conditions of the theorem are satisfied for any value of k; so that *this conclusion still holds when Σa_n is summable* (k).

(II) As a second example I take

$$\varphi_n(t) = \frac{ne^{-nt}(1-e^{-t})}{1-e^{-nt}};$$

I leave it to the reader to verify that the conditions of Theorem 3 are satisfied.

Writing q for e^{-t} we obtain the following theorem: *if* Σa_n *is summable* (k) *to sum* s

$$\lim_{q=1}(1-q)\sum_1^\infty \frac{n a_n q^n}{1-q^n} = s.$$

For example

$$\lim_{q=1}(1-q)\sum_1^\infty \frac{(-)^{n-1} n^{s+1} q^n}{1-q^n} = 1^s - 2^s + 3^s - \cdots,$$

since the divergent series on the right is summable by mean values for any value of s.

If s is an even positive integer

$$1^s - 2^s + 3^s - \cdots = 0,$$

while if s is an odd positive integer $2t+1$

$$1^s - 2^s + 3^s - \cdots = (-)^t \frac{2^{2t+2}-1}{2t+2} B_{t+1}.^{*)}$$

If s is not integral the sum of the series is

$$-(2^{s+1}-1)\zeta(-s)$$

where $\zeta(s)$ is the Riemann ζ-function.

(III) As a final example I shall consider the series

$$F(x) = x^{\chi(0)} - x^{\chi(1)} + x^{\chi(2)} - \cdots$$

where $\chi(n)$ is a function of n, positive and integral for all positive integral values of n, and tending to infinity with n. The simplest and most interesting series of this kind (after the ordinary geometrical series) are

$$\begin{aligned}
&1 - x + x^4 - x^9 + \cdots + (-)^n x^{n^2} + \cdots\\
(1)\qquad &1 - x + x^8 - x^{27} + \cdots + (-)^n x^{n^3} + \cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

All the series of this type may however be dealt with under Theorem 2, which shows that each of them tends for $x=1$ to the limit $\frac{1}{2}$. The cases next in interest ar those of

$$\begin{aligned}
&x - x^2 + x^4 - x^8 + \cdots + (-)^n x^{2^n} + \cdots\\
(2)\qquad &x - x^3 + x^9 - x^{27} + \cdots + (-)^n x^{3^n} + \cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

I shall therefore examine the series

$$\sum (-)^n x^{2^n}$$

*) Cambridge Philosophical Transactions, Vol. XIX, p. 306.

from the point of view of Theorem 3. The argument may be applied, *mutatis mutandis*, to other and less simple series; but some of the steps present difficulties which are greater in such cases.

In the first place it is easy to verify that condition (IV) is satisfied. For we can show without difficulty that $\Delta^2\varphi_n = x^{2^n} - 2x^{2^{n+1}} + x^{2^{n+2}}$, when x has any value between 0 and 1, vanishes for one value of n, which tends to ∞ as x tends to 1. It is positive for larger and negative for smaller values of n. Hence condition (IV) is satisfied.

On the other hand condition (V), which requires

$$n\Delta\varphi_n < K$$

is not satisfied. For if

$$x^{2^n} = \beta \quad (0<\beta<1),$$
$$\Delta\varphi_n = \beta(1-\beta),$$
$$n\Delta\varphi_n > Kn.$$

The conditions of the theorem are therefore not satisfied. And it is evident that condition (V) will not be satisfied for any of the series (2). On the other hand all the conditions may be shown to be satisfied by any of the series (1), so that these series may be dealt with either under 2 or under 3.

13. It is interesting to observe that the failure of our proof in the case of the series (2) is not due to any inadequacy in the theorem, but to the fact that it is no longer true that the series tend to the limit $\frac{1}{2}$. In fact *the series*

$$F(x) = x - x^a + x^{a^2} - \cdots,$$

where a is a positive integer greater than 1, *oscillates between finite limits of indetermination for* $x=1$.

To prove this we observe in the first place that if $F(x)$ tends to a limit for $x=1$, that limit can only be $\frac{1}{2}$, since

(1) $$F(x) + F(x^a) = x.$$

In the second place we can find a solution of the equation (1) which is regular about the point $x=1$ and tends to $\frac{1}{2}$ as x approaches 1 along the real axis. Such a solution is

$$\Phi(x) = \sum_0^\infty \frac{\left\{\log\left(\frac{1}{x}\right)\right\}^n}{n!(1+a^n)},$$

since

$$\Phi(x) + \Phi(x^a) = \sum_0^\infty \frac{\left\{\log\left(\frac{1}{x}\right)\right\}^n}{n!} = x.$$

Now the function

$$\mathrm{X}(x) = F(x) - \Phi(x)$$

is a solution of the equation

$$f(x) + f(x^a) = 0 \tag{2}$$

which cannot be identically equal to zero, since the function $F(x)$ has the unit circle as a line of essential singularities while $\Phi(x)$ is an integral function of $\log\left(\frac{1}{x}\right)$.

But

$$\mathrm{X}(x) = -\,\mathrm{X}(x^a) = \mathrm{X}(x^{a^2}) = \mathrm{X}(x^{a^{2\mu}})$$

for all positive integral values of μ.

Or if

$$\mathrm{X}(x) = \Theta\left(\log\log\frac{1}{x}\right),$$

$$\Theta\left(\log\log\frac{1}{x}\right) = \Theta\left(\log\log\frac{1}{x} + 2\mu\log a\right).$$

Hence $\mathrm{X}(x)$ *is a periodic function of* $\log\log\frac{1}{x}$, *with a period*

$$2\log a,$$

and

$$x - x^a + x^{a^2} - \cdots$$

is the sum of (1) an integral function of $\log\left(\frac{1}{x}\right)$, which tends to $\frac{1}{2}$ for $x = 1$, and (2) a periodic function of $\log\log\left(\frac{1}{x}\right)$, which oscillates for $x = 1$, since $\log\log\frac{1}{x}$ tends to $-\infty$ as x tends to 1. Therefore *the series* $x - x^a + x^{a^2} - \cdots$ *oscillates for* $x = 1$ *between finite limits of indetermination.**)

14. After the series (2) the most interesting case of § 12, (III) is probably that of the series

$$f(x) = x - x^{1!} + x^{2!} - x^{3!} + \cdots,$$

and it may easily be shown that this series also oscillates for $x = 1$ between finite limits of indetermination, viz. 0 and 1.

Thus the general view which we have obtained of the behaviour of series of the type

$$x^{\chi(0)} - x^{\chi(1)} + y^{\chi(2)} - \cdots$$

for $x = 1$, $\chi(n)$ being an increasing function of n of a regular and simple type, such as

$$n,\ n^2,\ n^3,\ \cdots,\ 2^n,\ 3^n,\ \cdots,\ n!,\ (n!)^2,\ \cdots$$

is this:

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*) The simple proof given above was shown me by Mr. J. H. Maclagan-Wedderburn. I had originally obtained the result by means of a contour integral.

If the increase *(croissance)* of $\chi(n)$ is not too rapid the series tends for $x = 1$ to the limit $\frac{1}{2}$, the sum of the divergent series

$$1 - 1 + 1 - \cdots.$$

When the increase of $\chi(n)$ passes a certain limit the series oscillates for $x = 1$. The swing of the oscillations is greater the greater the increase of $\chi(n)$. If the increase of $\chi(n)$ is greater than or equal to that of $n!$ the swing covers the whole interval $(0, 1)$. Beyond these limits it obviously cannot go.

15. The theorems proved in this paper are no doubt capable of further generalisation. I have allowed my investigations to come to an end at this point because it seemed to me that the results which I had obtained were sufficient to enable us to deal effectively with such simple and interesting cases as present themselves naturally in analysis, and because it seems that when the conditions of no one of the three theorems are satisfied it is generally best to attempt to establish the inequality $|R_\mu| < K\varepsilon$ of § 12 (on which the required proof really hangs) *ab initio* in each particular case.*)

*) I hope at some future date to work out more systematically some of the applications of these theorems to the elliptic and elliptic modular functions, the nature of which has been indicated in a general way in some of my illustrations. Considerations of space prevent my attempting this in the present paper. I may however refer to Riemann's fragment ‚Über die Grenzfälle der elliptischen Modulfunctionen' (Werke, p. 427) and Dedekind's notes (ibid. p. 438), and to papers by H. J. S. Smith ('On some discontinuous series considered by Riemann, Messenger, Vol. XI, p. 1) and by the present writer ('Note on the limiting values of the Elliptic Modular Functions', Quarterly Journal, vol. XXXIV, p. 76). Independently of me, Prof. Bromwich had proved a number of theorems of which the following is an example: *if* φ_n, $\Delta\varphi_n$, $\Delta^2\varphi_n \geqq 0$, $\lim_{t=0} \varphi_n(t) = 1$, *then*

$$\lim_{t=0} \varphi_0 - (1+p)\varphi_1 + \frac{(1+p)(2+p)}{1\cdot 2}\varphi_2 - \cdots = 2^{-p-1}.$$

This theorem is of course a special case of Theorem I of this paper: but it can be proved independently in a very simple and elementary manner, and it enables us at once to assign the limits of a number of interesting series.

CORRECTIONS

p. 85, *footnote*. For Vol. III read Vol. IV. The reference is to 1907, 2.

pp. 90–2. Throughout § 12, for Theorem 2 read Theorem II.

p. 94, *line* 16. For § 12 read § 11.

COMMENTS

Theorems I, II, and 3 were extended by Bromwich,† who replaced conditions (II) and (III) of Theorem I by

$$\text{(II)}' \ \sum n^k|\Delta^{k+1}\phi_n(t)| < K, \qquad \text{(III)}' \ n^k\phi_n(t) \to 0 \text{ as } n \to \infty.$$

He also gave the corresponding result with Cesàro's definition of summability in place of Hölder's. Bromwich (loc. cit.) proved the equivalence of $(C, 2)$ and $(H, 2)$ summability, but only learnt during the course of his investigations that Knopp‡ had proved that (H, k) summability implies (C, k) summability. The equivalence of (H, k) and (C, k) summability was established by Schnee.§

Kojima‖ and Hurwitz†† completed Bromwich's theorem in the form: *the conditions* (II)$'$, (III)$''$: $n^k|\phi_n(t)| < M(t)$, *and* (IV) *are necessary and sufficient for* $F(t) = \sum a_n \phi_n(t)$ *to be convergent and* $F(t) \to s$ *as* $t \to 0$, *whenever* $\sum a_n$ *is summable* (C, k) *to* s.

Chapman‡‡ extended Bromwich's version to non-integral orders of summability. The corresponding extension of the Kojima–Hurwitz theorem (due to Hurwitz) is given in Moore, p. 45.

In example (I) of § 12, it is stated that (H, k) summability implies Lindelöf summability; see D.S., p. 77. Since it is now known that (H, k) is equivalent to $(R, n \log n, k)$, the result may be obtained from Riesz's theorem stated in 1913, 3, § 2; see H.R., Theorem 28. Example (II) states that (H, k) summability implies Lambert summability; details are given in 1914, 5, and also in D.S., pp. 372–3, where the result is deduced from Bromwich's theorem. The value of the (H, k) sum of $1^s - 2^s + 3^s - \dots$ is quoted from the Borel sum in 1904, 3, p. 306; the value is correct since the methods are consistent; see D.S., Theorem 147.

The results in §§ 13–14 are also given in 1907, 5.

† *Math. Annalen* 65 (1908), 350–69.

‡ *Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze*, Inaugural Dissertation, Berlin (1907).

§ *Math. Annalen* 67 (1909), 110–25.

‖ *Tôhoku Math. J.* 12 (1917), 291–326.

†† See Moore, pp. 45–6; Hurwitz's contribution was not published.

‡‡ *Proc. London Math. Soc.* (2), 9 (1911), 369–409.

GENERALISATION OF A THEOREM IN THE THEORY OF DIVERGENT SERIES

By G. H. Hardy.

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1. In a paper recently printed in these *Proceedings** I proved the following theorem† :—*If*

(1) *Σa_n is a series summable by Césaro's method of mean values, i.e., if*

$$(s_0+s_1+\ldots+s_n)/(n+1),$$

where

$$s_n = a_0+a_1+\ldots+a_n,$$

tends to a finite limit as n tends to infinity ;

(2) *f_n is a function of n which, together with its first and second differences*

$$f_n-f_{n+1}, \qquad f_n-2f_{n+1}+f_{n+2},$$

is positive for all values of n ;

then the series $\Sigma a_n f_n$ is also summable.

Further, if f_n is also a function of a variable x, and the condition (2) *is satisfied throughout a certain interval of values of x, say* (0, 1), *and f_0 has a finite upper limit throughout this interval,‡ then the series $\Sigma a_n f_n$ is* **uniformly** *summable throughout the interval : and if every f_n is a continuous function of x, the sum of the series is also a continuous function of x.*

I also stated (*l.c.*, p. 267) that I had no doubt of the truth of an obvious generalisation of this theorem. Suppose that the first of the quantities

$$s_n^1 = \frac{s_0+s_1+\ldots+s_n}{n+1},$$

$$s_n^2 = \frac{s_0^1+s_1^1+\ldots+s_n^1}{n+1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 247.

† *L.c.*, p. 256.

‡ It is obvious that the same is true of f_n.

which tends to a limit as n tends to infinity, is s_n^k. Then the series Σa_n may be said to be *summable* (Hk).*

Then it is natural to suppose that the theorem may be generalised by supposing Σa_n to be summable (Hk), and the $k+1$ sets of differences

$$\Delta f_n,\ \Delta^2 f_n,\ \dots,\ \Delta^{k+1} f_n,$$

to be positive. But when I wrote my former paper I had not been able to overcome the considerable algebraical difficulties which appeared to be involved in the proof of this theorem.

On the other hand, the theorem which I had proved was not sufficient to deal with all the interesting particular cases which actually arise when we try to make applications of it (*v.* p. 264 of my former paper). I was therefore led to consider in greater detail the most interesting particular case, viz., that in which the f_n's are such that $\Sigma a_n f_n$ is *convergent* for all points of (0, 1) except $x = 0$, and

$$\lim_{x\to 0} f_n = 1,$$

for all values of n; and I obtained three theorems† which were sufficiently general for the purposes of the applications which I had in view. Mr. Bromwich then proved a more general theorem which included all these theorems and also some very similar theorems arrived at independently, for the case of $k = 1$, by Dr. C. N. Moore.‡

It is mainly owing to suggestions derived from these latter investigations that I have since been able to prove a theorem which, so far as I know, includes all the theorems which have been referred to. This theorem stands to the generalisation contemplated in my former paper in the same relation which Mr. Bromwich's theorem bears to the first of the theorems which I proved in the *Math. Annalen*: that is to say, the condition

$$f_n,\ \Delta f_n,\ \Delta^2 f_n,\ \dots,\ \Delta^{k+1} f_n \geqslant 0$$

is replaced by the more general condition that

$$\Sigma n^k \,|\, \Delta^{k+1} f_n \,|$$

* This extension of Césaro's method is due (implicitly) to Hölder, *Math. Annalen*, Bd. XX., p. 535.

† "Some Theorems concerning Infinite Series," *Math. Annalen*, Bd. LXIV., p. 77.

‡ Moore, *Trans. Amer. Math. Soc.*, Vol. VIII., p. 299; Bromwich, *Math. Annalen*, Bd. LXV., pp. 359 and 362.

is convergent, or (when f_n is a function of x) that

$$\sum_{\nu=0}^{n} \nu^k \,|\, \Delta^{k+1} f_\nu \,| < K$$

for all values of x and n.

2. There are two alternative definitions of the sum of a divergent series on mean value lines when Césaro's original definition fails. One is Hölder's definition stated above, which defines *summability* (Hk). But Césaro himself gave a somewhat similar definition.* Let

$$A_n^k = \frac{(n+1)(n+2)\ldots(n+k)}{k!}$$

—which we may, in the ordinary continental notation, write in the form

$$A_n^k = \binom{n+k}{k},$$

—and let $$S_n^k = A_n^k a_0 + A_{n-1}^k a_1 + \ldots + A_0^k a_n.$$

And suppose that, as n tends to infinity,

$$S_n^k / A_n^k$$

tends to a limit. Then we shall say that Σa_n is *summable* (Ck).

For $k = 1$ Hölder's and Césaro's definitions are identical. That this is so for $k = 2$ has been proved by Mr. Bromwich.† In all ordinary cases (as applied, *e.g.*, to the series $1^s - 2^s + 3^s - \ldots$) the two definitions lead to the same result: and it has been proved by K. Knopp‡ that Césaro's definition *includes* Hölder's—*i.e.*, that *if* a series is summable (Hk) it is also summable (Ck), and the sums agree. It is not unlikely that Césaro's definition is more general: it is conceivable that the two always cover the same ground. But Césaro's definition should certainly be adopted as the standard one; for it is *at least* equally general, and is far more easy to work with in practice, owing to the fact that the expression of a_n in terms of the sums S_n^k is as simple as the reverse equation, whereas the expression of a_n in terms of s_n^k is complicated and clumsy. The contrast appears very clearly when Mr. Bromwich's work, with Césaro's definition, is contrasted with his own, or mine, with Hölder's.

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* Bromwich, *Infinite Series*, pp. 311 *et seq.*

† See pp. 363–5 of his paper in the *Math. Annalen* already quoted.

‡ *Grenzwerte von Reihen u. s. w.*, Inaugural Dissertation, Berlin, 1907, p. 19.

3. The first part of the theorem is as follows :—

THEOREM A.—*If* Σa_n *is summable* (Ck) *and*

$$\Sigma n^k \mid \Delta^{k+1} f_n \mid$$

is convergent, then $\Sigma a_n f_n$ *is summable* (Ck). *Further, its sum is equal to that of the series*

$$\Sigma S_n^k \Delta^{k+1} f_n,$$

which is absolutely convergent.

We note as a matter of minor detail that, if Σa_n is summable (Hk), it is also summable (Ck), and so $\Sigma a_n f_n$ is summable (Ck) : but we cannot affirm that the latter series is summable (Hk), except for $k = 1, 2$.

That $\Sigma S_n^k \Delta^{k+1} f_n$ is absolutely convergent follows at once from the fact that S_n^k / n^k tends to a limit as $n \to \infty$.

Some Algebraical Preliminaries.

4. We denote the sum

$$A_n^k a_0 f_0 + A_{n-1}^k a_1 f_1 + \ldots + A_0^k a_n f_n$$

formed from $\Sigma a_n f_n$, as S_n^k is formed from Σa_n, by T_n^k : and we proceed to express T_n^k in terms of

$$S_0^k,\ S_1^k,\ \ldots,\ S_n^k,$$

and the differences of the functions f_n. We have

$$(1) \qquad a_n = S_n^k - \binom{k+1}{1} S_{n-1}^k + \binom{k+1}{2} S_{n-2}^k - \ldots + (-)^{k+1} S_{n-k-1}^k .\text{*}$$

Thus

$$(2) \qquad T_n^k = \sum_{\nu=0}^{n} A_{n-\nu}^k f_\nu \sum_{r=0}^{k+1} (-)^r \binom{k+1}{r} S_{\nu-r}^k .$$

This expression, as it stands, involves a certain number of terms S_j^k with negative suffixes j : these must be considered to be defined as being equal to zero. In this formula for T_n^k the coefficient of S_j^k is

$$\sum_{\nu=j}^{j+k+1} (-)^{\nu-j} \binom{k+1}{\nu-j} A_{n-\nu}^k f_\nu,$$

or

$$\sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i}.$$

---

* It is easy to see (Bromwich, *Infinite Series*, *l.c.*) that

$$\Sigma S_n^k x^n \equiv (1-x)^{-(k+1)} \Sigma a_n x^n,$$

and

$$\Sigma a_n x^n \equiv (1-x)^{k+1} \Sigma S_n^k x^n.$$

If this expression contains any terms for which $j+i>n$, they may simply be omitted. Thus, with this proviso, (2) may be written in the form

(3) $$T_n^k = \sum_{j=0}^{n} S_j^k \sum_{i=0}^{k+1} (-)^i \binom{k+1}{i} A_{n-j-i}^k f_{j+i} = \sum_{j=0}^{n} a_j S_j^k,$$

say.

5. Now

(4) $$a_j = \sum_{i=0}^{k+1} (-)^i \beta_{j+i} f_{j+i},$$

where $$\beta_{j+i} = \binom{k+1}{i} A_{n-j-i}^k.$$

From this it follows that

(5) $$a_j = \sum_{i=0}^{k+1} (-)^i \gamma_{j+i} \Delta^{k+1-i} f_{j+i},$$

where

(6) $$\gamma_{j+i} = \beta_{j+i} - \binom{k-i+2}{1} \beta_{j+i-1} + \binom{k-i+3}{2} \beta_{j+i-2} - \dots$$

$$= \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \beta_{j+i-\nu}.$$

To verify this result substitute for γ_{j+i} in the expression (5), and pick out the coefficient of $\beta_{j+\lambda}$. We find this coefficient to be $(-1)^\lambda$ times

$$\binom{k-\lambda+1}{0} \Delta^{k-\lambda+1} f_{j+\lambda} + \binom{k-\lambda+1}{1} \Delta^{k-\lambda} f_{j+\lambda+1}$$

$$+ \binom{k-\lambda+1}{2} \Delta^{k-\lambda-1} f_{j+\lambda+2} + \dots = \sum_{i=\lambda}^{k+1} \binom{k-\lambda+1}{i-\lambda} \Delta^{k-i+1} f_{j+i},$$

and it is easy to see that this reduces to $f_{j+\lambda}$.* Thus

(8) $$\gamma_{j+i} = \sum_{\nu=0}^{i} (-)^\nu \binom{k-i+1+\nu}{\nu} \binom{k+1}{i-\nu} A_{n-j-i+\nu}^k.$$

* The simplest proof is probably by means of symbolical operators. Let E denote the operation which, when performed on f_n, changes it into f_{n+1}. The expression above, on writing $i = \lambda + \mu$, becomes

$$\sum_{\mu=0}^{k+1-\lambda} \binom{k+1-\lambda}{\mu} \Delta^{k+1-\lambda-\mu} E^\mu f_{j+\lambda} = \left(1+\frac{E}{\Delta}\right)^{k+1-\lambda} \Delta^{k+1-\lambda} f_{j+\lambda} = (\Delta+E)^{k+1-\lambda} f_{j+\lambda}.$$

But $$(\Delta+E) f_n = f_n - f_{n+1} + f_{n+1} = f_n,$$

whence the result.

But this expression may be simplified considerably. For

$$\binom{k-i+1+\nu}{\nu}\binom{k+1}{i-\nu}A^k_{n-j-i+\nu}$$

$$=\frac{(k-i+1+\nu)!}{(k-i+1)!\,\nu!}\,\frac{(k+1)!}{(i-\nu)!\,(k-i+1+\nu)!}\,\frac{(n-j-i+\nu+k)!}{k!\,(n-j-i+\nu)!}$$

$$=\binom{k+1}{i}\binom{i}{\nu}\binom{n-j-i+\nu+k}{k},$$

and so

(9) $$\gamma_{j+i}=\binom{k+1}{i}\sum_{\nu=0}^{i}(-)^{\nu}\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}.$$

But $$\sum_{\nu=0}^{i}(-)^{\nu}\binom{i}{\nu}\binom{n-j-i+\nu+k}{k}$$

is the coefficient of t^k in

$$\sum_{\nu=0}^{i}(-)^{\nu}\binom{i}{\nu}(1+t)^{n-j-i+\nu+k},$$

or $$(1+t)^{n-j-i+k}\{1-(1+t)\}^{i},$$

or $$(-)^{i}\,t^{i}\,(1+t)^{n-j-i+k};$$

and is therefore equal to $(-)^{i}\binom{n-j-i+k}{k-i}$,

if $0\leqslant i\leqslant k$, and to zero if $i=k+1$. Thus

(10) $$\gamma_{j+i}=(-)^{i}\binom{k+1}{i}\binom{n-j-i+k}{k-i}.$$

Hence

(11) $$a_j=\sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

and

(12) $$T_n^k=\sum_{j=0}^{n}S_j^k\sum_{i=0}^{k}\binom{k+1}{i}\binom{n-j-i+k}{k-i}\Delta^{k+1-i}f_{j+i},$$

with the proviso, we may repeat, that if $j+i>n$ we must write 0 for f_{j+i}. This formula is the end of our algebraical transformations.*

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* It has been suggested to me that these transformations should be capable of being simplified, and I do not doubt that this is so; but I have not been able to effect any appreciable simplification.

6. Suppose, *e.g.*, that $k = 1$. Then (12) becomes

$$(12_1) \qquad T_n^1 = \sum_{j=0}^{n} S_j^1 \{(n-j+1)\Delta^2 f_j + 2\Delta f_{j+1}\},$$

which is easily verified. If $k = 2$, (12) becomes

$$(12_2) \quad T_n^2 = \sum_{j=0}^{n} S_j^2 \left\{\frac{(n-j+2)(n-j+1)}{2}\Delta^3 f_j + 3(n-j+1)\Delta^2 f_{j+1} + 3\Delta f_{j+2}\right\},$$

and so on.

7. We can now proceed to the proof of our theorem. We suppose that

$$\Sigma n^k \,|\Delta^{k+1} f_n|$$

is convergent. If this is so the same is true, as has been shown by Mr. Bromwich,* of all the series

$$\Sigma n^{k-\lambda} \,|\Delta^{k+1-\lambda} f_n| \quad (\lambda = 0, 1, \ldots, k).$$

We have to show that in these circumstances

$$\lim (T_n^k / A_n^k) = \sum_0^\infty S_n^k \Delta^{k+1} f_n.$$

8. We consider first the terms in T_n^k for which $i = 0$. These give

$${}_0T_n^k = \sum_{j=0}^{n} \binom{n-j+k}{k} S_j^k \Delta^{k+1} f_j.$$

Now $\displaystyle \binom{n-j+k}{k} - A_n^k = \frac{1}{k!}\{(n-j+1)(n-j+2)\ldots(n-j+k)$

$$- (n+1)(n+2)\ldots(n+k)\},$$

which is negative and numerically less than

$$Kn^{k-1},$$

where K is a constant. Thus

$$\frac{{}_0T_n^k}{A_n^k} = \sum_{j=0}^{n} S_j^k \Delta^{k+1} f_j + R_0,$$

where

$$|R_0| < \frac{K}{n} \sum_{j=0}^{n} |\Delta^{k+1} f_j|;$$

and so

$$(13) \qquad \lim_{n\to\infty} \left(\frac{{}_0T_n^k}{A_n^k}\right) = \sum_{j=0}^{\infty} S_n^k \Delta^{k+1} f_j.$$

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* *Math. Annalen, l.c.*, p. 361.

9. Next we consider

$$_iT_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} S_j^k \, \Delta^{k+1-i} f_{j+i}.$$

Since

$$\binom{n-j-i+k}{k-i} = (n-j+1)(n-j+2) \ldots (n-j+k-i)/(k-i)! < Kn^{k-i},$$

it follows that
$$\left| \frac{_iT_n^k}{A_n^k} \right| < \frac{K}{n^i} \sum_{j=0}^{n} | \Delta^{k+1-i} f_{j+1} | ;$$

and therefore

(14) $$\lim_{n \to \infty} \left(\frac{_iT_n^k}{A_n^k} \right) = 0.$$

From (13) and (14) it follows that

(15) $$\lim (T_n^k / A_n^k) = \sum_{j=0}^{\infty} S_j^k \, \Delta^{k+1} f_j,$$

which establishes the theorem.

10. THEOREM B.—*If, in addition,*

$$\sum_0^n \nu^k | \Delta^{k+1} f_\nu | < K^*$$

for all values of n and x, then the series

$$\Sigma \, S_j^k \, \Delta^{k+1} f_j$$

is **uniformly** *convergent.*

Let S be the sum (Ck) of the series Σa_n: and let $\Sigma a_n'$ be the series for which
$$a_0' = a_0 - S, \quad a_n' = a_n \quad (n > 0),$$

so that $S' = 0$. Then

$$\sum_{j=m}^{m'} S_j^k \, \Delta^{k+1} f_j = S \sum_m^{m'} \Delta^{k+1} f_j + \sum_m^{m'} S_j'^k \, \Delta^{k+1} f_j = \sigma_1 + \sigma_2,$$

say. Choose m so that for $j \geqslant m$,

$$| S_j'^k / A_j^k | < \epsilon.$$

* Mr. Bromwich (*l.c.*, p. 361) has proved that the same is then true of

$$\sum_0^n \nu^{k-\lambda} \, | \Delta^{k+1-\lambda} f_\nu | \quad (\lambda = 0, 1, \ldots, k).$$

Then

(16) $$|\sigma_2| < 2\epsilon K.$$

Also

(17) $$\left|\sum_m^{m'} \Delta^{k+1} f_j\right| < \frac{1}{m^k}\sum_m^{m'} j^k \,|\Delta^{k+1} f_j| < \frac{K}{m^k},$$

and from (16) and (17) the theorem follows.

COROLLARIES.—(α) *If every* f_n *is continuous, the sum of the series* $\Sigma a_n f_n$ *is continuous.*

(β) *If all the differences*

$$f_n,\ \Delta f_n,\ \ldots,\ \Delta^{k+1} f_n$$

are positive, the condition $\sum_0^n n^k \Delta^{k+1} f_n < K$

is certainly satisfied, and the conclusions of the theorem apply.

The proof of this will be found in Lemma A of my paper in the *Math. Annalen* quoted above.

11. *Applications.*—I have already stated that the very general theorems proved by Messrs. Fejér, Moore, and Bromwich, and myself, with especial reference to a particular case, enable us to deal effectively enough with the majority of interesting special applications which occur naturally in analysis. It would therefore be futile to give any considerable number of illustrations here. In the paper cited above* I pointed out the kind of case in which a more general theorem of the kind here proved is necessary. A simple example is given by supposing

$$f_n = \frac{1}{(a+nx)^s} \quad (s > 0).$$

If the series Σa_n is summable (Ck) it follows that

$$\Sigma \frac{a_n}{(a+nx)^s}$$

is uniformly summable (Ck) in any interval $(0, \xi)$. Thus, *e.g.*,

$$\Sigma \frac{(-)^n n^t}{(a+nx)^s},$$

* *L.c.*, p. 85.

where $0 < t < k$, is uniformly summable (Ck) and is a continuous function of x for $x = +0$. In order to deal with this by my former theorems it was necessary to suppose $s > k+1$, while Mr. Bromwich's theorem required $s > k$—the series being then *convergent* except for $s = 0$.

Even in the theorem here proved, however, it must be observed that f_n is what Dr. Moore has called a *convergence factor:* its introduction into the series Σa_n makes that series, if not convergent, at any rate *more summable.* The series

$$\Sigma (-)^n n^t (a+nx)^s \quad (s > 0),$$

in which f_n is a *divergence* factor, and $\Sigma a_n f_n$ *less* summable than Σa_n, falls outside the scope of any theorem hitherto proved, though, of course, it may be dealt with easily enough by special devices.

The Theorems A, B, however, seem to me interesting less on account of any of their applications than as a contribution to the abstract theory of divergent series, and as marking something like the limit of what may reasonably be expected to be proved concerning the introduction of convergence factors into series summable by the method of mean values.

COMMENTS

In Theorem A an extra condition $f_n \to 0$ should be inserted. In 1910, 1 (addendum at the foot of pp. 279–81) Hardy explains that he had intended to assume this throughout the paper, as is implied by calling f_n a 'convergence factor' in § 11. The first part of the statement of Theorem A still holds if the extra hypothesis is any of the conditions $f_n = O(1), f_n \to \lim$, or $\sum |\Delta f_n| < \infty$, which are equivalent when the stated hypothesis holds. The case $k = 1$ (with the last alternative) is included in Theorem 2.a 1 of 1907, 2. The case $k = 0$ (where the extra condition is not required) is Dedekind's refinement of Abel's test for convergence (see the Comments on 1907, 2). If $\lim f_n \neq 0$, the sum in the second part of the statement of Theorem A must be increased by $S \lim f_n$, where S is the (C, k)-sum of $\sum a_n$.

A companion theorem, in which $\sum a_n$ is bounded (C, k) and $\lim f_n = 0$, was obtained independently by Bohr.† The theorems are known jointly as the Bohr–Hardy theorem.

Some criticisms of Theorem A were made by Bohr in his Dissertation,‡ and Hardy gives a corrected version in 1910, 1; see also D.S., Theorem 71. One of these criticisms was that, in order to account for the error in omitting certain of the f_n, it is necessary to use the condition $f_n \to 0$. However, it may be remarked that the error due to these terms is in fact zero. For formula (3) may be written

$$T_n^k = \sum_{r=0}^{n} A_{n-r}^k a_r f_r = \sum_{j=0}^{n} S_j^k \Delta^{k+1}(A_{n-j}^k f_j) = \sum_{j=0}^{n} \alpha_j S_j^k,$$

where A_n^k is defined, for $n \geqslant 0$, by the formal expansion

$$(1-x)^{-k-1} = \sum_{0}^{\infty} A_n^k x^n,$$

and $A_n^k = 0$ for $n < 0$. In this formula, the values of the f_j with $j > n$ are irrelevant, since they are absent from the first sum and occur with zero coefficients in the second. This agrees with Hardy's remark that 'they may simply be omitted'. The passage from (3) to (11) may be presented briefly in the form

$$\begin{aligned} \alpha_j = \Delta^{k+1}(A_{n-j}^k f_j) &= \sum_{i=0}^{k+1} \binom{k+1}{i} \Delta^i A_{n-j}^k \Delta^{k+1-i} f_{j+i} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} A_{n-j}^{k-i} \Delta^{k+1-i} f_{j+i} \\ &= \sum_{i=0}^{k+1} (-1)^i \gamma_{j+i} \Delta^{k+1-i} f_{j+i}. \end{aligned}$$

The value of the last sum must also be independent of the values of the f_j with $j > n$. Hardy's proviso after (12) that 'if $j+i > n$ we must write 0 for f_{j+i}' is just a reminder that this has already been done. Since there is no error if the f_j are now replaced, the condition $f_n \to 0$ need not be used in this part of the proof. The quotation from Bromwich§ requires only the existence of $\lim f_n$. The full force of the condition $f_n \to 0$ is needed in the final limit, at the point where $S \lim f_n$ would enter.

There is one qualification to these remarks. In Hardy's evaluation of γ_{j+i} it is enough to assume (in view of the proviso) that $j+i \leqslant n$. But if the f_j with $j > n$ are replaced, it is necessary to observe that, for $j = n$, $i = k+1$, we obtain

$$(-1)^i \gamma_{j+i} = (-1)^{k+1} \gamma_{(n)+(k+1)} = A_0^{-1} = 1,$$

† *Comptes rendus* 148 (1909), 75–80; *Bidrag* . . ., pp. 61–8, English translation, pp. 53–8; see the list of abbreviated titles.

‡ *Bidrag* . . ., pp. 68–70, English translation, pp. 59–60.

§ *Math. Annalen* 65 (1908), 350–69.

so that a term $S_n^k f_{n+k+1}$ must be added to the right-hand side of (12). This is confirmed by $(12)_1$ and $(12)_2$, where simplification shows that there are redundant terms $-S_n^1 f_{n+2}$ and $-S_n^2 f_{n+3}$ respectively. This extra term is easily overlooked, in view of the fact that the equation

$$A_n^k = A_k^n, \quad \text{i.e. } \binom{n+k}{k} = \binom{n+k}{n},$$

which holds for $n, k = 0, 1, \ldots$, breaks down when $n = 0, k = -1$; cf. D.S., Theorems 72 and 73, where the same oversight occurs.

Fekete|| completed Hardy's theorem in the form: *necessary and sufficient conditions for $\sum a_n f_n$ to be summable (C, k), whenever $\sum a_n$ is summable (C, k), are*

$$\sum n^k |\Delta^{k+1} f_n| < \infty \quad \textit{and} \quad \sum |\Delta f_n| < \infty.$$

Schur†† stated a general theorem which, in particular, completes Bohr's theorem in the form: *necessary and sufficient conditions for $\sum a_n f_n$ to be summable (C, k), whenever $\sum a_n$ is bounded (C, k), are*

$$\sum n^k |\Delta^{k+1} f_n| < \infty \quad \textit{and} \quad f_n \to 0.$$

Hardy's Theorem B, § 10, also needs correction. The extra hypothesis $f_n \to 0$ is again intended, and in the conclusion, S_j^k should be replaced by $S_j'^k = S_j^k - SA_j^k$, where S is the (C, k)-sum of $\sum a_n$. The proof is discussed and corrected in 1910, 1, where Hardy proves the Corollary (α), § 10, that: *if, in addition, $f_n(x)$ is continuous, then the (C, k)-sum of $\sum a_n f_n$ is continuous.*

Bohr‡‡ proved independently that: *if $\sum n^k |\Delta^{k+1} f_n|$ converges uniformly, $|f_n(x)| < K_n$ for each n and all x, and $f_n \to 0$, and if $\sum a_n$ is bounded (C, k), then $\sum a_n f_n$ is uniformly summable (C, k) to the value $\sum S_n^k \Delta^{k+1} f_n$, which is a uniformly convergent series.*

Similarly, by first writing $\displaystyle\sum_0^n a_r = S + \sum_0^n a_r'$,

we obtain the result that: *if* $\sum n^k |\Delta^{k+1} f_n| < K$ *for all x, $|f_n(x)| < K$ for all n and x, and if $\sum a_n$ is summable (C, k) to S, then $\sum a_n f_n$ is uniformly summable (C, k) to the value*

$$Sf_\infty + \sum S_n^k \Delta^{k+1} f_n = Sf_0 + \sum S_n'^k \Delta^{k+1} f_n,$$

where the last series is uniformly convergent. This is a slight extension of a theorem of Ferrar,§§ and includes Theorem B and Corollary (α), as amended in 1910, 1. The case $k = 1$ is equivalent to Theorem 2 *a* 1 of 1907, 2, where it is intended that $|f_0(x)| < K$.

The Bohr–Hardy theorem was extended to non-integral orders of summability by Andersen,|||| and many further results on convergence and summability factors have been given by other writers; for references see D.S., p. 146, and the books††† of Moore and Zeller. The term 'convergence factor' was first used in this connection by Moore‡‡‡ in 1907. The notation (C, k) and (H, k) was introduced in the present paper.

|| *Math. és Termés. Ért.* 35 (1917), 309–24.

†† *J. für die reine u. angew. Math.* 151 (1921), 79–111; see Bosanquet, *J. London Math. Soc.* 20 (1945), 39–48.

‡‡ *Nachr. v. d. Königlichen Ges. d. Wiss. Göttingen.* (1909), 247–62, *Collected Works*, Vol. I; *Bidrag* . . ., p. 72, English translation, p. 62.

§§ *Proc. London Math. Soc.* (2), 27 (1928), 541–8. Ferrar assumes that $\Sigma n^k |\Delta^{k+1} f_n|$ converges uniformly.

|||| *Studier* . . .; *Proc. London Math. Soc.* (2), 27 (1928), 39–71.

††† See the Introduction.

‡‡‡ *Trans. American Math. Soc.* 8 (1907), 299–330.

THE MULTIPLICATION OF CONDITIONALLY CONVERGENT SERIES

By G. H. Hardy.

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1. Although much has been written concerning the multiplication of series according to Cauchy's rule, the last word has not yet been said upon the subject, and a number of interesting questions connected with it remain unanswered. In this paper I prove a few simple theorems which I believe to be new. In § 4 I prove that a sufficient condition for the multiplication of two convergent series Σa_n, Σb_n is that na_n and nb_n should each tend to zero as n tends to infinity. In § 8 I generalise this result by showing (by the aid of slightly more elaborate analysis) that it is sufficient that the absolute values of na_n and nb_n should have an upper limit. In § 7 I establish a generalisation of a somewhat different kind, showing that the conditions

$$n\phi(n)\,a_n \to 0, \qquad \frac{nb_n}{\phi(n)} \to 0,$$

where $\phi(n)$ is one of a general class of functions of which $\log n$ is typical, are sufficient.

I have also (§ 13) stated and indicated the proofs of some corresponding theorems for integrals, and I have added (§§ 12, 10) a generalisation of Mertens' theorem and new proofs of some results of Pringsheim's concerning series of a special form. I have thought it worth while to add this last section, although it contains no new results, because the class of series to which it refers is the most natural and important of all, and because, so far as I know, the results have never yet been proved with anything like the simplicity which is desirable and attainable.

I wish to state explicitly that I have not proved, either positively or negatively, but particularly negatively, as much as I think ought to be capable of proof. In § 11 I indicate some questions which seem to me of considerable interest, but which I am at present unable to answer.

2. I shall adopt the notation of Mr. Bromwich's *Infinite Series* (pp. 82 *et seq.*); *i.e.*, I shall denote by A, B, C the series

$$a_1+a_2+a_3+\ldots,\quad b_1+b_2+b_3+\ldots,\quad c_1+c_2+c_3+\ldots,$$

where
$$c_n = a_1 b_n+a_2 b_{n-1}+\ldots+a_n b_1.$$

I shall also use the letters A, B, C in equations or inequalities to denote the *sums* of the series, when they are convergent; and I shall denote the sums of the first n terms of the series by A_n, B_n, C_n, so that, *e.g.*,

$$A_n = a_1+a_2+\ldots+a_n.$$

3. The classical results in connection with the multiplication of series are the following:—

(1) **Abel's Theorem.**—*If all three series are convergent, then* $C = AB$.

(2) **Cauchy's Theorem.**—*If A and B are absolutely convergent, then C is absolutely convergent.*

(3) **Mertens' Theorem.**—*If A is absolutely and B conditionally convergent, then C is convergent.*

In addition to these results, a number of theorems have been proved by Pringsheim, Voss, and Cajori.* These relate to the case in which A and B are conditionally convergent, but one at least becomes absolutely convergent when its terms are associated in certain groups, the number in each group being less than some fixed number. I shall return to some of the simplest and most important of these theorems later on.

4. THEOREM A.—*If A and B are convergent, and*

$$na_n \to 0, \qquad nb_n \to 0$$

as $n \to \infty$, then C is convergent.

The proof is very simple. For

$$C_n = a_1 B_n+a_2 B_{n-1}+\ldots+a_n B_1 = a_1 B_n+a_2 B_{n-1}+\ldots+a_N B_{n+1-N}$$
$$+a_{N+1} B_{n-N}+a_{N+2} B_{n-N-1}+\ldots+a_n B_1.$$

Applying Abel's partial summation lemma to the first line, we obtain

$$C_n-A_N B_{n+1-N} = A_1 b_n+A_2 b_{n-1}+\ldots+A_{N-1} b_{n+2-N}$$
$$+B_1 a_n+B_2 a_{n-1}+\ldots+B_{n-N} a_{N+1}.$$

* For references, see Bromwich, *Infinite Series*, p. 87.

If N is such a function of n that N and $n-N$ tend to infinity with n, then

$$A_N B_{n+1-N} \to AB. \tag{1}$$

This is certainly the case if $Gn < N < Hn$, where G and H are constants, and $0 < G < H < 1$. But then

$$| A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < K(N-1)\beta,$$

$$| B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} | < K(n-N)\alpha,$$

where K is a constant, and α and β are the greatest of the moduli of

$$a_{N+1},\ a_{N+2},\ \ldots,\ a_n \qquad b_{n+2-N},\ b_{n+3-N},\ \ldots,\ b_n$$

respectively. In virtue of the restriction imposed upon N, we have

$$N-1 < \lambda n, \qquad n-N < \lambda n,$$

where λ is a constant. And we can choose n_0 so that

$$| n\alpha | < \epsilon/\lambda K, \qquad | n\beta | < \epsilon/\lambda K,$$

for $n \geqslant n_0$. It follows that for $n \geqslant n_0$, we have

$$\begin{cases} | A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} | < \epsilon, \\ | B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1} | < \epsilon, \end{cases} \tag{2}$$

and from (1) and (2) the conclusion follows.

5. This theorem is not of very wide application, the range of series which are only conditionally convergent, and yet satisfy the condition $na_n \to 0$, being of course comparatively narrow. The simplest of such series are those of the type

$$\frac{1}{\phi(1)} - \frac{1}{2\phi(2)} + \frac{1}{3\phi(3)} - \ldots,$$

where $\phi(n)$ is any function which tends steadily to infinity with n, but (like $\log n$ or $\log n \log\log n$) so slowly that the series is not absolutely convergent. Or, again, the series

$$\Sigma n^{-1-\alpha i} \quad (\alpha \gtrless 0)$$

is known* to oscillate finitely, so that, if $\phi(n)$ is any function which tends steadily to infinity with n, the series

$$\Sigma \frac{1}{n^{1+\alpha i}\,\phi(n)}, \quad \Sigma \frac{\cos(\alpha \log n)}{n\phi(n)}, \quad \Sigma \frac{\sin(\alpha \log n)}{n\phi(n)}$$

are convergent. This result may be extended (as in Mr. Bromwich's paper printed earlier in this volume) to such series as

$$\Sigma \frac{\begin{matrix}\cos\\ \sin\end{matrix}(\alpha \log_{k+1} n)}{n \log n \log_2 n \ldots \log_k n\, \phi(n)},$$

where

$$\log_2 n = \log\log n, \quad \log_3 n = \log\log_2 n, \quad \ldots,$$

and generally to series of the type

$$\Sigma \frac{f'(n)}{\phi(n)} \begin{matrix}\cos\\ \sin\end{matrix} \{f(n)\},$$

* See Landau, *Crelle*, Bd. cxxv., pp. 105–7, for references in connection with this series.

where $f(n)$ is a function of n such that $f(n)$, $f'(n)$ are monotonic, $f(n) \to \infty$, $f'(n) \to 0$, and

$$\Sigma \{f'(n)\}^2$$

is convergent. Another interesting type is

$$\Sigma \frac{\Gamma(i+n)}{\Gamma(1+n)} \frac{1}{\phi(n)}.$$

The theorem, however, seems to me of some interest in spite of its comparatively narrow range of applicability, on account of the simplicity of the conditions and the fact that no use whatever is made of the notion of absolute convergence. All of Pringsheim's theorems depend on the possibility of securing absolute convergence in one at least of the series A, B by the insertion of brackets in some prescribed manner.

6. Series for which $na_n \to 0$ have another interesting property first discovered by Tauber.* The converse of Abel's theorem on the continuity of power series holds for them—that is to say, the convergence of Σa_n may be *deduced* from the equations

$$\lim na_n = 0, \qquad \lim_{n \to 1} \Sigma a_n x^n = A.$$

The fact that the simplest proof of Abel's theorem on the multiplication of series is derived from his theorem on the continuity of power series suggests that Theorem A might be deduced from Tauber's theorem. But this proves not to be the case, for the equations

$$\lim na_n = 0, \qquad \lim nb_n = 0,$$

do *not* involve

$$\lim nc_n = 0.$$

Suppose, *e.g.*, that

$$a_n = b_n = \frac{(-1)^n}{(n+1)\sqrt{\{\log(n+1)\}}},$$

so that

$$c_n = (-1)^n \sum_1^n \frac{1}{(r+1)(n+2-r)\sqrt{\{\log(r+1)\log(n+2-r)\}}}.$$

It is easy to see that, if n is odd, the value of r which makes $\log(r+1)\log(n+2-r)$ greatest is $r = \frac{1}{2}(n+1)$, so that

$$c_n > \frac{1}{\log\{\frac{1}{2}(n+3)\}} \sum_1^n \frac{1}{(r+1)(n+2-r)} = \frac{2}{(n+3)\log\{\frac{1}{2}(n+3)\}} \sum_2^{n+1} \frac{1}{r} > \frac{K}{n},$$

and nc_n certainly does not tend to zero. In fact this line of argument suffices to prove that C is convergent only when the more stringent conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

are satisfied.

7. Theorem A may be generalised as follows. It is easy to verify that if $\psi(n)$ is any function of the form

(1) $$(\log n)^\alpha (\log\log n)^\beta (\log\log\log n)^\gamma \ldots$$

which tends to infinity with n, then

$$\lim_{n \to \infty} \frac{\psi\left\{\frac{n}{\psi(n)}\right\}}{\psi(n)} = 1;$$

we may indeed replace the $\psi(n)$ which occurs inside the curly bracket by any other function of n of the same type as $\psi(n)$.

* For references see Bromwich, *Infinite Series*, p. 251.

THEOREM B.—*If A and B are convergent, and*

$$n\psi(n)\,a_n \to 0, \qquad \frac{nb_n}{\psi(n)} \to 0,$$

where $\psi(n)$ is any function of n of the form (1), *then C is convergent.*

We have, as in § 1 above,

$$C_n - A_N B_{n+1-N} = A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N}$$
$$+ B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1},$$

and
$$|C_n - A_N B_{n+1-N}| < K\{(n-N)\alpha + (N-1)\beta\},$$

where α and β are the greatest of the moduli of $a_{N+1}, a_{N+2}, \ldots, a_n$ and $b_{n+2-N}, b_{n+3-N}, \ldots, b_n$ respectively.

We choose N to be of the same order of greatness as $n/\psi(n)$. Then given ϵ, we can choose n so that

$$\alpha < \frac{\epsilon}{(N+1)\,\psi(N+1)}, \qquad \beta < \frac{\epsilon\psi(n+2-N)}{n+2-N},$$

and so
$$|C_n - A_N B_{n+1-N}| < K\epsilon\left\{\frac{n}{N\psi(N)} + \frac{N\psi(n)}{n}\right\}$$

$$< K\epsilon\left[1 + \frac{\psi(n)}{\psi\left\{\dfrac{n}{\psi(n)}\right\}}\right] < K\epsilon.*$$

From this the theorem follows. The simplest and most interesting case is that in which

$$n(\log n)^a\, a_n \to 0, \qquad \frac{nb_n}{(\log n)^a} \to 0,$$

where $0 \leqslant a \leqslant 1$ (if $a > 1$ the first series is absolutely convergent and the result is a mere corollary from Mertens' theorem).

8. Another generalisation of Theorem A, in a somewhat different direction, is the following :—

THEOREM C.—*If A and B are convergent, and*

$$|na_n| < K, \qquad |nb_n| < K,$$

for all values of n, then C is convergent.

---

* Of course K is not the same constant in all these inequalities.

It is known* that

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n} = AB.$$

It is also known† that, if a series Σc_n is such that $(C_1+C_2+\ldots+C_n)/n$ has a limit as $n \to \infty$, then the necessary and sufficient condition for the convergence of the series is

$$\lim \frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = 0:$$

this indeed follows at once from the identity

$$\frac{c_1+2c_2+3c_3+\ldots+nc_n}{n} = \frac{n+1}{n} C_n - \frac{C_1+C_2+\ldots+C_n}{n}.$$

Let us denote the sums

$$a_1+2a_2+\ldots+na_n, \quad b_1+2b_2+\ldots+nb_n, \quad c_1+2c_2+\ldots+nc_n$$

by $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_n$ respectively. It is easy to verify the identity

$$\begin{aligned}\bar{C}_n+C_n = \; & a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1 \\ & +b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1.\end{aligned}$$

Also $$C_1+C_2+\ldots+C_n = n(AB+\gamma_n),$$

where $\gamma_n \to 0$ as $n \to \infty$, and so

$$\frac{C_n}{n} = \frac{AB}{n} + \gamma_n - \frac{n-1}{n}\gamma_{n-1} \to 0.$$

It follows that the necessary and sufficient condition for the convergence of C is that

(1) $$(X+Y)/n \to 0,$$

where

(2) $$\begin{cases} X = a_1\bar{B}_n+a_2\bar{B}_{n-1}+\ldots+a_n\bar{B}_1, \\ Y = b_1\bar{A}_n+b_2\bar{A}_{n-1}+\ldots+b_n\bar{A}_1. \end{cases}$$

---

* Bromwich, *Infinite Series*, p. 83.

† This result is due to Tauber and Pringsheim. See Bromwich, *Infinite Series*, p. 251, for references.

This condition can be written in a variety of different forms. Thus, applying Abel's lemma to X and Y, we obtain

$$(3) \qquad \begin{cases} X = b_1 A_n + 2b_2 A_{n-1} + \ldots + nb_n A_1, \\ Y = a_1 B_n + 2a_2 B_{n-1} + \ldots + na_n B_1. \end{cases}$$

Further, if we put $\quad A_n = A + \epsilon_n, \quad B_n = B + \eta_n,$

so that $\epsilon_n \to 0$, $\eta_n \to 0$, we see that

$$X = A\bar{B}_n + b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \ldots + nb_n \epsilon_1,$$

$$Y = B\bar{A}_n + a_1 \eta_n + 2a_2 \eta_{n-1} + \ldots + na_n \eta_1.$$

Since $\bar{A}_n/n$, $\bar{B}_n/n$ each tend to zero, we see that the necessary and sufficient condition for the convergence of C is that

$$(4) \qquad (X' + Y')/n \to 0,$$

where

$$(5) \quad X' = b_1 \epsilon_n + 2b_2 \epsilon_{n-1} + \ldots + nb_n \epsilon_1, \quad Y' = a_1 \eta_n + 2a_2 \eta_{n-1} + \ldots + na_n \eta_1.$$

But, if $|na_n| < K$ and $|nb_n| < K$, it is clear that

$$\left|\frac{X'}{n}\right| < K \frac{|\epsilon_1| + |\epsilon_2| + \ldots + |\epsilon_n|}{n} \to 0,$$

and similarly $\qquad |Y'/n| \to 0.$

Hence the theorem is established.

9. The simplest example of the use of this theorem is obtained by applying it to the series

$$\pm \frac{1}{a} \pm \frac{1}{a+b} \pm \frac{1}{a+2b} \pm \ldots \pm \frac{1}{a+nb} \pm \ldots.$$

We see that *any* two series of this type, whatever be the law of arrangements of the signs, may be multiplied together, provided only they are convergent. A simple example is obtained by squaring the series

$$(1) \qquad 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} - \ldots,$$

in which the number of terms in each group of signs increases by one at each step. That the series is convergent is easily proved by observing that if we subtract from it the series

$$(1) \qquad 1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{7} - \ldots$$

we obtain an absolutely convergent series, and that the series (2) is convergent and equal to

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu (\nu+1)}{\frac{1}{2}\nu(\nu+1)+1}.$$

The corresponding series in which the numbers of terms in the groups are 1^k, 2^k, 3^k, ..., where k is any positive integer, is also convergent. On the other hand, if the numbers are k, k^2, k^3, ..., the series oscillates, behaving very much like the oscillatory series

$$\Sigma \frac{\cos(a \log n)}{n}, \quad \Sigma \frac{\sin(a \log n)}{n}.$$

10. It will be convenient to give at this stage the simple proof of some of Pringsheim's results to which I alluded in § 1. The most important case, and the only one which I shall consider here, is that in which

$$a_n = (-1)^{n-1} a_n, \quad b_n = (-1)^{n-1} \beta_n,$$

where a_n and β_n are positive and decreasing. The generalisations of Cajori are rather artificial, and it seems to me worth while to establish the really important results in as simple a way as possible; and Pringsheim's own proofs are far from being the simplest possible.*

Pringsheim's results may be stated thus: *if* a_n, β_n *tend steadily to zero, we have the following alternative sets of conditions for the multiplication of*

$$\Sigma(-1)^{n-1} a_n, \quad \Sigma(-1)^{n-1} \beta_n,$$

by Cauchy's rule :—

(1) *it is* **necessary and sufficient** *that*

$$\gamma_n = |c_n| = a_1\beta_n + a_2\beta_{n-1} + \ldots + a_n\beta_1 \to 0;$$

(2) *it is* **necessary and sufficient** *that*

$$(a_1+a_2+\ldots+a_n)\beta_n \to 0, \quad (\beta_1+\beta_2+\ldots+\beta_n)a_n \to 0;$$

(3) *it is* **sufficient but not necessary** *that*

$$\Sigma a_n \beta_n$$

should be convergent ;

---

* A simpler proof of one of them is given by Mr. Bromwich, *Infinite Series*, pp. 86, 87. Even this proof does not seem to me as simple as it may be made.

(4) *it is* **necessary but not sufficient** *that*

$$\Sigma(\alpha_n\beta_n)^{1+s}$$

should be convergent for any positive value of s.

These results may be proved as follows. We observe first that, if

$$A_n = A+(-1)^n \rho_n, \qquad B_n = B+(-1)^n \sigma_n,$$

we have $$0 < \rho_n < a_{n+1}, \qquad 0 < \sigma_n < \beta_{n+1}.$$

Also $$C_n = a_1 B_n + a_2 B_{n-1} + \ldots + a_n B_1,$$

$$(-1)^n (C_n - A_n B) = a_1\sigma_n - a_2\sigma_{n-1} + \ldots + (-1)^{n-1} a_n \sigma_1,$$

and so $$|C_n - A_n B| < a_1\beta_{n+1} + a_2\beta_n + \ldots + a_n\beta_2 = \gamma_{n+1} - a_{n+1}\beta_1.$$

From this it follows that the condition $\gamma_n \to 0$ is *sufficient* to ensure $C_n \to AB$, and that the condition is *necessary* is obvious. This establishes Pringsheim's theorem (1).

Again $$\gamma_n = a_1\beta_n + a_2\beta_{n-1} + \ldots + a_n\beta_1 > (a_1 + \ldots + a_n)\beta_n,$$

and similarly $$\gamma_n > (\beta_1 + \ldots + \beta_n) a_n.$$

Hence the conditions (2) are *necessary.*

Also, if $\nu = \frac{1}{2}(n+1)$ or $\frac{1}{2}n$, according as n is odd or even, we have

$$\gamma_n = a_1\beta_n + a_2\beta_{n-1} + \ldots + a_n\beta_1 < (a_1 + a_2 + \ldots + a_\nu)\beta_{n+1-\nu} + (\beta_1 + \beta_2 + \ldots + \beta_{n-\nu}) a_{\nu+1},$$

and from this it follows that the conditions (2) are *sufficient.*

Finally, if $\Sigma a_n \beta_n$ is convergent, we can choose μ_0 so that

$$a_\mu\beta_\mu + a_{\mu+1}\beta_{\mu+1} + \ldots + a_\nu\beta_\nu < \epsilon \quad (\mu_0 < \mu < \nu),$$

and, *a fortiori,* $$(a_\mu + a_{\mu+1} + \ldots + a_\nu)\beta_\nu < \epsilon \quad (\mu_0 < \mu < \nu).$$

But when μ is fixed we can choose ν_0, so that

$$(a_1 + a_2 + \ldots + a_{\mu-1})\beta_\nu < \epsilon \qquad (\nu_0 < \nu),$$

and so $$(a_1 + a_2 + \ldots + a_\nu)\beta_\nu < 2\epsilon \quad (\nu_0 < \nu).$$

Similarly, we can prove that the second of the conditions (2) is satisfied. Hence condition (3) is *sufficient;* that it is not necessary has been shown by Pringsheim by an example.*

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* The example is given by $a_n = \beta_n = \{(n+1)\log(n+1)\}^{-\frac{1}{2}}$.

Finally, as regards (4), I have nothing to add to Pringsheim's own proof. Since

$$(a_1+a_2+\ldots+a_n)\,\beta_n > na_n\beta_n,$$

the condition

$$na_n\beta_n \to 0$$

is *necessary*. Thus

$$n^{1+s}(a_n\beta_n)^{1+s} \to 0,$$

and so $\Sigma(a_n\beta_n)^{1+s}$ is convergent; *i.e.*, (4) is a *necessary* condition.

11. Theorems A, B, and C, taken in connection with Pringsheim's theorems, suggest questions of some interest to which I am unable at present to give a definite answer.

Let us, for simplicity, consider the special problem of the multiplication of the two series

$$\pm 1^{-s} \pm 2^{-s} \pm 3^{-s} \pm \ldots, \qquad \pm 1^{-t} \pm 2^{-t} \pm 3^{-t} \pm \ldots,$$

where all that is known about the signs of the terms is that they are such as to ensure the convergence of each series.

If $0 < s \leqslant \frac{1}{2}$, $0 < t \leqslant \frac{1}{2}$, or more generally, if s, t and $s+t$ are all positive and not greater than unity, we can certainly choose the signs so that A and B are convergent and C oscillatory. It is enough to take the alternating series $1^{-s}-2^{-s}+\ldots$, $1^{-t}-2^{-t}+\ldots$. The modulus γ_n of the n-th term of the product series is

$$\sum_{r=1}^{n} r^{-s}(n+1-r)^{-t},$$

which tends to infinity with n, if $s+t<1$, and to the finite limit*

$$\int_0^1 \frac{dx}{x^s(1-x)^{1-s}} = \frac{\pi}{\sin s\pi},$$

if $s+t=1$.

On the other hand, if $s=1$, $t=1$, Theorem C shows that the product series is convergent for *all* arrangements of the signs. But the argument by which it was proved does not appear to be capable of extension.

Now let us consider such a case as that in which $s=t=\frac{3}{4}$, or $s=\frac{1}{2}$, $t=1$. Then either (*a*) the product series is always convergent, or (*b*) it is possible to choose the signs so that the product series is oscillatory. My own opinion is that (*b*) is true; *i.e.*, that when $s+t>1$, but at least one

* In connection with the representation of infinite integrals as the limits of finite sums, see a paper by Mr. Bromwich and myself, *Quarterly Journal*, Vol. XXXIX., p. 222.

of s and t is less than 1, we can make A and B convergent and C oscillatory by a proper choice of signs. But I am unable to support this conclusion by an actual example. I wish merely to point out the considerable margin of uncertainty that still remains. In all such cases as these, of course, Pringsheim's results show that the product of the *alternating* series is convergent.

It is easy to see that examples of the kind desired are not likely to be very readily found. For the conditions

$$\sqrt{n}\, a_n \to 0, \qquad \sqrt{n}\, b_n \to 0$$

are sufficient to ensure $$c_n \to 0,\text{*}$$

since $|c^n|$ can never be greater than in the alternating case. Moreover, the series Σc_n is *in any case* summable by Cesàro's mean value, *i.e.*,

$$\lim_{n\to\infty} \frac{C_1+C_2+\ldots+C_n}{n}$$

exists. Now series whose n-th term tends to zero, and which are summable, but not convergent, certainly exist—examples are given by the series

$$\Sigma\frac{\sin\sqrt{n}}{\sqrt{n}}, \quad \Sigma\frac{\cos\sqrt{n}}{\sqrt{n}}, \quad \Sigma\frac{(-1)^{[\sqrt{n}]}}{\sqrt{n}}.$$

But such examples are not particularly obvious, much less is it obvious how to construct examples in which the general term is of the form of the general term of the product of two convergent series.

12. I take this opportunity of also stating the following generalisation of Mertens' theorem, which I have not seen before, although it is not strictly relevant to the main purpose of the paper.

If A is absolutely convergent, and B is a finitely oscillating series whose n-th term tends to zero, then C is a finitely oscillating series; and if the limits of oscillation of B are β_1 and β_2, those of C are $A\beta_1$ and $A\beta_2$.

To prove this, we go back to the equation

$$\begin{aligned} C_n - A_N B_{n+1-N} = {} & A_1 b_n + A_2 b_{n-1} + \ldots + A_{N-1} b_{n+2-N} \\ & + B_1 a_n + B_2 a_{n-1} + \ldots + B_{n-N} a_{N+1}. \end{aligned}$$

Let us suppose that A is absolutely convergent, and that $|B_\nu| < K$ for all values of ν.

---

* It will be remembered (§ 6) that the conditions

$$n\sqrt{(\log n)}\, a_n \to 0, \qquad n\sqrt{(\log n)}\, b_n \to 0$$

ensure $nc_n \to 0$.

First choose N_0 so that

(1) $$|a_{N+1}|+|a_{N+2}|+\ldots<\epsilon/K,$$

for $N \geqslant N_0$. *A fortiori*, we have also

(2) $$|A-A_N|<\epsilon/K.$$

When any value of N greater than N_0 has been determined, we can choose n_0 so that

(3) $$|A_1b_n+A_2b_{n-1}+\ldots+A_{N-1}b_{n+2-N}|<\epsilon,$$

for $n \geqslant n_0$. From (1), (2), and (3) it follows that

$$|C_n-A_NB_{n+1-N}|<2\epsilon,$$

$$|C_n-AB_{n+1-N}|<3\epsilon,$$

for $n \geqslant n_0$, which establishes the result. In the particular case in which $\beta_1=\beta_2$, we obtain Mertens' theorem. It should be observed that the theorem is *not* true if the condition $b_n \to 0$ is removed. Suppose, for example, that $a_n>0$, and form the product of

$$a_1+a_2+a_3+\ldots, \quad 1-1+1-\ldots.$$

We easily see that $$C_{2n}=a_2+a_4+\ldots+a_{2n},$$

$$C_{2n+1}=a_1+a_3+\ldots+a_{2n+1},$$

so that C oscillates, but not between the limits prescribed by the theorem. In particular the product of

$$\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots, \quad 1-1+1-\ldots,$$

converges to the sum 1.

13. I shall conclude by stating the theorems for integrals which are analogous to some of those for series discussed in the preceding pages. But, as these theorems are of much less importance, I shall only outline the proofs.

Suppose that $a(x)$ and $b(x)$ are continuous functions, such that

$$\int_0^\infty a(x)\,dx, \quad \int_0^\infty b(x)\,dx$$

are convergent and have the values A, B. And let

$$c(x)=\int_0^x a(t)\,b(x-t)\,dt=\int_0^x a(x-t)\,b(t)\,dt.$$

$$A(x)=\int_0^x a(t)\,dt, \quad B(x)=\int_0^x b(t)\,dt, \quad C(x)=\int_0^x c(t)\,dt.$$

Then it is easy to prove the formulæ

$$C(x) = \int_0^x A(t)\, b(x-t)\, dt = \int_0^x A(x-t)\, b(t)\, dt$$

$$= \int_0^x a(t) B(x-t)\, dt = \int_0^x a(x-t) B(t)\, dt,$$

$$\int_0^x C(t)\, dt = \int_0^x A(t) B(x-t)\, dt = \int_0^x A(x-t) B(t)\, dt.$$

It is moreover easy to prove that, if $A(x)$ and $B(x)$ tend, as $x \to \infty$, to limits A and B, then

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x A(t)\, B(x-t)\, dt = AB.$$

It follows that :—

(1) *If* $$\int_0^\infty a(x)\, dx = A, \qquad \int_0^\infty b(x)\, dx = B,$$

then $$\lim_{x\to\infty} \frac{1}{x} \int_0^x dt \int_0^t du \int_0^u a(w)\, b(u-w)\, dw = AB.$$

This is the analogue of Cesàro's theorem that

$$(C_1+C_2+\ldots+C_n)/n \to AB,$$

whenever A and B are convergent.

From this the analogue of Abel's theorem follows at once; viz.,

(2) *If* $$\int_0^\infty dx \int_0^x a(t)\, b(x-t)\, dt$$

is convergent, its value is AB.

There is no difficulty whatever in establishing the analogues of Cauchy's and Mertens' theorems, viz., that

(3) *If A and B are absolutely convergent, so is C;*

(4) *If A is absolutely and B conditionally convergent, C is (absolutely or conditionally) convergent.*

Corresponding to Theorem A we have

(5) *If A and B are convergent, and $xa(x) \to 0$, $xb(x) \to 0$, as $x \to \infty$, then C is convergent.*

Corresponding to Theorem B, we have

(6) *If* $\phi(x) = (\log x)^{\alpha} (\log_2 x)^{\beta} \dots (\log_k x)^{\kappa} \to \infty$ *with* x, *and*

$$x\phi(x)\, a(x) \to 0, \qquad \frac{xb(x)}{\phi(x)} \to 0,$$

then C *is convergent.*

Finally, we can show that the necessary and sufficient conditions for the convergence of

$$\int_0^\infty f(x)\, dx,$$

are

$$\text{(i.)} \quad \frac{1}{x}\int_0^x dt \int_0^t f(u)\, du \to 0,$$

$$\text{(ii.)} \quad \frac{1}{x}\int_0^x tf(t)\, dt \to 0;$$

and from this we can deduce the analogue of Theorem C, viz.

(7) *If* $|\, xa(x) \,| < K$, $|\, xb(x) \,| < K$, *then* C *is convergent.*

CORRECTIONS

p. 414, *line* 4. For § 1 read § 4.

p. 416, *last line*. For (1) read (2).

p. 418, *line* 5 (twice) and *line* 8. For $(-1)^n$ read $(-1)^{n-1}$

—— *line* 6 *up*. For 'choose ν_0' read 'choose $\nu_0 \geqslant \mu$'.

p. 420, *line* 11. For c^n read c_n.

COMMENTS

Theorem A may be stated in the form: *if* $\sum a_n$ *and* $\sum b_n$ *are summable* $(C, -1)$, *then* $\sum c_n$ *is convergent.*† Cesàro‡ showed that if $\sum a_n$ and $\sum b_n$ are summable (C, r) and (C, s) respectively, then $\sum c_n$ is summable $(C, r+s+1)$, where r, s are non-negative integers. The result was extended by Knopp§ and Chapman‖ to all real $r, s > -1$; see D.S., Theorem 164. The example in § 6 shows that Cesàro's theorem is false for $r = s = -1$. In D.S., Theorem 166, the example is modified to show that Cesàro's theorem is false for $r = -1$, $s \geqslant -1$.

In Theorem C the idea of replacing o by O in Tauber's condition occurs for the first time. This may have suggested Hardy's O-Tauberian theorem for (C, k) summability (1910, 3), which was presented for publication a year later; cf. 1912, 2, p. 398, last footnote.

Hardy and Littlewood show in 1913, § 47, that the hypotheses in Theorem C imply that $\sum a_n$ and $\sum b_n$ are summable $(C, -1+\delta)$, for every $\delta > 0$, so that, by the extension of Cesàro's theorem, $\sum c_n$ is summable $(C, -1+2\delta)$; see D.S., Theorem 169.

In answer to a question raised in § 11, Hardy and Littlewood sketch a proof in 1913, 2, § 49, that, whenever $0 < 1-a < b < 1-\frac{1}{2}a < 1$, the series $\sum n^{-b}e^{in^a}$ is convergent, while its 'square' is divergent. It follows that, if we write

$$C = \sum n^{-b} \cos n^a, \quad S = \sum n^{-b} \sin n^a,$$

the series C and S are convergent, while at least one of the Cauchy product series $C \times C$, $S \times S$ or $C \times S$ is divergent.

Extensions of Theorem C are given in 1912, 2 and 1927, 10 and in papers by Rosenblatt†† and Neder.‡‡ Rosenblatt§§ replaced o by O in Theorem B, and still sharper forms of Theorem B are given in 1944, 2.

† For the definition of summability $(C, -1)$ see D.S., p. 98; the notation was introduced by Young, *Proc. London Math. Soc.* (2), 17 (1918), 195–236 (209–10).

‡ *Bull. des sci. math.* (2), 14 (1890), 114–20.

§ *Sitz. d. Berliner Math. Ges.* 7 (1907), 1–12.

‖ *Proc. London Math. Soc.* (2), 9 (1911), 369–409.

†† *Jahresber. d. Deutschen Math.-Verein.* 23 (1914), 80–4.

‡‡ *Proc. London Math. Soc.* (2), 23 (1925), 172–84.

§§ *Bull. de l'Acad. Polonaise* (A), 1913, 603–31.

I. *Further researches in the Theory of Divergent Series and Integrals.*

By G. H. Hardy, M.A.

[*Received*, April 2, 1908. *Read*, May 18, 1908.]

§ 1. This paper is a continuation of one published in the *Quarterly Journal of Mathematics* in 1904*.

In § 16 of the paper referred to I said:

'The definitions of the previous sections are perhaps of most use in connection with double limit problems, such as differentiation under the integral sign. Their employment in such problems raises questions which demand a detailed treatment which I must reserve for the present.'

In the present paper I propose to consider some of these questions in greater detail.

A. *Generalised limits and integrals and infinite series.*

§ 2. Two of the most important among the double limit problems of ordinary analysis are the following:

(i) when is the limit of the sum of an infinite series equal to the sum of the limits of the terms of the series?

(ii) when is the integral of the sum of an infinite series equal to the sum of the integrals of the terms?

Or in symbols,

(i) when is $$\lim_{x\to a} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x\to a} f_n(x)?$$

(ii) when is $$\int_a^A dx \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \int_a^A f_n(x)\, dx\,?$$

The case which is of especial interest to us now is that in which in (i) $a=\infty$ and in (ii) $a=0$, $A=\infty$†. The two problems may then be regarded as substantially the same. For if we suppose that the series $\Sigma f_n(x)$ may be integrated term by term over any finite interval $(0, X)$, and write

$$F_n(x) = \int_0^x f_n(t)\, dt,$$

* 'Researches in the Theory of Divergent Series and Divergent Integrals,' *Q. J.*, vol. xxxv. pp. 22—66.

† Throughout this paper I suppose the lower limit of the integrals discussed to be zero: the limitation is of course apparent only.

the second problem takes the form—

(ii)′ when is
$$\lim_{X\to\infty} \sum_{n=0}^{\infty} F_n(X) = \sum_{n=0}^{\infty} \lim_{X\to\infty} F_n(X)?$$

—which is substantially the same problem as (i).

The problems which we have now to consider are—

(1) when is
$$\underset{x\to\infty}{L} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \underset{x\to\infty}{L} f_n(x)?$$

(2) when is
$$G\int_0^\infty dx \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} G\int_0^\infty f_n(x)\,dx?$$

—the symbols $\underset{x\to\infty}{L}$ and $G\int_0^\infty$ denoting the generalised limit and generalised integral according to the definitions of my former paper. In that paper I in the first instance defined $\underset{x\to\infty}{L} F(x)$ as being
$$\lim_{t\to\infty}\int_0^\infty e^{-x}F(tx)\,dx,$$
and $G\int_0^\infty f(x)\,dx$ as being
$$\lim_{t\to\infty}\int_0^\infty e^{-x/t}f(x)\,dx,$$
or (what is, at any rate in all cases of interest, the same thing)
$$\int_0^\infty dt\int_0^\infty xe^{-tx}f(x)\,dx;$$
and I showed that if, for all positive values of τ,
$$\lim_{x\to\infty} e^{-\tau x}f(x) = 0$$
and if
$$F(x) = \int_0^x f(t)\,dt,$$
then
$$\underset{x\to\infty}{L} F(x) = G\int_0^\infty f(x)\,dx.$$

In these circumstances the problems (1) and (2) are equivalent in the same sense as were (i) and (ii). I shall in what follows adopt (2) as the standard form of the problem, as it takes this form in the most interesting applications; and I shall for the present confine myself to the simple definitions recalled above. As I explained in my former paper, more powerful definitions may be given; but those just stated are easy to work with and are sufficient to deal with the most interesting and obvious cases. I shall, moreover, concern myself solely with the difficulties proper to the particular problems under consideration, ignoring those which affect equally the ordinary double limit problems of the Integral Calculus, such as those which arise from discontinuities of the subject of integration.

§ 3. The transformation expressed by the equation (2) is valid if
$$\lim_{t\to 0}\int_0^\infty e^{-\tau x}\,dx\,\Sigma f_n(x) = \lim_{t\to 0}\Sigma\int_0^\infty e^{-\tau x}f_n(x)\,dx$$
$$= \Sigma\lim_{t\to 0}\int_0^\infty e^{-\tau x}f_n(x)\,dx.$$

Now the first of these equations asserts that the series

$$\Sigma\, e^{-\tau x} f_n(x)$$

may be integrated term by term from $x=0$ to $x=\infty$. The second transformation asserts that the series

$$\sum_0^\infty \int_0^\infty e^{-\tau x} f_n(x)\,dx$$

is a continuous function of τ for $\tau=0$. Thus we obtain

Theorem I. *The equation*

$$G\int_0^\infty \left\{\sum_0^\infty f_n(x)\right\} dx = \sum_0^\infty G\int_0^\infty f_n(x)\,dx$$

will certainly be true, if

(i) *the series* $\Sigma\, e^{-\tau x} f_n(x) \quad (\tau > 0)$

can be integrated term by term over the interval $(0, \infty)$,

(ii) *the series* $\Sigma \int_0^\infty e^{-\tau x} f_n(x)\,dx$

is a continuous function of τ *for* $\tau = 0$.

$\tau > \tau_0 > 0$ but not $\tau < \tau_0$

Our problem is therefore reduced to the investigation of (1) the legitimacy of a certain ordinary term by term integration, and (2) the continuity of a certain infinite series.

It is useful to notice one case in which the first of the two conditions stated above is certainly fulfilled. This case is that in which

(*a*) $\Sigma f_n(x)$ is uniformly convergent over any finite range $(0, X)$,

(*b*) the integral $\int_0^\infty e^{-\tau x} \Sigma\, |f_n(x)|\,dx$

is convergent. For then, as I have proved in a note in the *Messenger of Mathematics* *, the integration term by term, from 0 to ∞, of the series $\Sigma e^{-\tau x} f_n(x)$ is certainly legitimate.

§ 4. By far the most interesting case is that in which

$$f_n(x) = a_n x^n \phi(x),$$

the series $\Sigma\, a_n x^n$ being convergent for all values of x. We have then to consider

(*a*) whether the series $\Sigma\, e^{-\tau x} \phi(x)\, a_n x^n$

may be integrated term by term from 0 to ∞,

(*b*) whether the series $\Sigma\, a_n \int_0^\infty e^{-\tau x} \phi(x)\, x^n\,dx$

is continuous for $\tau = 0$.

Let us first notice certain cases in which the first of these questions can certainly be answered in the affirmative.

* Vol. xxxv. p. 126. See also Bromwich, *ibid.*, vol. xxxvi. p. 1, and *Infinite Series*, pp. 448—455.

(1) It can certainly be answered in the affirmative if the integration is legitimate over any finite interval $(0, X)$, and

$$e^{-\tau x}\phi(x),\quad e^{-\tau x}\Sigma|a_n|x^n$$

each tend to zero as x tends to ∞, for any positive value of τ. For then the conditions stated at the end of the last section are satisfied. In particular it is fulfilled if $e^{-\tau x}\phi(x)$ tends to zero, for any positive value of τ, and $\Sigma a_n x^n$ is an integral function of order less than 1.

(2) It can certainly be answered in the affirmative if the integration is permissible over any finite interval $(0, X)$, and $e^{-\tau x}\phi(x)$ tends to zero for any positive value of τ, and $\Sigma\, n!\, a_n x^n$ is convergent for all values of x. For then, if X is large enough, $|\phi(x)| < e^{\frac{1}{2}\tau x}$ for $x \geqq X$, and

$$\left|\int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right| < \int_X^\infty e^{-\frac{1}{2}\tau x} x^n\, dx$$

$$= n!\left(\frac{2}{\tau}\right)^{n+1} e^{-\frac{1}{2}\tau X}\left\{1 + \tfrac{1}{2}\tau X + \frac{(\frac{1}{2}\tau X)^2}{2!} + \ldots + \frac{(\frac{1}{2}\tau X)^n}{n!}\right\},$$

which is always less than $$n!\left(\frac{2}{\tau}\right)^{n+1},$$

and, for any assigned values of τ and n, tends to zero as $X \to \infty$. Hence

$$\left|\sum_0^\infty a_n \int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right| < \left(\sum_0^N + \sum_{N+1}^\infty\right)|a_n|\left|\int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right|$$

$$< \sum_0^N |a_n| \int_X^\infty e^{-\frac{1}{2}\tau x} x^n\, dx + \sum_{N+1}^\infty n!\,|a_n|\left(\frac{2}{\tau}\right)^{n+1}.$$

We can now choose, first N so that the second sum is less than $\frac{1}{2}\epsilon$, and then X so that the first sum is less than $\frac{1}{2}\epsilon$; and hence we see that

$$\lim_{X\to\infty} \sum_0^\infty a_n \int_X^\infty e^{-\tau x} x^n \phi(x)\, dx = 0,$$

and this is precisely the condition that the integration over the whole range $(0, \infty)$ should be legitimate.

It should be remarked that the results just proved are by no means sufficient for the applications that we have in view. There are many interesting cases in which the result holds for all positive values of τ, but its correctness does not follow from anything that has yet been proved. If, e.g., $\phi(x) = e^{-mix}$, where $m > 0$, and $a_n = (-\sigma i)^n/n!$, where $\sigma > 0$, so that

$$\Sigma a_n x^n = e^{-\sigma x i},$$

the equation states that $$\int_0^\infty e^{-\{\tau + (m+\sigma)i\}}\, dx = \Sigma \frac{(-\sigma i)^n}{(\tau + mi)^{n+1}},$$

a result which is true for all positive values of τ, if $\sigma < m$. But the conditions (1) are not satisfied, since

$$\Sigma|a_n|x^n = e^{\sigma x};$$

and the conditions (2) are not satisfied, since $\Sigma\, n!\, a_n y^n$ is not an integral function of y.

§ 5. Before passing on I may make a few further remarks. In a former paper in these *Transactions** I proved that

$$\int_0^\infty e^{-x}(\Sigma a_n x^n)\,dx = \Sigma\, n!\, a_n$$

whenever the series on the right is convergent. It follows that

$$\int_0^\infty e^{-\tau x}(\Sigma a_n x^n)\,dx = \Sigma\, n!\, a_n \tau^{-n-1},$$

if $\tau > 0$ and the series on the right-hand side is convergent.

A more general result is the following.

If $\tau > 0$, $\mu > -1$, *the equations*

$$\int_0^\infty e^{-\tau x}(\Sigma a_n x^{\mu+n})\,dx = \Sigma a_n \int_0^\infty e^{-\tau x} x^{\mu+n}\,dx$$

$$= \Sigma a_n \frac{\Gamma(\mu+n+1)}{\tau^{\mu+n+1}}$$

are certainly true whenever the last series is convergent.

Let
$$u_n = a_n \Gamma(\mu+n+1)\,\tau^{-\mu-n-1}.$$

Then, for any positive value of X,

$$\int_0^X e^{-\tau x}\left(\Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n x^{\mu+n}\right) dx = \Sigma \frac{\tau^{\mu+n+1} u_n}{\Gamma(\mu+n+1)} \int_0^X e^{-\tau x} x^{\mu+n}\,dx,$$

and what we have to prove is that

$$\lim_{X\to\infty} \Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx = 0.$$

Now

$$\frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx$$

$$= e^{-\tau X}\left\{\frac{(\tau X)^\mu}{\Gamma(\mu+1)} + \frac{(\tau X)^{\mu+1}}{\Gamma(\mu+2)} + \dots + \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}\right\} + \frac{\tau^\mu}{\Gamma(\mu)} \int_X^\infty e^{-\tau x} x^{\mu-1}\,dx,$$

and so
$$\Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx = S_1 + S_2,$$

where
$$S_1 = e^{-\tau X} \sum_{n=0}^\infty u_n \sum_{\lambda=0}^n \frac{(\tau X)^{\mu+\lambda}}{\Gamma(\mu+\lambda+1)},$$

$$S_2 = \frac{\tau^\mu}{\Gamma(\mu)} \left(\int_X^\infty e^{-\tau x} x^{\mu-1}\,dx\right)(\Sigma u_n).$$

Obviously $S_2 \to 0$ as $X \to \infty$. Also, as in my former proof, we have

$$S_1 = e^{-\tau X} \sum_{n=0}^\infty \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sum_{\lambda=n}^\infty u_\lambda = e^{-\tau X} \sum_{n=0}^\infty \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sigma_n,$$

say.

* Vol. XIX. p. 299.

Now if
$$\chi(x) = \sum_{n=0}^{\infty} \frac{x^{\mu+n}}{\Gamma(\mu+n+1)}$$
it is known* that
$$\chi(x) = e^x(1+\epsilon_x),$$
where $\epsilon_x \to 0$ as $x \to \infty$, and so, for all positive values of X,
$$\chi(\tau X) < K e^{\tau X},$$
and, *a fortiori*,
$$\sum_{N+1}^{\infty} \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} < K e^{\tau X}.$$
But
$$S_1 = e^{-\tau X} \sum_0^N \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sigma_n + e^{-\tau X} \sum_{N+1}^{\infty} \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sigma_n.$$
Choose N so that
$$|\sigma_n| < \epsilon/K \qquad (n > N).$$
Then
$$|S_1| < e^{-\tau X} \sum_0^N \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} + \epsilon,$$
and from this it follows immediately that $S_1 \to 0$ as $X \to \infty$. Thus our theorem is established.

The question is naturally suggested as to whether it is not always true that
$$\int_0^{\infty} e^{-\tau x} (\Sigma a_n x^n) \phi(x)\, dx = \Sigma a_n \int_0^{\infty} e^{-\tau x} x^n \phi(x)\, dx,$$
when $\tau > 0$, $e^{-\tau x}\phi(x) \to 0$ for any positive value of τ, and the series on the right-hand side is convergent. But it is easy to show by an example that this is not the case. Suppose
$$\phi(x) = e^{-mix} \qquad (m > 0)$$
(a case with which we shall be much concerned in the sequel). Then the question takes the form: *is*
$$\int_0^{\infty} e^{-(\tau+mi)x} (\Sigma a_n x^n)\, dx = \Sigma \frac{n!\, a_n}{(\tau+mi)^{n+1}}$$
whenever the last series is convergent? Now let
$$\tau + mi = y, \quad a_n = \frac{z^n}{n!}.$$
The result would be
$$\int_0^{\infty} e^{-(y-z)x}\, dx = \Sigma \frac{z^n}{y^{n+1}}.$$

The right-hand side is convergent if $|z| < |y|$, the left-hand side when $R(z) < R(y)$, and it is obvious that the first of these conditions does not imply the second.

§ 6. (3) As the sets of conditions (1) and (2) are not sufficiently general for the applications I have in view, I shall indicate a set of conditions of a different character, under which it is always possible to give an affirmative answer to the question (*a*) of § 4.

Let us suppose first that $e^{-\tau x}\phi(x) \to 0$ for any positive value of τ, and write
$$\psi(\tau) = \int_0^{\infty} e^{-\tau x} \phi(x)\, dx.$$
And further let us suppose that $\psi(\tau)$ is *an analytic function of τ, regular in the neighbourhood of the origin*: (*a fortiori* regular in any region throughout which the real part of τ is always positive).

* See e.g. *Proc. Lond. Math. Soc.*, N.S., vol. II. p. 405.

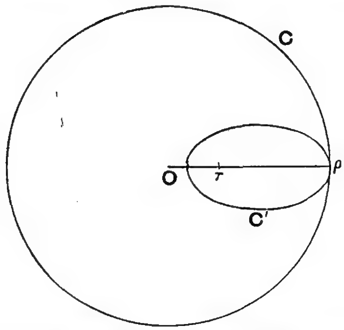
The integrals
$$\int_0^\infty e^{-\tau x} x^n \phi(x)\, dx$$
are all uniformly convergent in any interval (τ_0, τ_1), where $0 < \tau_0 < \tau_1$. Hence, for any positive value of τ, we have
$$\left(\frac{d}{d\tau}\right)^n \psi(\tau) = (-1)^n \int_0^\infty e^{-\tau x} x^n \phi(x)\, dx.$$

Also the integral on the right converges to a limit as $\tau \rightarrow 0$, and this limit is equal to $\psi^{(n)}(0)$: or in other words
$$G\int_0^\infty x^n \phi(x)\, dx = (-1)^n \psi^{(n)}(0).$$

Now let δ denote the distance from the origin of the nearest singularity of $\psi(\tau)$. Then if $\rho < \delta$ and the contour of integration is the circle C defined by $|u| = \rho$, we have
$$\left(\frac{d}{d\tau}\right)^n \psi(\tau) = \frac{n!}{2\pi i}\int \frac{\psi(u)\, du}{(u-\tau)^{n+1}}.$$

Finally let us suppose that *the series* $\Sigma n!\, a_n y^n$ *has a radius of convergence greater than* δ. Then, for sufficiently small values of τ, the series
$$\chi(u, \tau) = \Sigma \frac{(-1)^n n!\, a_n}{(u-\tau)^{n+1}},$$
is uniformly convergent along C. We have therefore
$$\begin{aligned} \frac{1}{2\pi i}\int \psi(u)\chi(u, \tau)\, du &= \Sigma \frac{(-1)^n n!\, a_n}{2\pi i}\int \frac{\psi(u)\, du}{(u-\tau)^{n+1}} \\ &= \Sigma (-1)^n a_n \left(\frac{d}{d\tau}\right)^n \psi(\tau) \\ &= \Sigma a_n \int_0^\infty e^{-\tau x} \phi(x)\, x^n\, dx. \end{aligned}$$

Now the only singularity of the subject of integration, within C, is $u = \tau$. We may therefore replace C by a contour C' such as is shown in the figure, cutting the positive real axis between $u = 0$ and $u = \tau$, say at $u = \gamma$. On C', u has its real part positive and greater than γ. Hence



$$\begin{aligned} &\frac{1}{2\pi i}\int_{C'} \psi(u)\chi(u, \tau)\, du \\ &\qquad = \frac{1}{2\pi i}\int_{C'} \chi(u, \tau)\, du \int_0^\infty e^{-u\xi}\phi(\xi)\, d\xi. \end{aligned}$$

In this repeated integral we may invert the order of integration. For, in the first place, this inversion is obviously justifiable when the upper limit ∞ is replaced by any positive number X. And, in the second place,
$$\left|\frac{1}{2\pi i}\int_{C'} \chi(u, \tau)\, du \int_X^\infty e^{-u\xi}\phi(\xi)\, d\xi\right| < K\int_X^\infty e^{-\gamma\xi}|\phi(\xi)|\, d\xi,$$
which may obviously be made as small as we please by sufficiently increasing X.

Hence $$\frac{1}{2\pi i}\int_{C'} \psi(u)\,\chi(u,\ \tau)\,du = \frac{1}{2\pi i}\int_0^\infty \phi(\xi)\,d\xi \int_{C'} e^{-u\xi}\chi(u,\ \tau)\,du.$$

But $$\frac{1}{2\pi i}\int_{C'} e^{-u\xi}\chi(u,\ \tau)\,du = \frac{1}{2\pi i}\int_{C} e^{-u\xi}\chi(u,\ \tau)\,du$$

$$= \frac{1}{2\pi i}\int_C e^{-u\xi}\left\{\Sigma\frac{(-1)^n n!\,a_n}{(u-\tau)^{n+1}}\right\}du$$

$$= \Sigma\frac{(-1)^n n!\,a_n}{2\pi i}\int_C \frac{e^{-u\xi}}{(u-\tau)^{n+1}}\,du$$

$$= \Sigma(-1)^n a_n\left(\frac{d}{d\tau}\right)^n e^{-\tau\xi}$$

$$= e^{-\tau\xi}\,\Sigma a_n \xi^n.$$

Hence finally $$\Sigma a_n\int_0^\infty e^{-\tau x}\phi(x)\,x^n\,dx = \int_0^\infty e^{-\tau\xi}\phi(\xi)\,\Sigma a_n\xi^n\,d\xi.$$

Thus the question (a) of § 4 may be answered affirmatively.

§ 7. I shall now pass on to the question (b), and show that if the conditions of § 6 are satisfied it also can be answered affirmatively. In order to prove this let us go back to the equation

$$\Sigma(-1)^n a_n\left(\frac{d}{d\tau}\right)^n\psi(\tau) = \frac{1}{2\pi i}\int_C \psi(u)\,du\,\Sigma\frac{(-1)^n n!\,a_n}{(u-\tau)^{n+1}}.$$

The series under the integral sign converges uniformly for all values of u on C and all values of τ such that $0 \leqq \tau \leqq \tau_0$. Hence each side of the equation is a continuous function of τ for $\tau = 0$, and the series

$$\Sigma(-1)^n a_n\psi^{(n)}(0) = \Sigma a_n\,G\int_0^\infty x^n\phi(x)\,dx,$$

is convergent and equal to

$$\lim_{\tau\to 0}\Sigma(-1)^n a_n\psi^{(n)}(\tau),$$

or to

$$\lim_{\tau\to 0}\Sigma a_n\int_0^\infty e^{-\tau x}x^n\phi(x)\,dx.$$

Thus the question (b) may also be answered in the affirmative.

§ 8. Thus we arrive at:

THEOREM II. *We may evaluate the generalised integral*

$$G\int_0^\infty \phi(x)\,F(x)\,dx,$$

by expanding $F(x)$ as a power series $\Sigma a_n x^n$ and taking the generalised integral term by term, provided

(1) *the function* $$\psi(\tau) = \int_0^\infty e^{-\tau x}\phi(x)\,dx$$
is regular at the origin,

(ii) *the series $\Sigma n!\,a_n y^n$ has a radius of convergence greater than $1/\delta$, where δ is the distance from the origin of the nearest singularity of $\psi(\tau)$.*

§ 9. Let us consider in particular the case in which

$$\phi(x) = e^{-mix} x^{\mu},$$

where $m > 0$, $\mu > -1$. Then

$$\psi(\tau) = \int_0^{\infty} e^{-(\tau+mi)x} x^{\mu} dx = \frac{\Gamma(\mu+1)}{(\tau+mi)^{\mu+1}},$$

where that branch of $(\tau+mi)^{-\mu-1}$ is chosen which reduces to

$$m^{-\mu-1} e^{-\frac{1}{2}(\mu+1)\pi i},$$

for $\tau = 0$. In this case $\delta = m$, and we shall certainly have

$$G\int_0^{\infty} e^{-mix} (\Sigma a_n x^n) x^{\mu} d\mu = \Sigma a_n G \int_0^{\infty} e^{-mix} x^{n+\mu} dx$$

$$= \Sigma a_n \Gamma(n+\mu+1) m^{-n-\mu-1} e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

if the series $\Sigma n!\, a_n y^n$ has a radius of convergence greater than $1/m$. This condition, however, may be reduced to a simpler form. For the radius of convergence of

$$\Sigma a_n \Gamma(n+\mu+1) e^{-\frac{1}{2}(n+\mu+1)\pi i} y^n$$

is the same as that of $\Sigma n!\, a_n y^n$. The integration is therefore certainly legitimate if its radius of convergence is greater than $1/m$. But we can go further and say that the integration is certainly legitimate if the radius of convergence is *as great as* $1/m$.

For let
$$\psi_1(\tau) = \int_0^{\infty} e^{-(\tau+\kappa+mi)x} x^{\mu} dx = \frac{\Gamma(\mu+1)}{(\tau+\kappa+mi)^{\mu+1}}.$$

The distance of the nearest singularity of $\psi_1(\tau)$ from the origin is

$$\sqrt{(m^2+\kappa^2)} > m,$$

and therefore
$$\int_0^{\infty} e^{-(\kappa+mi)x} (\Sigma a_n x^n) x^{\mu} dx = \Sigma a_n G \int_0^{\infty} e^{-(\kappa+mi)x} x^{n+\mu} dx$$

$$= \Sigma a_n \frac{\Gamma(n+\mu+1)}{(\kappa+mi)^{n+\mu+1}},$$

provided the radius of convergence of $\Sigma n!\, a_n y^n$ is greater than

$$1/\sqrt{(m^2+\kappa^2)},$$

and therefore certainly if it is equal to $1/m$.

But, by a well-known extension of Abel's theorem on the continuity of power-series

$$\lim_{\kappa\to 0} \Sigma a_n \frac{\Gamma(n+\mu+1)}{(\kappa+mi)^{n+\mu+1}} = \Sigma a_n \Gamma(n+\mu+1) m^{-n-\mu-1} e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

provided only the series on the right is convergent, or even if it is oscillatory, but summable by Cesàro's mean value process or one of its extensions.*

We have thus proved:

THEOREM III. *We may evaluate the generalised integral*

$$G\int_0^{\infty} e^{-mix} F(x) x^{\mu} dx,$$

* Bromwich, *Infinite Series*, pp. 210 *et seq.* and pp. 310 *et seq.*

by expanding $F(x)$ in a power-series $\Sigma a_n x^n$, and taking the generalised integral term by term, whenever this method of procedure leads to a series either convergent, or oscillatory and summable by mean values.

And we may add that *if $F(x)$ is an integral function of order less than 1 the second condition of Theorem II and the solitary condition of Theorem III will certainly be satisfied.*

§ 10. I shall now give some examples of the use of the Theorems II, III, starting with the latter, from which we conclude that if $F(x) = \Sigma a_n x^n$ then

$$G\int_0^\infty e^{-mix} F(x)\, x^\mu\, dx = \sum_0^\infty a_n \Gamma(n+\mu+1)\, m^{-n-\mu-1}\, e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

provided only the series on the right is convergent.

Supposing a_n real, and separating the real and imaginary parts, we obtain

$$G\int_0^\infty \cos mx\, F(x)\, x^\mu dx = -C\sin\tfrac{1}{2}\mu\pi - S\cos\tfrac{1}{2}\mu\pi,$$

$$G\int_0^\infty \sin mx\, F(x)\, x^\mu dx = -C\cos\tfrac{1}{2}\mu\pi + S\sin\tfrac{1}{2}\mu\pi,$$

where
$$C = \Sigma a_n \frac{\Gamma(n+\mu+1)}{m^{n+\mu+1}}\cos\tfrac{1}{2}n\pi,$$

$$S = \Sigma a_n \frac{\Gamma(n+\mu+1)}{m^{n+\mu+1}}\sin\tfrac{1}{2}n\pi,$$

or
$$C = \frac{\Gamma(\mu+1)}{m^{\mu+1}}\left\{a_0 - \frac{(\mu+1)(\mu+2)}{m^2}a_2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{m^4}a_4 - \dots\right\},$$

$$S = \frac{\Gamma(\mu+1)}{m^{\mu+1}}\left\{-\frac{(\mu+1)}{m}a_1 + \frac{(\mu+1)(\mu+2)(\mu+3)}{m^3}a_3 - \dots\right\}.$$

In particular, if $\mu = 0$,

$$G\int_0^\infty \cos mx\, F(x)\, dx = -\frac{a_1}{m^2} + \frac{3!\,a_3}{m^4} - \frac{5!\,a_5}{m^6} + \dots,$$

$$G\int_0^\infty \sin mx\, F(x)\, dx = \frac{a_0}{m} - \frac{2!\,a_2}{m^3} + \frac{4!\,a_4}{m^5} - \dots$$

By taking $\mu = 0$ and $F(x) = J_0(\sqrt{x})$ we obtain

$$\int_0^\infty J_0(\sqrt{x})\,{\cos \atop \sin}\, mx\, dx = \frac{1}{m}\,{\sin \atop \cos}\left(\frac{1}{4m}\right) \quad \dots\dots\dots(1);$$

by taking $F(x) = \cos\sqrt{x}$ and $\mu = -\frac{1}{2}$ we obtain

$$\int_0^\infty \frac{\cos\sqrt{x}}{\sqrt{x}}\cos mx\, dx = \sqrt{\left(\frac{\pi}{2m}\right)}\left(\cos\frac{1}{4m} + \sin\frac{1}{4m}\right) \quad \dots\dots\dots(2),$$

or
$$\int_0^\infty \cos x^2 \cos 2\mu x\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\cos\mu^2 + \sin\mu^2) \quad \dots\dots\dots(3);$$

and similarly
$$\int_0^\infty \sin x^2 \cos 2\mu x\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\cos\mu^2 - \sin\mu^2) \quad \dots\dots\dots(4).$$

These are cases in which $F(x)$ is an integral function of order <1, and the integrals on the left are all convergent in the ordinary sense.

More general results may be obtained by taking

$$x^\mu F(x) = x^k J_0(\sqrt{x}),\quad x^k \frac{\cos\sqrt{x}}{\sqrt{x}}$$

(where k is a positive integer); e.g.

$$G\int_0^\infty J_0(\sqrt{x})\, x^{2k}\cos mx\, dx = (-)^k \left(\frac{d}{dm}\right)^{2k}\left\{\frac{1}{m}\sin\frac{1}{4m}\right\} \quad\text{.................(5).}$$

§ 11. Taking $\mu = 0$ and

$$F(x) = J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \dots,$$

we obtain

$$\int_0^\infty \cos mx\, J_0(x)\, dx = 0 \quad\text{.....................................(6),}$$

$$\int_0^\infty \sin mx\, J_0(x)\, dx = \frac{1}{m} + \frac{1}{2}\frac{1}{m^3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{1}{m^5} + \dots$$

$$= \frac{1}{\sqrt{(m^2-1)}} \quad\text{..................................... (7),}$$

provided $m > 1$. The results agree with Weber's well-known formulae*.

If we take $\mu = k$, and $F(x) = J_0(x)$, we obtain formulae for

$$G\int_0^\infty x^k \begin{matrix}\cos\\ \sin\end{matrix}\, mx\, J_0(x)\, dx,$$

which agree with the results of formal differentiation of (6) and (7).

More generally we may take

$$x^\mu F(x) = x^{\rho-1} J_\alpha(x),$$

where $\rho + \alpha > 0$, and express

$$G\int_0^\infty x^{\rho-1} \begin{matrix}\cos\\ \sin\end{matrix}\, mx\, J_\alpha(x)\, dx$$

as a hypergeometric series. When $-\alpha < \rho < \frac{3}{2}$ we obtain a known expression of an ordinary integral. An interesting special case is that in which $\rho - 1 = \alpha$. In this case we find

$$G\int_0^\infty x^\alpha J_\alpha(x)\, e^{-mix}\, dx = \Sigma \frac{(-)^n}{2^{\alpha+2n}\, n!\, \Gamma(n+\alpha+1)}\, G\int_0^\infty e^{-mix}\, x^{2n+2\alpha}\, dx$$

$$= \Sigma \frac{(-)^n}{2^{\alpha+2n}\, n!\, \Gamma(n+\alpha+1)} \frac{\Gamma(2n+2\alpha+1)}{m^{2n+2\alpha+1}}\, e^{-\frac{1}{2}(2n+2\alpha+1)\pi i}.$$

Using the formula

$$\Gamma(\alpha)\,\Gamma(\alpha+\tfrac{1}{2}) = \Gamma(2\alpha)\, 2^{\frac{1}{2}-2\alpha}\sqrt{2\pi},$$

we can reduce this series to

$$\frac{2^\alpha\, \Gamma(\alpha+\frac{1}{2})\, e^{(-\alpha+\frac{1}{2})\pi i}}{m^{2\alpha+1}\sqrt{\pi}}\, \Sigma \frac{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})\dots(\alpha+n-\frac{1}{2})}{1 \cdot 2 \dots n}\left(\frac{1}{m^2}\right)^n$$

$$= \frac{2^\alpha e^{(-\alpha+\frac{1}{2})\pi i}\, \Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\,(m^2-1)^{\alpha+\frac{1}{2}}}.$$

* Gray and Mathews, *Bessel Functions*, p. 73.

Equating real and imaginary parts we obtain

$$G\int_0^\infty x^\alpha J_\alpha(x)\cos mx\,dx = -\frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}(m^2-1)^{\alpha+\frac{1}{2}}}\sin\alpha\pi \quad\ldots\ldots(8),$$

$$G\int_0^\infty x^\alpha J_\alpha(x)\sin mx\,dx = \frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}(m^2-1)^{\alpha+\frac{1}{2}}}\cos\alpha\pi \quad\ldots\ldots(9).$$

Since $$\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-\alpha)=\frac{\pi}{\cos\alpha\pi}$$

the second of these two formulae agrees with the formula

$$\int_0^\infty x^\alpha J_\alpha(x)\sin mx\,dx = \frac{2^\alpha\sqrt{\pi}}{\Gamma(\frac{1}{2}-\alpha)(m^2-1)^{\alpha+\frac{1}{2}}}, \qquad (n>1,\ -\tfrac{1}{2}<n<\tfrac{1}{2})$$

given by Sonine*. Our formulae are valid provided only $m>1$ and the integral is convergent at the lower limit, which requires $\alpha>-\frac{1}{2}$ for (8) and $\alpha>-1$ for (9). If $\alpha=-\frac{1}{2}$ the formula (9) becomes illusory and reduces to the well-known result

$$\int_0^\infty \frac{\cos x\sin mx}{x}\,dx=\tfrac{1}{2}\pi \qquad (m>1) \quad\ldots\ldots(10).$$

Another interesting pair of results is

$$\int_0^\infty J^\alpha(x)\,{\cos \atop \sin}\,(x\cosh\omega)\frac{dx}{x}={\cos \atop \sin}\,(\tfrac{1}{2}\alpha\pi)\,\frac{e^{-\alpha\omega}}{\alpha} \quad\ldots\ldots(11)$$

(valid for $\alpha>1$)†.

To prove these formulae we observe that if $m=\cosh\omega$

$$\int_0^\infty J^\alpha(x)\,e^{-mix}\frac{dx}{x}=\sum_{n=0}^\infty\frac{(-)^n\,2^{-\alpha-2n}}{n!\,\Gamma(\alpha+n+1)}\,G\int_0^\infty x^{\alpha-1+2n}\,e^{-mix}\,dx$$

$$=(2m)^{-\alpha}\,e^{-\frac{1}{2}\alpha\pi i}\sum_{n=0}^\infty\frac{\Gamma(\alpha+2n)}{n!\,\Gamma(\alpha+n+1)}(2m)^{-2n}$$

$$=\frac{(2m)^{-\alpha}e^{-\frac{1}{2}\alpha\pi i}}{\alpha}\left\{1+\alpha\left(\frac{1}{2m}\right)^2+\frac{\alpha(\alpha+3)}{1.2}\left(\frac{1}{2m}\right)^4+\frac{\alpha(\alpha+4)(\alpha+5)}{1.2.3}\left(\frac{1}{2m}\right)^6+\ldots\right\}$$

$$=\frac{m^{-\alpha}e^{-\frac{1}{2}\alpha\pi i}}{\alpha}\left\{1+\sqrt{\left(1-\frac{1}{m^2}\right)}\right\}^{-\alpha}$$

$$=e^{-\frac{1}{2}\alpha\pi i}\,\frac{e^{-\alpha\omega}}{\alpha},$$

the only condition required being $m>1$, which is satisfied.

§ 12. The following three examples are also instructive:

(i) $$G\int_0^\infty \cos mx\cos\lambda x\,dx=0+0+0+\ldots=0,$$

$$G\int_0^\infty \sin mx\cos\lambda x\,dx=\frac{1}{m}+\frac{\lambda^2}{m^3}+\frac{\lambda^4}{m^5}+\ldots=\frac{m}{m^2-\lambda^2},$$

provided $m>\lambda$—of course in this case the formulae are also true for $m<\lambda$;

* *Math. Annalen*, Bd. XVI. p. 39.

† For $\alpha>-2$ if the *sine* be taken: cf. Nielsen, *Handbuch der Cylinderfunktionen*, p. 197, and the analogous results of Schafheitlin, *Math. Annalen*, Bd. XXX. p. 171.

(ii) $$\int_0^\infty \frac{\sin mx \sin x}{x}\,dx = \Sigma \frac{1}{(2n+1)\,m^{2n+1}} = \tfrac{1}{2}\log\left(\frac{m+1}{m-1}\right);$$

(iii) $$\int_0^\infty \frac{\sin mx \sin x}{x^2}\,dx = \Sigma \frac{(-)^n}{(2n+1)!}\,G\int_0^\infty \sin mx\, x^{2n-1}\,dx$$
$$= \tfrac{1}{2}\pi,$$

all the integrals in this case vanishing save that for which $n=0$*. In these formulae we must have $m>1$. It might appear that the last formula should also hold for $m<1$, since the series

$$\tfrac{1}{2}\pi + 0 + 0 + 0 + \ldots$$

is always convergent. But our condition was that the series obtained by integrating

$$\int_0^\infty e^{-mix} F(x)\,dx$$

should converge, i.e. that the two series obtained by integrating

$$\int_0^\infty {\cos \atop \sin}\, mx\, F(x)\,dx$$

should *both* converge: and it is easy to see that in this case the series obtained by taking the cosine diverges for $m<1$. This must always be borne in mind. Otherwise we should be tempted to infer that

$$G\int_0^\infty \cos mx\, \phi(x^2)\,dx$$

(where ϕ is an integral function) is always zero, which is evidently not the case, as, e.g.,

$$\int_0^\infty \cos mx\, e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}m^2}.$$

In this case it will be found that the series for $\int_0^\infty \sin mx e^{-x}\,dx$ is divergent for all values of m.

§ 13. Let us consider next some applications of the more general theorem II. It is easy to prove that

$$\int_0^\infty J^\alpha(mx)\, e^{-\tau x} x^\rho\,dx = \frac{m^\alpha}{\tau^{\alpha+\rho+1}} \frac{\Gamma(\alpha+\rho+1)}{2^\alpha\,\Gamma(\alpha+1)} F\left(\frac{\alpha+\rho+1}{2}, \frac{\alpha+\rho+2}{2}, \alpha+1, -\frac{m^2}{\tau^2}\right) \ldots(12),$$

provided $\alpha+\rho>-1$, $\tau>m$. We have only to replace $J^\alpha(mx)$ by its expression as a series and integrate term by term. This formula fails us for small values of τ, but, by the help of a formula of Euler's connecting two hypergeometric functions, we can deduce from it †

$$\int_0^\infty J^\alpha(mx)\, e^{-\tau x} x^\rho\,dx$$
$$= \frac{\Gamma(\alpha+\rho+1)}{\Gamma(\alpha+1)} \frac{(\frac{1}{2}m)^\alpha}{(m^2+\tau^2)^{\frac{1}{2}(\alpha+\rho+1)}} F\left(\frac{\alpha+\rho+1}{2}, \frac{\alpha-\rho}{2}, \alpha+1, \frac{m^2}{m^2+\tau^2}\right) \ldots(13).$$

* We note that $$G\int_0^\infty x^{2n}\cos mx\,dx = G\int_0^\infty x^{2n+1}\sin mx\,dx = 0,$$
$$G\int_0^\infty x^{2n+1}\cos mx\,dx = (-)^{n+1}\frac{(2n+1)!}{m^{2n+2}},\quad G\int_0^\infty x^{2n}\sin mx\,dx = (-)^n \frac{2n!}{m^{2n+1}}.$$

† Hankel, *Math. Annalen*, Bd. VIII. p. 467. Nielsen, *Handbuch der Cylinderfunktionen*, pp. 185, 188.

In the limit for $\tau=0$ this equation becomes

$$G\int_0^\infty J^\alpha(mx)\,x^\rho\,dx=\frac{2^\rho}{m^{\rho+1}}\frac{\Gamma\{\frac{1}{2}(\alpha+\rho+1)\}}{\Gamma\{\frac{1}{2}(\alpha-\rho+1)\}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

This formula holds for $\alpha+\rho>-1$. If also $\rho<\frac{1}{2}$, the integral is convergent in the ordinary sense *.

§ 14. Now let us, in the general integral

$$G\int_0^\infty \phi(x)\,F(x)\,dx,$$

suppose

$$F=\Sigma\, a_n x^n,$$

and

$$\phi(x)=J^\alpha(mx)\,x^\rho.$$

Then

$$\psi(\tau)=\int_0^\infty e^{-\tau x}J^\alpha(mx)\,x^\rho\,dx$$

is, as we can see from the equation (13), a function of τ regular within the circle whose centre is the origin and whose radius is m.

From Theorem II we deduce that if the series $\Sigma\, n!\, a_n y^n$ has a radius of convergence greater than $1/m$, then

$$G\int_0^\infty J^\alpha(mx)x^\rho F(x)\,dx=\Sigma a_n\, G\int_0^\infty J^\alpha(mx)\,x^{n+\rho}\,dx$$

$$=\Sigma\frac{2^{\rho+n}}{m^{\rho+n+1}}a_n\frac{\Gamma\{\frac{1}{2}(\alpha+1+\rho+n)\}}{\Gamma\{\frac{1}{2}(\alpha+1-\rho-n)\}}.$$

Writing $\alpha-\beta$ for α and putting $m=1$ we obtain

$$G\int_0^\infty J^{\alpha-\beta}(x)\,x^\rho F(x)\,dx=2^\rho\,\Sigma\,2^n a_n\frac{\Gamma\{\frac{1}{2}(\alpha-\beta+1+\rho+n)\}}{\Gamma\{\frac{1}{2}(\alpha-\beta+1-\rho-n)\}}.$$

Now suppose

$$x^\rho F(x)=x^{-\gamma+\alpha+\beta}J^{\gamma-1}(xz)\qquad(0<z<1)$$

$$=x^{\alpha+\beta-1}\left(\frac{z}{2}\right)^{\gamma-1}\sum_{\nu=0}^{\infty}\frac{(-)^\nu(\frac{1}{2}xz)^{2\nu}}{\nu!\,\Gamma(\gamma+\nu)}.$$

Then we must take

$$\rho=\alpha+\beta-1,$$

$$a_{2\nu+1}=0,\quad a_{2\nu}=\left(\frac{z}{2}\right)^{2\nu+\gamma-1}\frac{(-)^\nu}{\nu!\,\Gamma(\gamma+\nu)},$$

and we find

$$G\int_0^\infty J^{\alpha-\beta}(x)\,J^{\gamma-1}(xz)\,x^{-\gamma+\alpha+\beta}\,dx$$

$$=2^{\alpha+\beta-1}\left(\frac{z}{2}\right)^{\gamma-1}\Sigma\,(-)^\nu\,2^{2\nu}\left(\frac{z}{2}\right)^{2\nu}\frac{\Gamma(\alpha+\nu)}{\nu!\,\Gamma(\gamma+\nu)\,\Gamma(1-\beta-\nu)}$$

$$=\frac{z^{\gamma-1}}{2^{\gamma-\alpha-\beta}}\frac{\sin\beta\pi}{\pi}\Sigma\frac{\Gamma(\alpha+\nu)\,\Gamma(\beta+\nu)}{\nu!\,\Gamma(\gamma+\nu)}z^{2\nu}$$

$$=\frac{z^{\gamma-1}}{2^{\gamma-\alpha-\beta}}\frac{\Gamma(\alpha)}{\Gamma(1-\beta)\,\Gamma(\gamma)}F(\alpha,\beta,\gamma,z^2)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15);$$

* Nielsen, *loc. cit.*, p. 189.

a formula which contains a very large number of interesting particular cases. Our proof involves only that the integral should be convergent at the lower limit, *i.e.* that $\alpha > 0$. If $\gamma - \alpha - \beta > -1$ the integral is convergent in the ordinary sense*.

§ 14. As a final example of the use of Theorem II let us consider the integrals

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx,$$

where m and a are positive, λ a positive integer, and $f(x)$ is an even function of x defined by a series $f(x) = \Sigma\, a_n x^{2n}$, convergent for all values of x.

Since
$$\frac{x^{2k}}{c^2+x^2} = (-1)^k c^{2k} \left\{ \frac{1}{c^2+x^2} - \frac{1}{c^2} + \frac{x^2}{c^4} - \ldots + (-1)^k \frac{x^{2k-2}}{c^{2k-2}} \right\},$$

we have
$$G\int_0^\infty \frac{x^{2k} \cos mx}{c^2+x^2}\, dx = (-1)^k c^{2k} \int_0^\infty \frac{\cos mx}{c^2+x^2}\, dx = (-1)^k \tfrac{1}{2}\pi c^{2k-1} e^{-mc},$$

$$G\int_0^\infty \frac{x^{2k+1} \sin mx}{c^2+x^2}\, dx = (-1)^k c^{2k} \int_0^\infty \frac{x \sin mx}{c^2+x^2}\, dx = (-1)^k \tfrac{1}{2}\pi c^{2k} e^{-mc}.$$

Hence, taking the divergent integral term by term, we obtain

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = \Sigma a_n\, G\int_0^\infty \frac{x^{2(n+\lambda)} \cos mx}{c^2+x^2}\, dx$$
$$= (-1)^\lambda \tfrac{1}{2}\pi c^{2\lambda-1} e^{-mc}\, \Sigma (-1)^n a_n c^{2n},$$

$$G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = \Sigma a_n\, G\int_0^\infty \frac{x^{2(n+\lambda)+1} \sin mx}{c^2+x^2}\, dx$$
$$= (-1)^\lambda \tfrac{1}{2}\pi c^{2\lambda} e^{-mc}\, \Sigma (-1)^\lambda a_n c^{2n}.$$

These results may be stated in the form

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = (-1)^\lambda \tfrac{1}{4}\pi c^{2\lambda-1} e^{-mc} \{f(ci) + f(-ci)\} \quad \text{...........(16)},$$

$$G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = (-1)^\lambda \tfrac{1}{4}\pi c^{2\lambda} e^{-mc} \{f(ci) + f(-ci)\} \quad \text{..............(17)}.$$

We have now to consider under what conditions this procedure is legitimate. In the first place

$$\theta(z) = \int_0^\infty \frac{e^{-zx}}{c^2+x^2} x^{2\lambda}\, dx$$

is known to be an analytic function of z, regular save at the origin and at infinity. It follows that

$$\psi(\tau) = \int_0^\infty \frac{e^{-(\tau+mi)x}}{c^2+x^2} x^{2\lambda}\, dx$$

has as its singularity nearest to the origin the point $\tau = -mi$, so that $\delta = m$.

* For special examples see Nielsen, *loc. cit.* pp. 191 *et seq.* If $\gamma - \alpha - \beta > 0$ the hypergeometric series converges for $z = 1$, and the result may be extended to this case (Nielsen, *loc. cit.* p. 194).

Thus we are justified in evaluating the integrals

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{\sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx,$$

in the manner adopted above, if the series

$$\Sigma\, 2n!\, a_n y^{2n}$$

has a radius of convergence greater than $1/m$. A similar argument can of course be applied to the integrals

$$G\int_0^\infty \frac{x\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{x\sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx.$$

It is however only in the two cases considered above that the result can be calculated in finite terms.

§ 15. So far we have considered only the particular case of the general theorem in which the integral is of the form

$$G\int_0^\infty \phi(x) f(x)\, dx,$$

where $f(x) = \Sigma a_n x^n$. Another interesting case is that in which $f(x)$ is a periodic function representable by a Fourier's series

$$\Sigma a_n e^{-2n\pi i x}.$$

If we suppose $f(x)$ *continuous*, it is known that

$$|a_n| < \frac{K}{n}.$$

If $f(x)$ is continuous except for a finite number of points x_ν in the interval $(0, 1)$, at which it has infinities of the type $A/(x-x_\nu)^\beta$, where $0<\beta<1$, it is known that

$$|a_n| < \frac{K}{n^{1-\beta}}.$$

We have to consider,

(1) whether $\displaystyle\int_0^\infty e^{-\tau x}\phi(x)\,\Sigma a_n e^{-2n\pi i x}\, dx = \Sigma a_n \int_0^\infty e^{-(\tau+2n\pi i)x}\phi(x)\, dx,$

(2) whether the last series is a continuous function of τ for $\tau = 0$.

I shall consider only the particular case in which

$$\phi(x) = x^{\alpha-1}. \qquad (\alpha > 0)$$

It is in this case not hard to show that the question (1) can be answered affirmatively. We have to prove that

$$\lim_{X\to\infty} \Sigma a_n \int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx = 0.$$

Now
$$\begin{aligned}\int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx &= X^{\alpha-1}\frac{e^{-(\tau+2n\pi i)X}}{\tau+2n\pi i} \\ &\quad + \frac{\alpha-1}{\tau+2n\pi i}\int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-2}\, dx \\ &= \lambda_n + \mu_n,\end{aligned}$$

say. But
$$|\lambda_n| < \frac{K}{n} X^{\alpha-1} e^{-\tau X}$$
and
$$|\mu_n| < \frac{K}{n} e^{-\frac{1}{2}\tau X} \int_X^\infty e^{-\frac{1}{2}\tau x} x^{\alpha-2}\, dx < \frac{K}{n} e^{-\frac{1}{2}\tau X},$$
and so
$$|\Sigma a_n \lambda_n| < K X^{\alpha-1} e^{-\tau X} \Sigma n^{\beta-2},$$
$$|\Sigma a_n \mu_n| < K e^{-\frac{1}{2}\tau X} \Sigma n^{\beta-2},$$
each of which tends to zero as $X \to \infty$.

The second question can also be answered in the affirmative if $\alpha \geqq 1$. For then the series
$$\Sigma a_n \int_0^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx = \Gamma(\alpha)\, \Sigma \frac{a_n}{(\tau+2n\pi i)^\alpha},$$
where that value of $(\tau+2n\pi i)^{-\alpha}$ is chosen which reduces to
$$(2n\pi)^{-\alpha} e^{-\frac{1}{2}\alpha\pi i}$$
for $\tau = 0$, has its terms numerically less than those of the series
$$K\Sigma n^{-\alpha-1+\beta},$$
and so is uniformly convergent for any interval $0 \leqq \tau \leqq \tau_0$.

Thus the equations
$$G\int_0^\infty x^{\alpha-1} f(x)\, dx = \sum_1^\infty a_n G \int_0^\infty x^{\alpha-1} e^{-2n\pi i x}\, dx$$
$$= \Gamma(\alpha)(2\pi)^{-\alpha} e^{-\frac{1}{2}\alpha\pi i} \sum_1^\infty \frac{a_n}{n^\alpha},$$
$$G\int_0^\infty x^{\alpha-1} \sum_1^\infty a_n \begin{matrix}\cos\\ \sin\end{matrix} 2n\pi x\, dx = \Gamma(\alpha)(2\pi)^{-\alpha} \begin{matrix}\cos\\ \sin\end{matrix} \tfrac{1}{2}\alpha\pi \sum_1^\infty \frac{a_n}{n^\alpha} \quad \text{..............(18)}$$
are certainly valid if $\alpha > 1$. On the other hand they are not necessarily valid if $0 < \alpha < 1$. Thus if $\alpha = \frac{1}{2}$ and $a_n = 1/\sqrt{n}$ we are led to the series
$$\Sigma \frac{1}{n},$$
which is divergent. In this case the integral also is divergent at the lower limit, since
$$e^{-2\pi i x} + \frac{e^{-4\pi i x}}{\sqrt{2}} + \frac{e^{-6\pi i x}}{\sqrt{3}} + \ldots$$
has an infinity of order $1/2$ for $x = 0$.

§ 16. Sufficient will have been said by now to show that, however difficult it may be in some cases to justify our procedure, the method of expansion and taking the generalised integrals of the separate terms is, in such cases as naturally occur in analysis, generally defensible; and *as a rule leads to correct results.* The reader will have no difficulty in constructing any number of further examples for himself, there being a large variety of integrals, of very different types, whose values are most easily determined in this way.

The process may be combined with Borel's method for the summation of a divergent series. This will probably be illustrated best by an example.

Consider the integral
$$G\int_0^\infty \frac{x^{a-1}e^{-mix}}{1+x}\,dx,$$
where a and m are positive. Expand $1/(1+x)$ in the series
$$1-x+x^2-\ldots$$
convergent if $0 \leqq x < 1$, summable if $x \geqq 1$. Taking the generalised integral term by term we obtain
$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\,m^{-a}\left\{1+a\left(\frac{i}{m}\right)+a(a+1)\left(\frac{i}{m}\right)^2+\ldots\right\}.$$
The series in brackets is divergent for all values of m. Its sum, according to Borel's definition, is
$$\int_0^\infty e^{-v}\,dv\left\{1+a\left(\frac{iv}{m}\right)+\frac{a(a+1)}{1\,.\,2}\left(\frac{iv}{m}\right)^2+\ldots\right\}.$$

The series under the sign of integration is itself divergent if $v > m$; but it is summable for all positive values of v, and its sum is known to be
$$\left(1-\frac{iv}{m}\right)^{-a}\text{*}.$$
Hence we are led to the result
$$G\int_0^\infty \frac{x^{a-1}e^{-mix}}{1+x}\,dx=\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\int_0^\infty \frac{e^{-v}\,dv}{(m-iv)^a}.$$

Whether our work can be justified is another matter. We shall see in a moment that to attempt to do so would involve considerable difficulties. The point is that the work *leads us to the result*, which is as a matter of fact correct and includes a number of interesting special cases. In particular if $m=0$, $a<1$, we obtain
$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx=\Gamma(a)\,\Gamma(1-a)=\frac{\pi}{\sin a\pi},$$
and if $a=1$ we obtain
$$\int_0^\infty \frac{e^{-mix}}{1+x}\,dx=\frac{1}{i}\int_0^\infty \frac{e^{-v}\,dv}{m-iv},$$
$$\int_0^\infty \frac{\cos mx}{1+x}\,dx=\int_0^\infty \frac{ve^{-v}\,dv}{m^2+v^2},\quad \int_0^\infty \frac{\sin mx}{1+x}\,dx=\int_0^\infty \frac{me^{-v}\,dv}{m^2+v^2},$$
a pair of formulae due originally to Cauchy†.

Let us consider what our transformations really involve. In summing the series
$$1+a\left(\frac{i}{m}\right)+a(a+1)\left(\frac{i}{m}\right)^2+\ldots$$
we had to use two repetitions of Borel's process: hence
$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\int_0^\infty \frac{e^{-v}\,dv}{(m-iv)^a}$$
is in reality the equivalent of
$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\,m^{-a}\int_0^\infty e^{-v}\,dv\int_0^\infty e^{-w}\,dw\,\Sigma\, a(a+1)\ldots(a+n-1)\,\frac{(ivw/m)^n}{(n!)^2},$$

* See, *e.g.*, Bromwich, *Infinite Series*, p. 302.

† 'Mémoire sur les Intégrales Définies,' *Oeuvres*, t. I. p. 377.

or of
$$\int_0^\infty e^{-v}\,dv \int_0^\infty e^{-w}\,dw\, \Sigma\Gamma(a+n)\, e^{-\frac{1}{2}(a+n)\pi i}\, m^{-a-n}\frac{(-vw)^n}{(n!)^2},$$

or of
$$\int_0^\infty e^{-v}\,dv \int_0^\infty e^{-w}\,dw \sum_{n=0}^{\infty}\frac{(-vw)^n}{(n!)^2}\lim_{\tau=0}\int_0^\infty e^{-(\tau+im)x}\,x^{a+n-1}\,dx\,;$$

and what we assert is that this is equal to
$$\lim_{\tau=0}\int_0^\infty e^{-(\tau+im)x}\,x^{a-1}\,dx \int_0^\infty e^{-v}\,dv \int_0^\infty e^{-w}\,dw \sum_{n=0}^{\infty}\frac{(-vwx)^n}{(n!)^2}.$$

Now each of these expressions is *an 8-ple repeated limit,* for an infinite integral
$$\int_0^\infty f(x)\,dx$$
is itself a repeated limit. Hence our work is in reality a shorthand representation of a multiple limit permutation of extreme complexity.

§ 17. As a further application of Borel's method let us notice the following. We obtained in § 6 the two formulae
$$G\int_0^\infty F(x)\cos mx\,dx = -\frac{a_1}{m^2}+\frac{3!\,a_3}{m^4}-\frac{5!\,a_5}{m^6}+\ldots,$$
$$G\int_0^\infty F(x)\sin mx\,dx = \frac{a_0}{m}-\frac{2!\,a_2}{m^3}+\frac{4!\,a_4}{m^5}-\ldots,$$
where $F(x)=\Sigma a_n x^n$. If we sum the series on the right by Borel's method we obtain
$$\int_0^\infty e^{-v}\psi_1(v)\,dv,\quad \int_0^\infty e^{-v}\psi_2(v)\,dv,$$
where
$$m\psi_1(v) = -a_1\left(\frac{v}{m}\right)+a_3\left(\frac{v}{m}\right)^2-\ldots = \tfrac{1}{2}i\left\{F\left(\frac{iv}{m}\right)-F\left(-\frac{iv}{m}\right)\right\},$$
$$m\psi_2(v) = a_1-a_2\left(\frac{v}{m}\right)^2+\ldots = \tfrac{1}{2}\left\{F\left(\frac{iv}{m}\right)+F\left(-\frac{iv}{m}\right)\right\}.$$

We thus obtain the formulae
$$G\int_0^\infty F(x)\cos mx\,dx = \frac{i}{2m}\int_0^\infty e^{-v}\left\{F\left(\frac{iv}{m}\right)-F\left(-\frac{iv}{m}\right)\right\}dv,$$
$$G\int_0^\infty F(x)\sin mx\,dx = \frac{1}{2m}\int_0^\infty e^{-v}\left\{F\left(\frac{iv}{m}\right)+F\left(-\frac{iv}{m}\right)\right\}dv,$$
or
$$G\int_0^\infty F(x)\,e^{mix}\,dx = \frac{i}{m}\int_0^\infty e^{-v}F\left(\frac{iv}{m}\right)dv = i\int_0^\infty e^{-m\eta}F(i\eta)\,d\eta.$$
This is exactly the formula which we obtain by integrating
$$\int F(x)\,e^{mix}\,dx \qquad (n>0)$$
round the contour formed by the positive parts of the real and imaginary axes and a very large quadrant of a circle, supposing the integrals along the axes convergent in the ordinary sense, and the curvilinear part of the integral evanescent in the limit.

This fact suggests that we must expect errors if $\Sigma a_n x^n$ has only a finite radius of convergence (though summable for all positive values of x), and $F(x)$ has poles situated within or on this contour. It also suggests that when $F(x)$ is integral we may be liable to error when the order of $F(x)$ is not less than unity or at any rate when it is not true that $|e^{mix} F(x)| \to 0$ as $|x| \to \infty$, for any value of $am.x$ between 0 and $\frac{1}{2}\pi$. It is instructive to consider from this general point of view, no less than from that of the precise theorems of the earlier sections, the results

$$\int_0^\infty \cos mx\, J_0(\sqrt{x})\, dx = \frac{1}{4m^2} - \frac{1}{3!}\frac{1}{4^3 m^4} + \frac{1}{5!}\frac{1}{4^5 m^6} - \dots$$

(valid for all positive values of m),

$$\int_0^\infty \cos mx\, J_0(x)\, dx = 0 + 0 + 0 + \dots,$$

$$\int_0^\infty \sin mx\, J_0(x)\, dx = \frac{1}{m} + \frac{1}{2}\frac{1}{m^3} + \frac{1.3}{2.4}\frac{1}{m^5} + \dots,$$

$$\int_0^\infty \cos mx\, e^{-x} dx = \frac{1}{m^2} - \frac{1}{m^4} + \frac{1}{m^6} - \dots,$$

$$\int_0^\infty \sin mx\, e^{-x} dx = \frac{1}{m} - \frac{1}{m^3} + \frac{1}{m^5} - \dots$$

(valid for $m > 1$ only),

and

$$\int_0^\infty \cos mx\, e^{-x^2} dx = 0 + 0 + 0 + \dots,$$

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = 0 + 0 + 0 + \dots$$

(valid for no value of m).

B. *Continuity of generalised integrals which contain a continuous parameter.*

§ 18. I shall now consider the generalised integral

$$G\int_0^\infty f(x, \alpha)\, dx \quad \text{...(1)},$$

and the question of its continuity for a particular value of α, which we may suppose to be zero.

The integral (1) will be continuous for $\alpha = 0$ if

$$\left.\begin{aligned} \lim_{\alpha\to 0} G\int_0^\infty f(x,\alpha)\, dx &= \lim_{\alpha\to 0}\lim_{\tau\to 0}\int_0^\infty e^{-\tau x} f(x,\alpha)\, dx \\ &= \lim_{\tau\to 0}\lim_{\alpha\to 0}\int_0^\infty e^{-\tau x} f(x,\alpha)\, dx \\ &= \lim_{\tau\to 0}\int_0^\infty e^{-\tau x} f(x,0)\, dx \\ &= G\int_0^\infty f(x,0)\, dx \end{aligned}\right\} \quad \text{.....................(2)}.$$

The applications of this transformation do not appear to be so interesting as those of the transformation of § 2. I shall therefore not discuss its legitimacy in great detail.

The most interesting case appears to be the following. Suppose that $f(x, \alpha)$ is a continuous function of both variables throughout the rectangle

$$(0, \alpha_0;\ 0, X),$$

where α_0 is some positive value of α, and X any positive value of X. Suppose also that for $0 \leqq \alpha \leqq \alpha_0$, and all positive values of x,

$$|f(x, \alpha)| < Hx^K,$$

where H and K are constants.

Further suppose that

$$\int_0^\infty f(x, \alpha)\, dx$$

is convergent for $\alpha > 0$, and that

$$G \int_0^\infty f(x, 0)\, dx$$

is summable*.

Then

$$\lim_{\tau \to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \int_0^\infty f(x, \alpha)\, dx,$$

$$\lim_{\tau \to 0} \int_0^\infty e^{-\tau x} f(x, 0)\, dx = G \int_0^\infty f(x, 0)\, dx,$$

so that the first and last steps of the argument expressed by the equations (2) are justified. Further, it is easy to see that, for any particular positive value of τ, the integral

$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx$$

is uniformly convergent throughout the interval $(0, \alpha_0)$. Hence

$$\lim_{\alpha \to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \int_0^\infty e^{-\tau x} f(x, 0)\, dx.$$

Thus the last step but one of the argument is justified. Thus if we write

$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \phi(\tau, \alpha)$$

the whole question reduces to the question whether

$$\lim_{\alpha \to 0} \lim_{\tau \to 0} \phi(\tau, \alpha) = \lim_{\tau \to 0} \lim_{\alpha \to 0} \phi(\tau, \alpha) \quad \text{.............................(3).}$$

We may notice that we are already assured of the existence of the second repeated limit and of the inner limit on the left-hand side.

Now Mr Bromwich† has enunciated the following necessary and sufficient conditions for the truth of (3):—

(i) the simple limits

$$\phi(\tau) = \lim_{\alpha \to 0} \phi(\tau, \alpha),$$

$$\phi(\alpha) = \lim_{\tau \to 0} \phi(\tau, \alpha)$$

exist,

* It seems better, on account of the ambiguity of the uses of the term *divergent*, to call a generalised integral *summable* than *convergent*.

† *Proc. Lond. Math. Soc.*, N.S., vol. I. p. 184, and vol. VI. p. 119. See also Hobson, *Proc. Lond. Math. Soc.*, N.S., vol. v. p. 225, and *Theory of Functions of a Real Variable*, pp. 303—311 and 464—467.

(ii) the repeated limit

$$\lim_{\tau \to 0} \phi(\tau)$$

exists,

(iii) given ϵ and τ_0 (each positive but as small as we please) we can choose a positive value of τ less than τ_0, and a positive value of α_0, so that

$$|\phi(\alpha) - \phi(\tau, \alpha)| < \epsilon$$

for the one value τ and every positive α less than α_0.

In the present case the first two conditions are certainly satisfied, and the third is equivalent to

$$\left| \int_0^\infty f(x, \alpha)(1 - e^{-\tau x})\, dx \right| < \epsilon$$

for one positive τ less than τ_0 and every positive α less than α_0.

We have therefore:

THEOREM IV. *If $f(x, \alpha)$ is a function of x and α continuous for $0 \leqq x \leqq X$, $0 \leqq \alpha \leqq \alpha_0$, however great X may be, and numerically less than Hx^K for all these values of x and α; if moreover $\int_0^\infty f(x, \alpha)\, dx$ $(\alpha > 0)$, is convergent, and $G \int_0^\infty f(x, 0)\, dx$ summable; and if finally, however small be ϵ and τ_0, we can find τ so that $0 < \tau < \tau_0$ and*

$$\left| \int_0^\infty f(x, \alpha)(1 - e^{-\tau x})\, dx \right| < \epsilon, \qquad (0 < \alpha \leqq \alpha_0)$$

then

$$\lim_{\alpha \to 0} \int_0^\infty f(x, \alpha)\, dx = G \int_0^\infty f(x, 0)\, dx.$$

§ 19. The most interesting case of this theorem is the following:

Let
$$f(x, \alpha) = \phi(\alpha x)\, e^{-mix} x^\mu,$$
where $m > 0$, $\mu > -1$, and $\phi(u)$ is a function of u which possesses the following properties:

(1) as $u \to \infty$, $\phi(u) \to 0$, and that more rapidly than any power of u;

(2) $\phi(u)$ has continuous derivatives $\phi'(u), \phi''(u), \ldots \phi^{(n+1)}(u)$, where $n > \mu$;

(3) the integral

$$\int_0^\infty \phi^{(n+1)}(u)\, u^\mu du$$

is absolutely convergent.

Then it can be shown that the conditions of Theorem IV are satisfied: but the proof rests on a number of preliminary results.

§ 20. Let
$$\chi_0(x, \tau) = e^{-(\tau + mi)x} x^\mu,$$
where $\tau > 0$. Further let

$$\chi_1(x, \tau) = \int_x^\infty \chi_0(t, \tau)\, dt,$$

$$\chi_2(x, \tau) = \int_x^\infty \chi_1(t, \tau)\, dt,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then it is easy to prove, by a repeated integration by parts, that, if $a = \tau + mi$, we have

$$\chi_1(x, \tau) = \frac{e^{-ax} x^\mu}{a} \left\{1 + \frac{\mu}{ax} + \frac{\mu(\mu-1)}{(ax)^2} + \dots + \frac{\mu(\mu-1)\dots(\mu-r+1)}{(ax)^r}\right\} + \frac{\mu(\mu-1)\dots(\mu-r)}{a^r}\int_x^\infty e^{-at} t^{\mu-r-1} dt \quad \dots(4),$$

where r is any positive integer (we may set aside the case in which μ is a positive integer, when $\chi_1(x, \tau)$ can be found in finite terms). In fact $\chi_1(x, \tau)$ has the asymptotic expansion

$$\frac{e^{-ax} x^\mu}{a}\left\{1 + \frac{\mu}{ax} + \frac{\mu(\mu-1)}{(ax)^2} + \dots\right\} \quad \dots\dots\dots\dots(5).$$

We can find an expression of the same kind for $\chi_k(x, \tau)$, k being any positive integer. For it is easy to prove* that

$$\begin{aligned}\chi_k(x, \tau) &= \frac{1}{(k-1)!}\int_x^\infty e^{-at} t^\mu (t-x)^{k-1} dt \\ &= \sum_{\lambda=0}^{k-1} \frac{(-x)^\lambda}{\lambda!(k-1-\lambda)!}\int_x^\infty e^{-at} t^{\mu+k-1-\lambda}\, dt \\ &= \sum_{\lambda=0}^{k-1} \frac{(-x)^\lambda}{\lambda!(k-1-\lambda)!}\left[\frac{e^{-ax} x^{\mu+k-1-\lambda}}{a}\left\{1 + \frac{\mu+k-1-\lambda}{ax} + \dots + \frac{(\mu+k-1-\lambda)(\mu+k-2-\lambda)\dots(\mu+k-r-\lambda)}{(ax)^r}\right\}\right. \\ &\qquad \left. + \frac{(\mu+k-1-\lambda)\dots(\mu+k-r-1-\lambda)}{a^r}\int_x^\infty e^{-at} t^{\mu+k-r-2-\lambda} dt\right] \quad \dots(6).\end{aligned}$$

This furnishes for $\chi_k(x, \tau)$ the asymptotic expansion

$$\frac{e^{-ax} x^{\mu+k-1}}{a}\sum \frac{A_\nu}{(ax)^\nu},$$

where $$A_\nu = \sum_{\lambda=0}^{k-1} \frac{(-1)^\lambda}{\lambda!(k-1-\lambda)!}(\mu+k-1-\lambda)(\mu+k-2-\lambda)\dots(\mu+k-\nu-\lambda).$$

It is, however, easy to prove that

$$A_0 = A_1 = \dots = A_{k-2} = 0,\ A_{k-1} = 1.$$

For $$\begin{aligned} A_\nu &= \frac{1}{(k-1)!}\sum_{\lambda=0}^{k-1}(-1)^\lambda \binom{k-1}{\lambda}\left[\left(\frac{d}{dx}\right)^\nu x^{\mu+k-1-\lambda}\right]_{x=1} \\ &= \frac{1}{(k-1)!}\left[\left(\frac{d}{dx}\right)^\nu x^{\mu+k-1}\left(1-\frac{1}{x}\right)^{k-1}\right]_{x=1} \\ &= \frac{1}{(k-1)!}\left[\left(\frac{d}{dx}\right)^\nu x^\mu (x-1)^{k-1}\right]_{x=1},\end{aligned}$$

from which the result follows at once. Thus for large values of x

$$\chi_k(x, \tau) \sim \frac{e^{-(\tau+mi)x} x^\mu}{(\tau+mi)^k},$$

and $$|\chi_k(x, \tau)| < Kx^\mu$$

for all positive values of x and τ.

Now let $$\chi_0(x, 0) = \lim_{\tau \to 0} \chi_0(x, \tau) = e^{-mix} x^\mu.$$

* Jordan, *Cours d'Analyse*, t. III. p. 59.

Then (see p. 59 of my paper in the *Quarterly Journal* already quoted) we know that

$$\chi_1(x, 0) = G\int_x^\infty \chi_0(t, 0)\,dt$$

is summable, and that

$$\chi_1(x, 0) = \lim_{\tau\to 0} \chi_1(x, \tau).$$

Now suppose, in the equation (4), that $r > \mu$, and make τ tend to zero. Clearly we obtain in the limit

$$\chi_1(x, 0) = \frac{e^{-mix}x^\mu}{mi}\left\{1 + \frac{\mu}{mix} + \frac{\mu(\mu-1)}{(mix)^2} + \dots \right.$$
$$\left. + \frac{\mu(\mu-1)\dots(\mu-r+1)}{(mix)^r}\right\} + \frac{\mu(\mu-1)\dots(\mu-r)}{(mi)^r}\int_x^\infty e^{-mit}t^{\mu-r-1}\,dt \dots (4)'.$$

Since $r > \mu$, we can determine a positive value of ϵ such that

$$\mu - r - 1 + \epsilon < -1,$$

and

$$\left|\int_x^\infty e^{-mit}t^{\mu-r-1}\,dt\right| = \left|x^{\mu-r-1+\epsilon}\int_x^\xi e^{-mit}t^{-\epsilon}\,dt\right| < Kx^{-1-\alpha}. \qquad (\xi > x,\ \alpha > 0)$$

It follows that the last term in the equation (4)′ possesses an absolutely convergent integral up to ∞. The other terms on the right-hand side possess generalised integrals up to ∞. Hence $\chi_1(x, 0)$ possesses a generalised integral up to ∞, and we may write

$$\chi_2(x, 0) = G\int_x^\infty \chi_1(t, 0)\,dt.$$

It is, moreover, not difficult to prove that

$$\lim_{t\to 0} \chi_2(x, \tau) = \chi_2(x, 0).$$

This point, however, does require proof, for

$$\lim_{\tau\to 0} \chi_2(x, \tau) = \lim_{\tau\to 0}\int_x^\infty \chi_1(t, \tau)\,dt$$

and

$$\chi_2(x, 0) = \lim_{\tau\to 0}\int_x^\infty e^{-\tau t}\chi_1(t, 0)\,dt$$

are not defined in the same manner. But it is easy to show that the two limits are the same. For

$$\lim_{\tau\to 0} \chi_2(x, \tau) = \lim_{\tau\to 0}\int_x^\infty \chi_1(t, \tau)\,dt$$
$$= \lim_{\tau\to 0}\left[\int_x^\infty\left\{\sum_{s=0}^{r} \frac{\mu(\mu-1)\dots(\mu-s+1)}{a^{s+1}} e^{-at}t^{\mu-s}\right.\right.$$
$$\left.\left. + \frac{\mu(\mu-1)\dots(\mu-r)}{a^{r+1}}\int_t^\infty e^{-au}u^{\mu-r-1}\,du\right\}\right]$$

and

$$\chi_2(x, 0) = \lim_{\tau\to 0}\int_x^\infty e^{-\tau t}\chi_1(t, 0)\,dt$$
$$= \lim_{\tau\to 0}\left[\int_x^\infty\left\{\sum_{s=0}^{r} \frac{\mu(\mu-1)\dots(\mu-s+1)}{(mi)^{s+1}} e^{-at}t^{\mu-s}\right.\right.$$
$$\left.\left. + \frac{\mu(\mu-1)\dots(\mu-r)}{(mi)^{r+1}} e^{-\tau t}\int_t^\infty e^{-miu}u^{\mu-r-1}\,du\right\}\right].$$

If we suppose $r > \mu$, the integral in the last line is absolutely convergent in the ordinary sense. And all that we have to show is that

$$\lim_{\tau \to 0} \int_x^\infty dt \int_t^\infty e^{-(\tau+mi)u} u^{\mu-r-1}\, du = \lim_{\tau \to 0} \int_x^\infty e^{-rt}\, dt \int_t^\infty e^{-miu} u^{\mu-r-1} du\,;$$

and since r may be as large as we like, this is very easily proved. For if $r > \mu + 1$ it follows, by comparison with the integral $\int_x^\infty dt \int_t^\infty u^{\mu-r-1} du$, that each of the integrals in question is uniformly convergent in an interval including $\tau = 0$, so that they have the common limit

$$\int_x^\infty dt \int_t^\infty e^{-miu} u^{\mu-r-1} du.$$

Thus $\chi_2(x, \tau) \to \chi_2(x, 0)$ as $\tau \to 0$; and we can now prove, precisely on the lines of the deduction of (1′) from 1, that $\chi_2(x, 0)$ possesses an asymptotic expansion precisely similar to that of $\chi_2(x, \tau)$, τ being replaced by 0, i.e. mi written for $\tau + mi$. It is clear that we can proceed indefinitely in this way, and so establish the existence of a series of functions

$$\chi_0(x, 0), \quad \chi_1(x, 0) = G \int_x^\infty \chi_0(t, 0)\, dt, \ldots, \quad \chi_k(x, 0) = G \int_x^\infty \chi_{k-1}(t, 0)\, dt, \ldots,$$

such that
$$\lim_{\tau \to 0} \chi_k(x, \tau) = \chi_k(x, 0),$$

and $\chi_k(x, 0)$ possesses an asymptotic expansion deducible from that of $\chi_k(x, \tau)$ by merely replacing τ in it by zero. In particular $\chi_k(x, 0)$ satisfies the same inequality

$$|\chi_k(x, 0)| < K x^\mu$$

that we found to be satisfied by $\chi_k(x, \tau)$.

It may be observed that
$$\chi_k(0, \tau), \quad \chi_k(0, 0)$$
can be found in finite terms. For if we integrate by parts and observe that

$$\lim_{x \to \infty} x^\alpha \chi_\nu(x, \tau) = 0 \qquad (\tau > 0)$$

for all values of α and ν, we see that

$$\begin{aligned}
\chi_k(0, \tau) &= \int_0^\infty \chi_{k-1}(x, \tau)\, dx = \int_0^\infty x \chi_{k-2}(x, \tau)\, dx \\
&= \int_0^\infty \frac{x^2}{2!} \chi_{k-3}(x, \tau)\, dx = \ldots \\
&= \int_0^\infty \frac{x^{k-1}}{(k-1)!} \chi_0(x, \tau)\, dx = \frac{1}{(k-1)!} \int_0^\infty e^{-(\tau+mi)x} x^{\mu+k-1} dx \\
&= \frac{\Gamma(\mu+k)}{\Gamma(k)} (\tau + mi)^{-\mu-k}.
\end{aligned}$$

Hence also
$$\chi_k(0, 0) = \frac{\Gamma(\mu+k)}{\Gamma(k)} m^{-\mu-k} e^{-\frac{1}{2}(\mu+k)\pi i}.$$

§ 21. We are now in a position to establish the result enunciated in § 19. For

$$\begin{aligned}\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx &= \int_0^\infty \phi(\alpha x)(1-e^{-\tau x})\,e^{-mix}x^\mu\,dx \\ &= \int_0^\infty \phi(\alpha x)\{\chi_0(x, 0) - \chi_0(x, \tau)\}\,dx \\ &= \phi(0)\{\chi_1(0, 0) - \chi_1(0, \tau)\} + \int_0^\infty \alpha\phi'(\alpha x)\{\chi_1(x, 0) - \chi_1(x, \tau)\}\,dx \\ &= \sum_{r=0}^{n} \alpha^r \phi^{(r)}(0)\{\chi_{r+1}(0, 0) - \chi_{r+1}(0, \tau)\} \\ &\qquad + \int_0^\infty \alpha^{n+1}\phi^{(n+1)}(\alpha x)\{\chi_{n+1}(x, 0) - \chi_{n+1}(x, \tau)\}\,dx.\end{aligned}$$

But the last integral is in absolute value less than

$$K\int_0^\infty \alpha^{n+1}|\phi^{(n+1)}(\alpha x)|\,x^\mu dx = K\alpha^{n-\mu}\int_0^\infty |\phi^{(n+1)}(u)|\,u^\mu du,$$

so that

$$\left|\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx\right| < \sum_{r=0}^{n} \alpha^r|\phi^{(r)}(0)|\,|\chi_{r+1}(0, 0) - \chi_{r+1}(0, \tau)| + K\alpha^{n-\mu};$$

and so the condition of Theorem IV is clearly satisfied.

Hence we obtain

THEOREM V. *If $\phi(u)$ is a function of u which tends to zero, as $u \to \infty$, more rapidly than any power of u, and has continuous derivatives $\phi'(u), \phi''(u), \dots \phi^{(n+1)}(u)$, and*

$$\int_0^\infty \phi^{n(+1)}(u)\,u^\mu du$$

is absolutely convergent, then

$$\lim_{\alpha\to 0}\int_0^\infty e^{-mix}x^\mu\phi(\alpha x)\,dx = \phi(0)\,G\int_0^\infty e^{-mix}x^\mu dx = \phi(0)\,\Gamma(\mu+1)\,m^{-\mu-1}e^{-\frac{1}{2}(\mu+1)\pi i},$$

provided $m > 0$, $\mu > -1$.

Examples of the preceding theorem are given by supposing $\phi(u) = e^{-u}$ (in which case the result is obvious), $\phi(u) = e^{-u^2}$, $\phi(u) = \operatorname{sech} u$, etc.

§ 22. The case in which $\mu = 0$ is of especial interest. In this case we have

$$\begin{aligned}\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx &= \int_0^\infty \phi(\alpha x)\{e^{-mix} - e^{-(\tau+mi)x}\}\,dx \\ &= \left(\frac{1}{mi} - \frac{1}{\tau+mi}\right)\phi(0) + \int_0^\infty \alpha\phi'(\alpha x)\left\{\frac{e^{-mix}}{mi} - \frac{e^{-(\tau+mi)x}}{\tau+mi}\right\}dx \\ &= \left(\frac{1}{mi} - \frac{1}{\tau+mi}\right)\phi(0) + \left\{\frac{1}{(mi)^2} - \frac{1}{(\tau+mi)^2}\right\}\alpha\phi'(0) \\ &\qquad + \int_0^\infty \alpha^2\phi''(\alpha x)\left\{\frac{e^{-mix}}{(mi)^2} - \frac{e^{-(\tau+mi)x}}{(\tau+mi)^2}\right\}dx,\end{aligned}$$

all that we have assumed so far being that $\phi'(u)$ and $\phi''(u)$ are continuous and tend to zero as $u \to \infty$, and that the original integral is convergent. If in addition

$$\int_0^\infty \phi''(u)\, du$$

is absolutely convergent, it follows as in the general case that the conditions of Theorem IV are satisfied, and that

$$\lim_{\alpha \to 0} \int_0^\infty \cos mx\, \phi(\alpha x)\, dx = 0 \quad \text{.................................(7)},$$

$$\lim_{\alpha \to 0} \int_0^\infty \sin mx\, \phi(\alpha x)\, dx = \frac{1}{m}\phi(0) \quad \text{........................(8)}.$$

A particular case in which $\int_0^\infty \phi''(u)\, du$ is certainly absolutely convergent is that in which $\phi''(u)$ changes sign only a finite number of times.

As examples we have

$$\lim_{\alpha \to 0} \int_0^\infty \cos mx\, e^{-(\alpha x)^2} dx = \lim_{\alpha \to 0} \frac{\sqrt{\pi}}{2\alpha} e^{-(m/2\alpha)^2} = 0,$$

$$\lim_{\alpha \to 0} \int_0^\infty \sin mx\, e^{-(\alpha x)^2} dx = \lim_{\alpha \to 0} \frac{1}{\alpha} e^{-(m/2\alpha)^2} \int_0^{m/2\alpha} e^{t^2} dt = \frac{1}{m},$$

$$\lim_{\alpha \to 0} \int_0^\infty \frac{\cos mx}{1 + \alpha^2 x^2}\, dx = \lim_{\alpha \to 0} \frac{\pi}{2\alpha} e^{-m/\alpha} = 0,$$

$$\lim_{\alpha \to 0} \int_0^\infty \frac{\sin mx}{1 + \alpha x}\, dx = \lim_{\alpha \to 0} \left\{ \frac{1}{\alpha} \cos\left(\frac{m}{\alpha}\right) \int_{m/\alpha}^\infty \frac{\sin u}{u}\, du - \frac{1}{\alpha} \sin\left(\frac{m}{\alpha}\right) \int_{m/\alpha}^\infty \frac{\cos u}{u}\, du \right\}$$

$$= \lim_{\alpha \to 0} \left\{ \frac{1}{\alpha} \cos^2\left(\frac{m}{\alpha}\right) \cdot \frac{\alpha}{m} + \frac{1}{\alpha} \sin^2\left(\frac{m}{\alpha}\right) \cdot \frac{\alpha}{m} \right\} = \frac{1}{m}.$$

It is also instructive to notice that the result of the theorem is true when

$$\phi(u) = \frac{\sin u}{u}, \quad J_0(\sqrt{u}), \quad J_0(u),$$

since

$$\int_0^\infty \frac{\sin \alpha x}{\alpha x} \cos mx\, dx = 0, \qquad \int_0^\infty \frac{\sin \alpha x}{\alpha x} \sin mx\, dx = \frac{1}{2\alpha} \log\left(\frac{m+\alpha}{m-\alpha}\right),$$

$$\int_0^\infty J_0(\sqrt{\alpha x}) \cos mx = \frac{1}{m} \sin\left(\frac{\alpha}{4m}\right), \qquad \int_0^\infty J_0(\sqrt{\alpha x}) \sin mx\, dx = \frac{1}{m} \cos\left(\frac{\alpha}{4m}\right),$$

$$\int_0^\infty J_0(\alpha x) \cos mx\, dx = 0, \qquad \int_0^\infty J_0(\alpha x) \sin mx\, dx = \frac{1}{\sqrt{(m^2 - \alpha^2)}},$$

which tend to the prescribed limits as $\alpha \to 0$. But in these cases the conditions which we have laid down are not satisfied, the integral $\int_0^\infty \phi''(u)\, du$ not being absolutely convergent.

In the case of the integrals

$$\int_0^\infty \cos(\alpha x)^2 \begin{matrix} \cos mx \\ \sin mx \end{matrix}\, dx = \frac{1}{2\alpha} \sqrt{(\tfrac{1}{2}\pi)} \left\{ \cos\left(\frac{m}{2\alpha}\right)^2 \pm \sin\left(\frac{m}{2\alpha}\right)^2 \right\}$$

the conditions are not satisfied and the result does not hold.

These examples naturally suggest that the conditions of this section may be generalised. Indeed a variety of generalisations of Theorems IV and V are naturally suggested: but I shall be content with investigating the simplest and most obvious cases.

Uniform summability and continuity.

§ 23. We have not, so far, used the idea of *uniform* summability of

$$G\int_0^\infty f(x, \alpha)\, dx \quad \text{.......................................}(9).$$

We shall naturally say that the integral is uniformly summable if

$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx \quad \text{.......................................}(10)$$

tends to a limit as $\tau \to 0$, uniformly for all values of α in question.

If, as in Theorem IV, $f(x, \alpha)$ is a continuous function of both variables, and

$$|f(x, \alpha)| < H_x{}^K,$$

the integral (10) is for any positive τ, uniformly convergent and continuous: and so (9) is also continuous.

This test is, however, less general than that of Theorem IV, at any rate in the only case to which that theorem applies, viz. that in which

$$\int_0^\infty f(x, \alpha)\, dx$$

is convergent for $\alpha > 0$. For if

$$\lim_{\tau \to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \int_0^\infty f(x, \alpha)\, dx$$

uniformly for all positive values of α, we can, given τ, find τ_0 so that

$$\left| \int_0^\infty f(x, \alpha)\,(1 - e^{-\tau x})\, dx \right| < \epsilon,$$

for all positive values of τ *less than* τ_0 *and all positive values of* α, and this is more than the test of Theorem IV demands. At the same time this more stringent test is satisfied in the cases considered in the preceding paragraphs, and such an integral as

$$G\int_0^\infty \phi(\alpha x)\, e^{-mix} x^\mu\, dx$$

is uniformly summable throughout the interval $0 \leqq \tau \leqq \tau_0$.

§ 24. Before passing on to other questions I may point out the simplest example of a *discontinuous* generalised integral, viz.

$$\begin{aligned} G\int_0^\infty \alpha \sin \alpha x\, dx &= 1 \qquad (\alpha \neq 0) \\ &= 0. \qquad (\alpha = 0) \end{aligned}$$

It is easy to see that in this case the condition for uniform summability is not satisfied, since

$$\int_0^\infty \alpha e^{-\tau x} \sin \alpha x\, dx = \frac{\alpha^2}{\alpha^2 + \tau^2}$$

has the limit 1 if $\alpha \neq 0$ and the limit 0 if $\alpha = 0$. Also to make

$$1 - \frac{\alpha^2}{\alpha^2 + \tau^2} = \frac{\tau^2}{\alpha^2 + \tau^2} < \epsilon$$

we must take $\tau < \alpha\sqrt{\{\epsilon/(1-\epsilon)\}}$, which cannot be effected by a choice of τ independent of α. Similarly

$$G\int_0^\infty \alpha^{\mu+1} \cos \alpha x\, x^\mu\, dx = \Gamma(\mu+1) \cos \tfrac{1}{2}\mu\pi, \quad G\int_0^\infty \alpha^{\mu+1} \sin \alpha x\, x^\mu\, dx = \Gamma(\mu+1) \sin \tfrac{1}{2}\mu\pi$$

are discontinuous for $\alpha = 0$.

The integral
$$G\int_0^\infty \alpha \cos \alpha x\, dx = 0$$

is continuous, but not uniformly convergent. For

$$\int_0^\infty \alpha e^{-\tau x} \cos \alpha x\, dx = \frac{\alpha\tau}{\alpha^2 + \tau^2},$$

which has the limit zero for all values of α, but does not approach its limit uniformly.

C. *Differentiation with respect to a parameter.*

§ 25. Let us next consider the equation

$$\frac{d}{d\alpha} G\int_0^\infty f(x, \alpha)\, dx = G\int_0^\infty \frac{\partial f}{\partial \alpha}\, dx.$$

This rests upon the equations

$$\begin{aligned}
\frac{d}{d\alpha} G\int_0^\infty f(x, \alpha)\, dx &= \frac{d}{d\alpha} \lim_{\tau\to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx \\
&= \lim_{\tau\to 0} \frac{d}{d\alpha} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx \\
&= \lim_{\tau\to 0} \int_0^\infty e^{-\tau x} \frac{\partial f}{\partial \alpha}\, dx \\
&= G\int_0^\infty \frac{\partial f}{\partial \alpha}\, dx.
\end{aligned}$$

It is however, in dealing with the question of differentiation, more convenient to adopt the alternative form of the definition of the generalised integral, viz.*

$$G\int_0^\infty f(x)\, dx = \int_0^\infty dt \int_0^\infty x e^{-tx} f(x)\, dx,$$

which is equivalent to the definition hitherto followed in all cases of any practical interest. For if, as we shall throughout suppose,

$$\lim_{x\to\infty} e^{-\tau x} f(x) = 0$$

$f(x) = x^{q-p+1}$

for any positive value of τ, it is easy to see that

$q > p - 1$

$$\begin{aligned}
\int_\tau^T dt \int_0^\infty x e^{-tx} f(x)\, dx &= \int_0^\infty x f(x)\, dx \int_\tau^T e^{-tx}\, dx \\
&= \int_0^\infty (e^{-\tau x} - e^{-Tx}) f(x)\, dx,
\end{aligned}$$

and
$$\int_0^\infty dt \int_0^\infty x e^{-tx} f(x)\, dx = \lim_{\tau\to 0} \int_0^\infty e^{-\tau x} f(x)\, dx,$$

if, and only if, the latter limit exists.

* *Quarterly Journal, loc. cit.* p. 50.

The transformation which we have to justify is then

$$\frac{d}{d\alpha} G \int_0^\infty f(x, \alpha)\, dx = \frac{d}{d\alpha} \int_0^\infty dt \int_0^\infty xe^{-tx} f(x, \alpha)\, dx$$
$$= \int_0^\infty dt \frac{\partial}{\partial\alpha} \int_0^\infty xe^{-tx} f(x, \alpha)\, dx$$
$$= \int_0^\infty dt \int_0^\infty xe^{-tx} \frac{\partial f}{\partial\alpha}\, dx$$
$$= G \int_0^\infty \frac{\partial f}{\partial\alpha}\, dx,$$

which rests upon a double application of Leibniz's theorem.

Let us suppose now that $\dfrac{\partial f}{\partial\alpha}$ is a continuous function of x and α and that

$$e^{-\tau x} \frac{\partial f}{\partial\alpha},$$

where τ has any positive value, tends to zero, as $x \to \infty$, uniformly throughout an interval

$$(\alpha_0 - H,\ \alpha_0 + H).$$

Then
$$\int_0^\infty xe^{-tx} \frac{\partial f}{\partial\alpha}\, dx$$

is uniformly convergent throughout the region

$$0 < \tau_0 \leqq \tau, \quad \alpha_0 - H \leqq \alpha \leqq \alpha_0 + H.$$

For
$$\left|\frac{\partial f}{\partial\alpha}\right| < Ke^{\frac{1}{2}\tau_0 x},$$

and so
$$\left|\int_X^{X'} xe^{-tx} \frac{\partial f}{\partial\alpha}\, dx\right| < \int_X^{X'} xe^{-\frac{1}{2}\tau_0 x}\, dx.$$

From this it follows (1) that, for any positive value of t,

$$\frac{\partial}{\partial\alpha} \int_0^\infty xe^{-tx} f(x, \alpha)\, dx = \int_0^\infty xe^{-tx} \frac{\partial f}{\partial\alpha}\, dx,$$

(2) that each side of this equation is a continuous function of t and α throughout the region of values just defined. Now let

$$F(t, \alpha) = \int_0^\infty xe^{-tx} f(x, \alpha)\, dx.$$

Then $\dfrac{\partial F}{\partial\alpha}$ is a continuous function of both variables throughout the region; and so a sufficient further condition for the truth of the equation

$$\frac{d}{d\alpha} \int_0^\infty F(t, \alpha)\, dt = \int_0^\infty \frac{\partial F}{\partial\alpha}\, dt,$$

is that the latter integral should be uniformly convergent, i.e. that it should be possible to make

$$\left|\int_\tau^{\tau_1} \frac{\partial F}{\partial\alpha}\, dt\right| < \epsilon, \quad \left|\int_{T_1}^{T} \frac{\partial F}{\partial\alpha}\, dt\right| < \epsilon,$$

for $0 < \tau < \tau_1$ and $T > T_1$ respectively, by a choice of τ_1 and T_1 independent of α.

Now if t_1 and t_2 are any positive values of t

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \alpha}\, dt = \int_{t_1}^{t_2} dt \int_0^\infty x e^{-tx} \frac{\partial f}{\partial \alpha}\, dx$$

$$= \int_0^\infty \left(e^{-t_1 x} - e^{-t_2 x}\right) \frac{\partial f}{\partial \alpha}\, dx,$$

the inversion of integrations being easily justified on the hypotheses that were made above concerning $\frac{\partial f}{\partial \alpha}$. Hence our conditions take the form

$$\left|\int_0^\infty \left(e^{-\tau x} - e^{-\tau_0 x}\right) \frac{\partial f}{\partial \alpha}\, dx\right| < \epsilon, \quad \left|\int_0^\infty \left(e^{-T_0 x} - e^{-Tx}\right) \frac{\partial f}{\partial \alpha}\, dx\right| < \epsilon.$$

The second condition can obviously be satisfied: the first can be satisfied if

$$\int_0^\infty e^{-tx} \frac{\partial f}{\partial \alpha}\, dx,$$

tends to a limit, as $t \to 0$, uniformly for $\alpha_0 - H \leqq \alpha \leqq \alpha_0 + H$. Hence we obtain

THEOREM VI. *If f and $\frac{\partial f}{\partial \alpha}$ are continuous functions of x and α, for $\alpha_0 - H \leqq \alpha \leqq \alpha_0 + H$ and all positive values of x; if further $e^{-\tau x} \frac{\partial f}{\partial x}$ tends uniformly to zero as $x \to \infty$, for any positive value of τ; if finally $G \int_0^\infty \frac{\partial f}{\partial \alpha}\, dx$ is uniformly summable, then*

$$\frac{d}{d\alpha}\, G \int_0^\infty f(x, \alpha)\, dx = G \int_0^\infty \frac{\partial f}{\partial \alpha}\, dx,$$

for $\alpha = \alpha_0$.

§ 26. *Examples of differentiation.* (i) The integrals

$$G \int_0^\infty x^{a-1} e^{-mix}\, dx, \quad G \int_0^\infty x^\alpha J_\alpha(x)\, e^{-mix}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

are summable if $a > 0$, $\alpha > -\frac{1}{2}$. Moreover, if n is any positive integer, the integrals

$$G \int_0^\infty x^{a+n-1} e^{-mix}\, dx, \quad G \int_0^\infty x^{\alpha+n} J_\alpha(x)\, e^{-mix}\, dx$$

are uniformly summable throughout any interval of values of m which does not include $m = 0$, as appears directly from the analysis by which they are evaluated. Hence the integrals (1) can be differentiated any number of times with respect to m, as may be immediately verified.

(ii) If $$I(a) = \int_0^\infty \frac{\cos \beta x - \cos a x}{x} f(\sin^2 x)\, dx,$$

where α and β are positive, and f is continuous, we find that

$$\frac{dI}{da} = G \int_0^\infty \sin a x\, f(\sin^2 x)\, dx$$

$$= \frac{1}{\sin \frac{1}{2} a\pi} \int_0^\pi \cos a x\, f(\sin^2 x)\, dx,$$

the uniform summability of the derived integral following from the analysis by which its value is found*.

In particular we find

$$\frac{d}{d\alpha}\int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\,dx = G\int_0^\infty \sin\alpha x\,dx = \frac{1}{\alpha},$$

$$\int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\,dx = \log\left(\frac{\alpha}{\beta}\right).$$

The result may be extended to cover the case in which f becomes infinite for certain values of x in such a way that

$$\int_0^\pi f(\sin^2 x)\,dx$$

is convergent.

Similarly, if
$$I(\alpha) = \int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\log x\,dx,$$

we find
$$\frac{dI}{d\alpha} = G\int_0^\infty \sin\alpha x\log x\,dx = -\frac{1}{\alpha}(\log\alpha + \gamma),$$

where γ is Euler's constant, and so

$$I = \gamma\log\left(\frac{\alpha}{\beta}\right) + \tfrac{1}{2}\{(\log\alpha)^2 - (\log\beta)^2\}.$$

(iii) If
$$I(\alpha) = \int_0^\infty \frac{\sin\alpha x}{x} f(\sin^2 x)\,dx,$$

$$\frac{dI}{d\alpha} = G\int_0^\infty \cos\alpha x\, f(\sin^2 x)\,dx = 0,$$

unless α is an even integer $2n$†. Hence $I(\alpha)$ is constant for $2n < \alpha < 2(n+1)$, and so

$$I(\alpha) = \int_0^\infty \frac{\sin(2n+1)x}{x} f(\sin^2 x)\,dx,$$

which is easily found to be equal to

$$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)x}{\sin x} f(\sin^2 x)\,dx.$$

In particular, if $0 < \alpha < 2$,

$$\int_0^\infty \frac{\sin\alpha x}{x} f(\sin^2 x)\,dx = \tfrac{1}{2}\pi\int_0^\pi f(\sin^2 x)\,dx.$$

(iv) Suppose (as in § 20) that

$$\chi_0(x,\tau) = e^{-(\tau+mi)x}x^\mu,$$

and consider the integral

$$I(\tau, m, \beta) = \int_0^\infty \chi_0(x,\tau)\,\phi(\beta x)\,dx \quad\text{..............................(2)},$$

where β is positive, and $\phi(u)$ is a function of u which has continuous derivatives of all orders, while

$$e^{-\tau u}\phi(u) \to 0,$$

* *Quarterly Journal, loc. cit.* pp. 55—58.

† *Quarterly Journal, loc. cit.*

as $u \to \infty$, for any positive value of τ. Then, integrating by parts, as in § 20, we find

$$I(\tau, m, \beta) = \sum_{s=0}^{\nu-1} \beta^s \phi^{(s)}(0) \chi_{s+1}(0, \tau) + \int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, \tau)\, dx \quad \ldots\ldots\ldots(3).$$

Now suppose that, for some value of ν, the integral

$$\int_0^\infty \phi^{(\nu)}(u)\, u^\mu\, du$$

is absolutely convergent. Then the integral

$$\int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, \tau)\, dx$$

is absolutely and uniformly convergent throughout any region of values of τ, m, and β defined by inequalities such as

$$0 \leqq \tau \leqq \tau_0, \quad 0 < m_0 < m, \quad 0 < \beta_0 < \beta,$$

and tends uniformly to the limit

$$\int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, 0)\, dx$$

as $\tau \to 0$, for all such values of m and β. Also

$$\chi_{s+1}(0, \tau) \to \chi_{s+1}(0, 0),$$

uniformly for all such values of m. Hence, under the conditions stated, the integral

$$G \int_0^\infty \chi_0(x, 0)\, \phi(\beta x)\, dx$$

is uniformly summable.

Thus, e.g., the integrals

$$G \int_0^\infty \frac{e^{-mix} x^\mu\, dx}{1 + \beta x}, \quad \int_0^\infty \frac{e^{-mix} x^\mu\, dx}{1 + \beta^2 x^2},$$

where $\mu > -1$, are uniformly summable for $m_0 < m$, $\beta_0 < \beta$, and the reader will easily write down any number of such examples. This result enables us to justify the processes used in the following examples.

(v) If $$I(m) = G \int_0^\infty \frac{e^{-mix} x^{\lambda-1}\, dx}{1+x},$$

where $\lambda > 0$, then $$\frac{dI}{dm} = -i\, G \int_0^\infty \frac{e^{-mix} x^\lambda\, dx}{1+x},$$

and $$\frac{dI}{dm} - iI = -i\, G \int_0^\infty e^{-mix} x^{\lambda-1}\, dx$$

$$= -i\, \Gamma(\lambda)\, m^{-\lambda} e^{-\frac{1}{2}\lambda\pi i},$$

an equation whose solution is

$$I = i\, \Gamma(\lambda)\, e^{(m - \frac{1}{2}\lambda\pi) i} \left(\int_m^\infty e^{-it} t^{-\lambda}\, dt + C \right).$$

It is easy to prove that $I \to 0$ as $m \to \infty$. Hence

$$G \int_0^\infty \frac{e^{-mix}\, x^{\lambda-1}\, dx}{1+x} = i\,\Gamma(\lambda)\, e^{(m-\frac{1}{2}\lambda\pi)i} \int_m^\infty e^{-it}\, t^{-\lambda}\, dt,$$

a result which is easily verified in special cases.

(vi) If
$$I(\alpha) = \int_0^\infty \frac{\cos \alpha x}{1+x^2} f(\sin^2 x)\, dx,$$

we find that
$$\frac{d^2 I}{d\alpha^2} - I = -G \int_0^\infty \cos \alpha x f(\sin^2 x)\, dx = 0,$$

unless α is an even integer.

Thus if
$$2n < \alpha < 2(n+1),$$

we have
$$I(\alpha) = A_n e^\alpha + B_n e^{-\alpha},$$

where A_n, B_n are given by the equations

$$I_{2n} = A_n e^{2n} + B_n e^{-2n},$$

$$I_{2n+2} = A_n e^{2n+2} + B_n e^{-2n-2}.$$

If $f(\sin^2 x) \equiv 1$, $\dfrac{d^2 I}{d\alpha^2} - I = 0$ for all values of α, so that

$$I = Ae^\alpha + Be^{-\alpha},$$

from which we deduce the well-known formula

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2}\, dx = \tfrac{1}{2}\pi e^{-\alpha}.$$

The same method may be applied to obtain the formula

$$J(\alpha) = \int_0^\infty \frac{\sin \alpha x}{1+x^2}\, dx = \tfrac{1}{2}\{e^{-\alpha}\, li\,(e^\alpha) - e^\alpha\, li\,(e^{-\alpha})\},$$

by means of the differential equation

$$\frac{d^2 I}{d\alpha^2} - I = -\frac{1}{\alpha}.$$

And it is evident that this method is capable of very general application to integrals of the form

$$\int_0^\infty \begin{matrix}\cos \\ \sin\end{matrix}\, \alpha x\, R(x)\, dx.$$

(vii)* If
$$I(\alpha) = \int_0^\infty \tanh \tfrac{1}{2}\pi x \sin \alpha x \frac{dx}{1+x^2},$$

we find
$$\frac{d^2 I}{d\alpha^2} - I = -G \int_0^\infty \tanh \tfrac{1}{2}\pi x \sin \alpha x dx$$

$$= -G \int_0^\infty \sin \alpha x dx + 2 \int_0^\infty \frac{\sin \alpha x}{e^{\pi x}+1}\, dx$$

$$= -\operatorname{cosech} \alpha,$$

and hence can deduce that

$$I(\alpha) = \tfrac{1}{2}\{e^{-\alpha} \log(e^{2\alpha} - 1) - e^\alpha \log(1 - e^{-2\alpha})\}.$$

Many other integrals of a similar type may be calculated in the same way.

* For the next two examples cf. Bromwich, *Infinite Series*, pp. 496—7.

(viii) Let $$P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n:$$

then the integral $$G\int_0^\infty x^{\mu-1} e^{iP(x)}\, dx \qquad (\mu > 0)$$

is uniformly convergent throughout the region defined by

$$0 < a_0{}^0 < a_0 < a_0{}^1, \quad a_r{}^0 < a_r < a_r{}^1.$$

This will be proved if we can show that, if X is any positive number, the integral

$$\int_X^\infty x^{\mu-1} e^{-\tau x + iP(x)}\, dx$$

tends uniformly to a limit as $\tau \to 0$.

Let $y = P(x)$. Then, if X is large enough, and $z = y^{1/n}$, we have, for $y \geqq Y = P(X)$, expansions of the forms

$$x = Az\left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots\right),$$

$$x^{\mu-1}\frac{dx}{dy} = Bz^{\mu-n}\left(1 + \frac{B_1}{z} + \frac{B_2}{z^2} + \dots\right),$$

$$e^{-\tau x} = e^{-\tau Az\left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots\right)}$$

$$= e^{-\tau Az}\left(1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \dots\right),$$

$$x^{\mu-1} e^{-\tau x}\frac{dx}{dy} = Dz^{\mu-n} e^{-\tau Az}\left(1 + \frac{D_1}{z} + \frac{D_2}{z^2} + \dots\right),$$

where $A > 0$, these series being absolutely and uniformly convergent for $y \geqq Y$ and for all values of the coefficients in question.

Now, if m is large enough, the integral

$$\int_Y^\infty e^{-\tau Az + iy} z^{\mu-n-m}\left(D_m + \frac{D_{m+1}}{z} + \dots\right) dy$$

is uniformly convergent and tends uniformly to a limit as $\tau \to 0$. Hence the problem is reduced to that of showing that

$$\int_Y^\infty y^\lambda e^{-\tau y^{1/n} + iy}\, dy$$

tends uniformly to a limit as $\tau \to 0$. But if

$$\psi_0(y) = y^\lambda e^{iy},$$

$$\psi_1(y) = G\int_y^\infty \psi_0(t)\, dt,$$

$$\psi_2(y) = G\int_y^\infty \psi_1(t)\, dt,$$

$$\dots\dots\dots\dots$$

we have $$\int_Y^\infty \psi_0(y) e^{-\tau y^{1/n}}\, dy = \sum_{s=0}^{\nu-1} \psi_{s+1}(Y)\left(\frac{d}{dY}\right)^s (e^{-\tau Y^{1/n}})$$
$$+ \int_Y^\infty \psi_\nu(y)\left(\frac{d}{dy}\right)^\nu (e^{-\tau y^{1/n}})\, dy.$$

Now (§ 20)
$$|\psi_\nu(y)| < Ky^\lambda$$
and
$$\left|\left(\frac{d}{dy}\right)^\nu e^{-\tau y^{1/n}}\right| < Ky^{-\nu\sigma},$$
where $\sigma = 1 - \frac{1}{n}$. By supposing ν large enough we can make $\lambda - \nu\left(1 - \frac{1}{n}\right)$ negative and as large as we like, and so ensure that
$$\int_Y^\infty \psi_\nu(y)\left(\frac{d}{dy}\right)^\nu (e^{-\tau y^{1/n}})\,dy$$
is uniformly convergent, for $0 \leqq \tau \leqq \tau_0$, and tends uniformly to a limit as $\tau \to 0$. And so
$$\int_Y^\infty y^\lambda e^{-\tau y^{1/n} + iy}\,dy$$
tends uniformly to the limit
$$\psi_1(Y) = G\int_Y^\infty y^\lambda e^{iy}\,dy.$$

The result enunciated originally is thus established.

Now let
$$I(\alpha) = \int_0^\infty e^{iP(x)}\,dx,$$
where
$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + \alpha x.$$
The integral is convergent if $x > 1$. Also
$$\frac{d^kI}{d\alpha^k} = i^k G\int_0^\infty x^k e^{iP(x)}\,dx,$$
these integrals being, as is easily proved, convergent in the ordinary sense if $k \leqq n-2$. Hence
$$na_0 i^{-(n-1)}\frac{d^{n-1}I}{d\alpha^{n-1}} + (n-1)a_1 i^{-(n-2)}\frac{d^{n-2}I}{d\alpha^{n-2}} + \dots$$
$$+ 2a_{n-2}i^{-1}\frac{dI}{d\alpha} + \alpha I = G\int_0^\infty P'(x)\,e^{iP(x)}\,dx = i;$$
since
$$\underset{x\to\infty}{L}\, e^{iP(x)} = \lim_{t\to\infty}\int_0^\infty e^{-x+iP(tx)}\,dx = \lim_{t\to\infty}\frac{1}{t}\int_0^\infty e^{-(x/t)+iP(x)}\,dx = \lim_{t\to\infty}\frac{1}{t}\int_0^\xi e^{iP(x)}\,dx,$$
and
$$\left|\int_0^\xi e^{iP(x)}\,dx\right| < K,$$
so that $\underset{x\to\infty}{L}\, e^{iP(x)} = 0$. Suppose in particular that $a_0 = 1$, $a_1 = a_2 = \dots = a_{n-2} = 0$. Then
$$ni^{-(n-1)}\frac{d^{n-1}I}{d\alpha^{n-1}} + \alpha I = i.$$

Thus, if $n = 3$, we see that
$$\int_0^\infty e^{i(x^3+\alpha x)}\,dx$$
satisfies the equation
$$-3\frac{d^2I}{d\alpha^2} + \alpha I = i,$$

or that $$\int_0^\infty \cos(x^3+\alpha x)\,dx, \quad \int_0^\infty \sin(x^3+\alpha x)\,dx,$$

satisfy the equations $$\frac{d^2I}{d\alpha^2}=\tfrac{1}{3}\alpha I, \quad \frac{d^2I}{d\alpha^2}-\tfrac{1}{3}\alpha I=-\tfrac{1}{3},$$

a result originally due to Stokes*.

D. *Integration of a generalised integral with respect to a parameter.*

§ 27. I shall consider finally the question of the integration of a generalised integral with respect to a parameter, as expressed by the equation

$$\int_\beta^\gamma d\alpha\, G\int_0^\infty f(x,\alpha)\,dx = G\int_0^\infty dx\int_\beta^\gamma f(x,\alpha)\,d\alpha \quad\text{.......................(1).}$$

This formula rests upon the transformations expressed by the equations

$$\left.\begin{aligned}\int_\beta^\gamma d\alpha\, G\int_0^\infty f(x,\alpha)\,dx &= \int_\beta^\gamma d\alpha\int_0^\infty dt\int_0^\infty xe^{-tx}f(x,\alpha)\,dx\\ &= \int_0^\infty dt\int_\beta^\gamma d\alpha\int_0^\infty xe^{-tx}f(x,\alpha)\,dx\\ &= \int_0^\infty dt\int_0^\infty xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,d\alpha\\ &= G\int_0^\infty dx\int_\beta^\gamma f(x,\alpha)\,dx\end{aligned}\right\}\quad\text{...............(2),}$$

i.e. on a repeated inversion of integrations.

I shall suppose (a) that $f(x,\alpha)$ is continuous throughout the region $0\leqq x\leqq X$, $\beta\leqq\alpha\leqq\gamma$, for any value of X however large, (b) that, for any positive value of τ,

$$e^{-\tau x}f(x,\alpha)\to 0$$

as $x\to\infty$, uniformly for all values of α in the interval (β,γ), and (c) that

$$G\int_0^\infty f(x,\alpha)\,dx$$

is uniformly summable throughout the same interval.

In the first place, the conditions (a) and (b) are sufficient to ensure that, for any positive value of t,

$$\int_\beta^\gamma d\alpha\int_0^\infty xe^{-tx}f(x,\alpha)\,dx = \int_0^\infty xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,d\alpha \quad\text{..................(3).}$$

For, however large X may be, we have

$$\int_\beta^\gamma d\alpha\int_0^X xe^{-tx}f(x,\alpha)\,dx = \int_0^X xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,dx \quad\text{..................(4).}$$

But we can choose K so that

$$|f(x,\alpha)|<Ke^{\frac{1}{3}tx}$$

* Stokes, *Math. Papers*, vol. II. p. 329 and vol. IV. pp. 77, 283; Stolz, *Grundzüge*, bd. III. p. 30; Bromwich, *Infinite Series*, p. 497. Since this paper has been in type Mr Bromwich has devised a shorter and simpler method of arriving at the results of this section, which will be printed shortly in the *Messenger of Mathematics*.

for all values of x and α in question, and then

$$\left|\int_\beta^\gamma d\alpha \int_X^{X'} xe^{-tx} f(x, \alpha)\, dx\right| < K(\gamma - \beta) \int_X^\infty xe^{-\frac{1}{2}tx}\, dx$$

$$= K(\gamma - \beta) \frac{e^{-\frac{1}{2}tX}}{(\frac{1}{2}t)^2}(1 + \tfrac{1}{2}tX)$$

for all values of X' greater than X. The last expression has the limit zero as $X \to \infty$. From this it follows that

$$\lim_{X\to\infty} \int_\beta^\gamma d\alpha \int_0^X xe^{-tx} f(x, \alpha)\, dx = \int_\beta^\gamma d\alpha \int_0^\infty xe^{-tx} f(x, \alpha)\, dx,$$

and so that (3) follows from (4).

Now let
$$F(t, \alpha) = \int_0^\infty xe^{-tx} f(x, \alpha)\, dx.$$

This integral, as is easily seen, converges uniformly with respect to t and α throughout any domain bounded by inequalities

$$0 < \tau \leqq t \leqq T,\ \beta \leqq \alpha \leqq \gamma,$$

and so is a continuous function of t and α throughout any such domain. Hence

$$\int_\beta^\gamma d\alpha \int_\tau^T F(t, \alpha)\, dt = \int_\tau^T dt \int_\beta^\gamma F(t, \alpha)\, d\alpha.$$

But
$$|f(x, \alpha)| < Ke^x,$$
and so, if $t > 1$,

$$\left|F(t, \alpha)\right| < K \int_0^\infty xe^{-(t-1)x}\, dx = \frac{K}{(t-1)^2}.$$

Thus, if $T' > T > 1$,

$$\left|\int_\beta^\gamma d\alpha \int_T^{T'} F(t, \alpha)\, dt\right| < (\gamma - \beta) K \int_T^{T'} \frac{dt}{(t-1)^2} < \frac{K}{T},$$

which tends to zero as $T \to \infty$. Hence

$$\int_\beta^\gamma d\alpha \int_\tau^\infty F(t, \alpha)\, dt = \lim_{T\to\infty} \int_\beta^\gamma \int_\tau^T = \lim_{T\to\infty} \int_\tau^T \int_\beta^\gamma = \int_\tau^\infty dt \int_\beta^\gamma F(t, \alpha)\, d\alpha.$$

I shall now prove that if the condition (c) is satisfied we may replace τ by 0 in this equation. To see this we observe that if $0 < \tau' < \tau$

$$\int_\beta^\gamma d\alpha \int_{\tau'}^\tau F(t, \alpha)\, dt = \int_{\beta'}^\gamma d\alpha \int_{\tau'}^\tau dt \int_0^\infty xe^{-tx} f(x, \alpha)\, dx$$

$$= \int_\beta^\gamma d\alpha \int_0^\infty (e^{-\tau' x} - e^{-\tau x}) f(x, \alpha)\, dx.$$

But if
$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx$$

converges uniformly to a limit, as $\tau \to 0$, we can, given ϵ, so choose τ_0 that

$$\left| \int_0^\infty (e^{-\tau' x} - e^{-\tau x}) f(x, \alpha)\, dx \right| < \epsilon$$

for $$0 < \tau' < \tau \leqq \tau_0,\ \beta \leqq \alpha \leqq \gamma,$$

and so $$\left| \int_\beta^\gamma d\alpha \int_{\tau'}^{\tau} F(t, \alpha)\, dt \right| < \epsilon$$

for all such values of τ', τ and α.

We can now state

THEOREM VII. *If (a) $f(x, \alpha)$ is continuous for $0 \leqq x \leqq X$, $\beta \leqq \alpha \leqq \gamma$, however large X may be; (b) for any positive value of* τ

$$e^{-\tau x} f(x, \alpha) \to 0$$

uniformly for $\beta \leqq \alpha \leqq \gamma$; and (c) the integral

$$G \int_0^\infty f(x, \alpha)\, dx$$

is uniformly summable for $\beta \leqq \alpha \leqq \gamma$, then

$$\int_\beta^\gamma d\alpha\; G \int_0^\infty f(x, \alpha)\, dx = G \int_0^\infty dx \int_\beta^\gamma f(x, \alpha)\, d\alpha.$$

§ 28. A particularly interesting special case of this theorem is one which leads us to certain extensions of Dirichlet's integral and Fourier's double integral, which are due to Sommerfeld*.

Suppose that $$f(x, \alpha) = f(\alpha)\, e^{-imx\alpha},$$

where $m > 0$. Then $G \int_0^\infty e^{-imx\alpha} dx$ is uniformly summable in the interval $\beta \leqq \alpha \leqq \gamma$ if β and γ have the same sign, say the positive. On this hypothesis we obtain the equations

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha)\, e^{-imx\alpha}\, d\alpha = \frac{1}{im} \int_\beta^\gamma \frac{f(\alpha)}{\alpha}\, d\alpha,$$

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx\alpha\, d\alpha = 0,$$

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \frac{1}{m} \int_\beta^\gamma \frac{f(\alpha)}{\alpha}\, d\alpha.$$

It is of course well known that if $f(\alpha)$ is monotonic as well as continuous (or, more generally, is a *fonction à variation bornée*) the integrals on the left-hand side of these equations are convergent in the ordinary sense. For

$$\int_0^X dx \int_\beta^\gamma f(\alpha) \cos mx\alpha\, d\alpha = \frac{1}{m} \int_\beta^\gamma f(\alpha) \frac{\sin mX\alpha}{\alpha}\, d\alpha,$$

$$\int_0^X dx \int_\beta^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \frac{1}{m} \int_\beta^\gamma f(\alpha) \frac{1 - \cos mX\alpha}{\alpha}\, d\alpha,$$

* *Die willkürlichen Funktionen in der Math. Physik*, Inaug. Diss., Königsberg, 1901. See also Carslaw, *Fourier's Series and Integrals*, p. 186. I owe these references to Mr Bromwich. The idea which is the base of Sommerfeld's work appears to go back to Cauchy; see *e.g.* his *Mémoire sur la Théorie des Ondes*, Note VI. (*Œuvres*, t. i, p. 133).

and the integrals on the right-hand side are known to have the limits

$$0, \quad \frac{1}{m}\int_{\beta}^{\gamma} \frac{f(\alpha)}{\alpha}\, d\alpha$$

as $X \to \infty$.

If $\beta = 0$ the formulae cease to be true, and in fact

$$G\int_0^\infty dx \int_0^\gamma f(\alpha) \cos mx\alpha\, d\alpha = \frac{\pi}{2m} f(0) \quad \text{.........................(5)},$$

whereas the corresponding sine integral is in general divergent.

It is very easy to establish the formula (5) on the assumption that $f(\alpha)$ is continuous, the only case contemplated in the general theorem. As however the result is one of considerable interest in itself, I shall adopt less restrictive hypotheses*. I shall suppose only that

(i) $f(\alpha)$ is integrable in any interval throughout which it is limited,

(ii) $f(+0)$ is determinate,

(iii) $\int_0^\gamma |f(\alpha)|\, d\alpha$ is convergent.

Then it is easy to see that

$$\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha) \cos mx\alpha\, d\alpha = \int_0^\gamma f(\alpha)\, d\alpha \int_0^\infty e^{-\tau x} \cos mx\alpha\, dx$$

$$= \int_0^\gamma \frac{\tau f(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2},$$

for any positive value of τ. For $e^{-\tau x} \cos mx\alpha f(\alpha)$ is an integrable function of the two variables x and α throughout any rectangle $(0, X; 0, \gamma)$, so that the equation certainly holds when ∞ is replaced by any positive number X, however large. And

$$\left| \int_0^\gamma f(\alpha)\, d\alpha \int_X^\infty e^{-\tau x} \cos mx\alpha\, dx \right| < \frac{1}{\tau} e^{-\tau X} \int_0^\gamma |f(\alpha)|\, d\alpha,$$

which tends to zero as $X \to \infty$.

Moreover, on the hypotheses which we have adopted,

$$\lim_{\tau \to 0} \int_0^\gamma \frac{\tau f(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2} = \frac{\pi}{2m} f(+0).$$

For let

$$f(\alpha) - f(+0) = \phi(\alpha)$$

so that $\phi(\alpha) \to 0$ with α. Then

$$\left| \int_0^\gamma \frac{\tau\phi(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2} \right| = \left| \left(\int_0^\delta + \int_\delta^\gamma \right) \frac{\tau\phi(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2} \right|$$

$$< \omega \int_0^\delta \frac{\tau\, d\alpha}{\tau^2 + m^2\alpha^2} + \frac{\tau}{\tau^2 + m^2\delta^2} \int_\delta^\gamma |\phi(\alpha)|\, dx$$

$$< \frac{\omega\pi}{2m} + \frac{\tau}{\tau^2 + m^2\delta^2} \int_0^\gamma |\phi(\alpha)|\, d\alpha,$$

where ω is the upper limit of $|\phi(\alpha)|$ in the interval $(0, \delta)$.

* The succeeding analysis is not essentially different from Sommerfeld's, but rather more general and direct.

Let $\delta = \tau^s$, where $0 < s < \frac{1}{2}$. Then ω and $\tau/(\tau^2 + m^2\delta^2)$ each tend to zero with τ, and so

$$\int_0^\gamma \frac{\tau\phi(\alpha)\, d\alpha}{\tau^2 + m^2 x^2} \to 0.$$

But
$$f(+0)\int_0^\gamma \frac{\tau\, d\alpha}{\tau^2 + m^2\alpha^2} = \frac{1}{m} f(+0)\int_0^{m\gamma/\tau} \frac{du}{1+u^2} \to \frac{\pi}{2m} f(0),$$

which establishes the result desired.

Similarly we can show that

$$\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \int_0^\gamma \frac{m\alpha f(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2}$$

in general tends to $+\infty$ or to $-\infty$ (according to the sign of $f(+0)$) as $\tau \to 0$.

If however $f(\alpha) = \alpha F(\alpha)$, and $F(\alpha) \to F(+0)$ as $\alpha \to 0$,

$$\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha) \sin mx\alpha\, d\alpha = m\int_0^\gamma \frac{\alpha^2 F(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2}$$
$$= \frac{1}{m}\left\{\int_0^\gamma F(\alpha)\, d\alpha - \tau\int_0^\gamma \frac{\tau F(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2}\right\} \to \frac{1}{m}\int_0^\gamma F(\alpha)\, d\alpha.$$

§ 29. The equation (5) expresses a generalisation of Dirichlet's integral much on the lines of Fejér's generalisation of Fourier's theorem, in which the 'conditions of Dirichlet' are removed and mere continuity (or integrability) assumed, and the Fourier's series, while possibly oscillatory, summable by Cesàro's method of mean values.

It is easy to obtain other generalisations on similar lines. For example, if $f(\alpha)$ satisfies conditions similar to those imposed in it in § 28, we have

$$\lim_{\tau \to 0} \int_0^\infty e^{-(\tau x)^2} dx \int_0^\gamma f(\alpha) \cos mx\alpha\, d\alpha = \lim_{\tau \to 0} \int_0^\gamma f(\alpha)\, d\alpha \int_0^\infty e^{-(\tau x)^2} \cos mx\alpha\, dx$$
$$= \frac{1}{2\tau}\sqrt{\pi}\int_0^\gamma f(\alpha)\, e^{-(m\alpha/2\tau)^2} d\alpha$$
$$= \frac{\pi}{2m} f(+0),$$

by a well-known theorem of Weierstrass*.

But a generalisation more precisely on Fejér's lines can be obtained by using the definition†

$$G\int_0^\infty \phi(x)\, dx = \lim_{x \to \infty} \frac{1}{x}\int_0^x dt \int_0^t \phi(u)\, du.$$

We have then to state conditions under which

$$\lim_{x \to \infty} \frac{1}{x}\int_0^x dt \int_0^t du \int_0^\gamma f(\alpha) \cos mu\alpha\, d\alpha = \frac{\pi}{2m} f(+0).$$

* This formula also is given by Sommerfeld (*loc. cit.*).

† *Quarterly Journal*, *loc. cit.*, p. 53. For a proof that, under very general conditions, this definition is included in our previous definition, see the same paper, p. 54, and C. N. Moore, *Trans. Amer. Math. Soc.*, vol. VIII. p. 299.

If $f(\alpha)$ satisfies the conditions employed above, we can invert the integrations, and obtain

$$\int_0^x dt \int_0^t du \int_0^\gamma f(\alpha) \cos mu\alpha \, d\alpha = \int_0^\gamma f(\alpha)\, d\alpha \int_0^x dt \int_0^t \cos mu\alpha \, du$$

$$= \int_0^\gamma f(\alpha) \frac{1 - \cos mx\alpha}{(m\alpha)^2} d\alpha,$$

so that

$$G \int_0^\infty dx \int_0^\gamma f(\alpha) \cos mx\alpha \, d\alpha = \lim_{x \to \infty} \int_0^\gamma \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} mx\alpha}\right)^2 f(\alpha)\, \tfrac{1}{2} x\, d\alpha,$$

if the latter limit exists. Since

$$\lim_{x \to \infty} \int_0^\gamma \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} mx\alpha}\right)^2 \tfrac{1}{2} x\, d\alpha = \lim_{x \to \infty} \frac{1}{m} \int_0^{\frac{1}{2} mx\gamma} \left(\frac{\sin u}{u}\right)^2 du$$

$$= \frac{1}{m} \int_0^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\pi}{2m},$$

it will be seen that what we have to prove is that

$$\frac{1}{x} \int_0^\gamma \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} m\alpha}\right)^2 \phi(\alpha)\, d\alpha \to 0,$$

where $\phi(\alpha) = f(\alpha) - f(+0)$ tends to zero with α.

Let ρ be a positive number less than γ, and let ω be the upper limit of $|\phi(x)|$ in the interval $(0, \rho)$. Then

$$\frac{1}{x} \left| \int_0^\rho \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} m\alpha}\right)^2 \phi(x)\, d\alpha \right| < \frac{\omega}{x} \int_0^\infty \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} m\alpha}\right)^2 d\alpha$$

$$< \frac{2\omega}{m} \int_0^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\omega\pi}{m},$$

and

$$\frac{1}{x} \left| \int_\rho^\gamma \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} m\alpha}\right)^2 \phi(\alpha)\, d\alpha \right| < \frac{4}{m^2 \rho^2 x} \int_0^\gamma |\phi(\alpha)|\, d\alpha.$$

Thus

$$\frac{1}{x} \left| \int_0^\gamma \left(\frac{\sin \frac{1}{2} mx\alpha}{\frac{1}{2} m\alpha}\right)^2 \phi(\alpha)\, d\alpha \right| < \frac{\omega\pi}{m} + \frac{4}{m^2 \rho^2 x} \int_0^\gamma |\phi(\alpha)|\, d\alpha,$$

and if we choose ρ so that

$$\rho \to 0, \quad \rho^2 x \to \infty,$$

as by taking $\rho = x^{-s}$, where $0 < s < \frac{1}{2}$, we see that the limit of the right-hand side is zero. Thus the result is established.

It is of course well known that there are continuous functions $f(\alpha)$ for which the equations

$$\lim_{X \to \infty} \int_0^\gamma f(\alpha) \frac{\sin mX\alpha}{\alpha} d\alpha = \tfrac{1}{2}\pi f(+0), \quad \int_0^\infty dx \int_0^\gamma f(\alpha) \cos(mx\alpha)\, d\alpha = \frac{\pi}{2m} f(+0)$$

do not hold. An example of such a function was given by Du Bois Reymond, and a simpler one by Schwarz*. The functions given by these writers are of a very complicated type and defined by an enumerable sequence of different formulae, in a corresponding

* See Hobson, *Theory of Functions of a real Variable*, pp. 701 *et seq.*, for references and further discussion.

sequence of intervals of values of x approaching the origin. In all such cases any of the generalised forms of Dirichlet's Theorem hold.

§ 30. (ii) Suppose

$$f(x, \alpha) = f(\alpha) \cos mx (\alpha - a).$$

Then

$$G \int_0^\infty \cos mx (\alpha - a)\, dx = 0$$

is uniformly convergent in any interval which does not include $x = \alpha$; and so

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = 0 \quad \text{..........................(6)},$$

if (β, γ) does not include $\alpha = a$.

On the other hand, if $\beta < \alpha < \gamma$,

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = \frac{\pi}{m} f(\alpha) \quad \text{........................(7)},$$

as may be shown by arguments precisely similar to those of § 28. When $f(\alpha)$, besides being continuous, satisfies Dirichlet's conditions, the sign of the generalised integral may be omitted, and the formulae reduce to Fourier's double integral formulae.

These formulae of course hold under wider conditions, and so do (6) and (7). It may be proved, precisely on the lines of § 28, that if $f(\alpha)$ satisfies the conditions there laid down,

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = \begin{cases} \frac{1}{2}\pi f(\beta + 0) & (a = \beta) \\ \frac{1}{2}\pi f(\gamma - 0) & (a = \gamma) \\ \frac{1}{2}\pi \{f(a-0) + f(a+0)\} & (\beta < \alpha < \gamma) \\ 0 \text{ (otherwise)}, & \end{cases}$$

a formula equivalent to Sommerfeld's principal result.

Again, precisely on the lines of § 29, we can show that the above equations remain valid when either of the definitions

$$G \int_0^\infty f(x)\, dx = \lim_{\tau \to 0} \int_0^\infty e^{-(\tau x)^2} f(x)\, dx,$$

$$G \int_0^\infty f(x)\, dx = \lim_{x \to \infty} \frac{1}{x} \int_0^x dt \int_0^t f(u)\, du,$$

is adopted.

§ 31. (iii) As a final illustration I shall consider Hankel's generalisation of Fourier's double integral theorem by means of Bessel functions. Hankel* first gave the formula

$$\int_0^\infty x J^\nu (ax)\, dx \int_\beta^\gamma f(\alpha) J^\nu (\alpha x)\, dx = \begin{matrix} 0 \\ f(a) \end{matrix} \quad \text{.....................(8)},$$

* *Math. Annalen*, bd. VIII. p. 482.

where β, γ, and a are positive, according as a does not or does fall inside the interval (β, γ) A rigorous proof that the value of the integral is

$$\tfrac{1}{2}\{f(a-0)+f(a+0)\} \qquad (\beta < a < \gamma),$$
$$\tfrac{1}{2}f(\beta+0) \qquad (\beta = a), \qquad \tfrac{1}{2}f(\gamma-0) \qquad (a=\gamma),$$
$$0 \qquad (a<\beta \text{ or } \gamma < a),$$

has been given by Nielsen*, it being assumed that

$$R(\nu) > -1,$$

and that $f(\alpha)$ satisfies Dirichlet's conditions.

A generalisation of Hankel's formula on the lines of §§ 29, 30 has been attempted and partly achieved by Sommerfeld in his dissertation already quoted. Sommerfeld shews that the formula holds for any integrable function in the form

$$\lim_{\tau\to 0}\int_0^\infty e^{-(\tau x)^2}\, xJ^\nu(ax)\,dx \int_\beta^\gamma f(\alpha)\,J^\nu(\alpha x)\,d\alpha = \left.\begin{matrix}\tfrac{1}{2}\{f(a-0)+f(a+0)\} \\ \tfrac{1}{2}f(\beta+0),\ \tfrac{1}{2}f(\gamma-0) \\ 0\end{matrix}\right\} \quad \text{.........(9)},$$

i.e. that

$$G\int_0^\infty xJ^\nu(ax)\,dx \int_\beta^\gamma f(\alpha)\,J^\nu(\alpha x)\,d\alpha = \tfrac{1}{2}\{f(a-0)+f(a+0)\}, \text{ etc.} \text{.........(10)},$$

if the definition of the generalised integral by means of the convergence factor $e^{-(\tau x)^2}$ is adopted.

But when the convergence factor $e^{-\tau x}$ is used Sommerfeld only succeeded in establishing the result for *integral* values of ν. I shall now prove that the formula holds for all values of ν whose real part is greater than -1.

§ 32. For this purpose we require the value of the integral

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\,J^\nu(\alpha x)\,dx \quad \text{.............................(11)},$$

where $R(\nu) > -1$ and τ, a, and α are positive. For this purpose we start from the formula†

$$\sum_{s=0}^\infty (\nu+s)\,P_s^\nu(\cos\theta)\,J^{\nu+s}(ax)\,J^{\nu+s}(\alpha x)$$
$$= \frac{(\tfrac{1}{2}a\alpha x)^\nu}{\Gamma(\nu)}(a^2 - 2a\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}\nu}\,J^\nu\{x\sqrt{(a^2-2a\alpha\cos\theta+\alpha^2)}\}.$$

It is easy to prove that we are justified in multiplying this series by $xe^{-\tau x}$ and integrating term by term from 0 to ∞. We thus obtain

$$\sum_{s=0}^\infty (\nu+s)\,P_s^\nu(\cos\theta)\,I_s$$
$$= \frac{(\tfrac{1}{2}a\alpha)^\nu}{\Gamma(\nu)}(a^2-2a\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}\nu}\int_0^\infty x^{\nu+1}e^{-\tau x}J^\nu\{x\sqrt{(a^2-2a\alpha\cos\theta+\alpha^2)}\}\,dx$$
$$= \frac{2\,(a\alpha)^\nu\,\Gamma(\nu+\tfrac{3}{2})}{\Gamma(\nu)\sqrt{\pi}}\,\frac{\tau}{(\tau^2+a^2+\alpha^2-2a\alpha\cos\theta)^{\nu+\frac{3}{2}}},$$

* *Cylinderfunktionen*, pp. 366—370.

† *ibid.* p. 280.

where
$$I_s = \int_0^\infty x e^{-\tau x} J^{\nu+s}(ax) J^{\nu+s}(\alpha x)\, dx.$$

Let us suppose for the present that $R(\nu) > -\frac{1}{2}$. Then it is easy to see that we can multiply this equation by
$$(\sin\theta)^{2\nu},$$
and integrate term by term from $\theta = 0$ to $\theta = \pi$. Since it is known that
$$\int_0^\pi (\sin\theta)^{2\nu} P_s^{\nu}(\cos\theta)\, d\theta = 0 \qquad (s > 0)$$
and
$$\int_0^\pi (\sin\theta)^{2\nu} d\theta = \frac{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}{\Gamma(\nu+1)},$$
we obtain
$$I_0 = \frac{(2\nu+1)(a\alpha)^\nu}{\pi} \int_0^\pi \frac{\tau(\sin\theta)^{2\nu} d\theta}{(\tau^2 + a^2 + \alpha^2 - 2a\alpha\cos\theta)^{\nu+\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Let us, in Sommerfeld's notation, write
$$A^2 = (\tau^2 + a^2 + \alpha^2)^2 - (2a\alpha)^2$$
$$= \{\tau^2 + (a+\alpha)^2\}\{\tau^2 + (a-\alpha)^2\},$$
$$\tau^2 + a^2 + \alpha^2 = A\xi, \quad 2a\alpha = A\sqrt{(\xi^2 - 1)}.$$

We then obtain
$$\int_0^\infty x e^{-\tau x} J^\nu(ax) J^\nu(\alpha x)\, dx = \frac{2^{-\nu}(2\nu+1)(\xi^2-1)^{\frac{1}{2}\nu}}{\pi A^{\frac{3}{2}}} \int_0^\pi \frac{\tau(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu+\frac{3}{2}}} \ldots\ldots (13).$$

Now Hobson* has given the formula
$$(\xi^2-1)^{\frac{1}{2}\nu} \int_0^\pi \frac{(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu-n}}$$
$$= \frac{\Gamma(n-\nu+1)}{\Gamma(n+\nu+1)} 2^\nu \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \left\{P_n^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_n^\nu(\xi)\right\},$$
and also the formulae†
$$\frac{\Gamma(n-\nu+1)}{\Gamma(n+\nu+1)} \left\{P_n^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_n^\nu(\xi)\right\} = P_n^{-\nu}(\xi),$$
$$P_n^{-\nu}(\xi) = P_{-n-1}^{-\nu}(\xi).$$

If in these formulae we put $n = -\frac{3}{2}$, we see that
$$(\xi^2-1)^{\frac{1}{2}\nu} \int_0^\pi \frac{(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu+\frac{3}{2}}}$$
$$= \frac{\Gamma(-\nu-\frac{1}{2})}{\Gamma(\nu-\frac{1}{2})} 2^\nu \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \left\{P_{-\frac{3}{2}}^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_{-\frac{3}{2}}^\nu(\xi)\right\}$$
$$= 2^\nu \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) P_{-\frac{3}{2}}^{-\nu}(\xi)$$
$$= 2^\nu \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) P_{\frac{1}{2}}^{-\nu}(\xi).$$

* *Phil. Trans. Roy. Soc.* (A), vol. CLXXXVII. p. 493. † *ibid.* pp. 462, 452.

Hence we deduce

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx = \frac{2\Gamma(\nu+\frac{3}{2})}{A^{\frac{3}{2}}\sqrt{\pi}}\,\tau\, P_{\frac{1}{2}}{}^{-\nu}(\xi) \quad \text{.....................(14).}$$

If however ν is integral we have also

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx$$

$$= \frac{2^{-\nu}(2\nu+1)}{\pi A^{\frac{3}{2}}} \cdot \frac{\Gamma(-\nu-\frac{1}{2})}{\Gamma(\nu-\frac{1}{2})} 2^\nu\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})\,\tau\, P_{-\frac{3}{2}}{}^{\nu}(\xi)$$

$$= \frac{2\Gamma(\frac{3}{2}-\nu)}{A^{\frac{3}{2}}\sqrt{\pi}}\,\tau P_{-\frac{3}{2}}{}^{\nu}(\xi) \quad \text{...(15).}$$

The function which Hobson denotes by $P_{-\frac{3}{2}}{}^{\nu}(\xi)$ would be denoted by Heine or Sommerfeld by

$$\sqrt{\left(\frac{2}{\pi}\right)}\frac{P_\nu{}^{\frac{1}{2}}(\xi)}{\Gamma(\frac{3}{2}-\nu)},$$

so that in this notation we obtain

$$\frac{\tau}{\pi}\left(\frac{2}{A}\right)^{\frac{3}{2}} P_\nu{}^{\frac{1}{2}}(\xi);$$

and this is the result obtained by Sommerfeld for integral values of ν.

The formula (14) has been proved on the assumption that $R(\nu) > -\frac{1}{2}$. Each side of the equation represents an analytic function of ν regular for all values of ν for which $R(\nu) > -1$, and the equation therefore holds for all such values.

§ 33. We have thus the formula

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx$$

$$= \frac{2\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \frac{\tau}{\{\tau^2+(a+\alpha)^2\}^{\frac{3}{4}}\{\tau^2+(a-\alpha)^2\}^{\frac{3}{4}}} P_{\frac{1}{2}}{}^{-\nu}\left\{\frac{\tau^2+a^2+\alpha^2}{[\tau^2+(a+\alpha)^2]^{\frac{1}{2}}[\tau^2+(a-\alpha)^2]^{\frac{1}{2}}}\right\} \text{...(16),}$$

and it is clear that, unless $a=\alpha$, this expression tends to zero with τ, and moreover does so uniformly in any interval of values of α which does not include $\alpha = a$. Hence

$$G\int_0^\infty x\, J^\nu(ax)\, J^\nu(\alpha x)\, dx = 0 \qquad (\alpha \neq a) \text{ .... (17);}$$

moreover the integral is uniformly summable in (β, γ) if that interval does not include a. In this case, therefore, by Theorem VII

$$G\int_0^\infty x\, J^\nu(ax) dx \int_\beta^\gamma f(\alpha)\, J^\nu(\alpha x)\, dx = 0.$$

If, however, $a=\alpha$, $A\to 0$ and $\xi\to\infty$ as $\tau\to 0$. In this case we require an asymptotic formula for $P_{\frac{1}{2}}{}^{-\nu}(\xi)$.

Now it is known* that

$$P_{\frac{1}{2}}{}^{-\nu}(\xi) = \frac{(2\xi)^{\frac{1}{2}}}{\Gamma(\nu+\frac{3}{2})\sqrt{\pi}} + \epsilon_\xi,$$

where

$$|\epsilon_\xi| < K\xi^{-\frac{3}{2}},$$

* Hobson, *loc. cit.*, p. 463. When $n=\frac{1}{2}$ the expansions there given become illusory, but the appropriate expansion can easily be deduced by a passage to the limit.

and, if $\beta > 0$, K is independent of ξ or α. Hence it follows that, if $\beta < a < \gamma$,

$$\lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,d\alpha\int_0^\infty xe^{-\tau x}J^\nu(ax)\,J^\nu(\alpha x)\,dx$$

$$= \lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,d\alpha\left\{\frac{2\Gamma(\nu+\frac{3}{2})}{A^{\frac{3}{2}}\sqrt{\pi}}\,\tau P_{\frac{1}{2}}{}^{-\nu}(\xi)\right\}$$

$$=\frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,\tau\xi^{\frac{1}{2}}A^{-\frac{3}{2}}\,d\alpha$$

$$=\frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^\gamma\frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}f(\alpha)\,d\alpha,$$

provided that the last limit exists.

We divide the range of integration into the two parts (β, a), (a, γ). Let us first evaluate the limit

$$\lim_{\tau\to 0}\int_\beta^a\frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,d\alpha.$$

The integral is equal to

$$\frac{(\tau^2+a^2+\beta^2)^{\frac{1}{2}}}{\tau^2+(a+\beta)^2}\,\text{arc tan}\left(\frac{a-\beta}{\tau}\right)+\int_\beta^a\text{arc tan}\left(\frac{a-\alpha}{\tau}\right)\frac{d}{d\alpha}\left\{\frac{(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\tau^2+(a+\alpha)^2}\right\}d\alpha.$$

It is easy to see that the last integral is continuous for $\tau=0$, so that the expression tends, as $\tau\to 0$, to the limit

$$\tfrac{1}{2}\pi\left[\frac{(a^2+\beta^2)^{\frac{1}{2}}}{(a+\beta)^2}+\int_\beta^a\frac{d}{d\alpha}\left\{\frac{(a^2+\alpha^2)^{\frac{1}{2}}}{(a+\alpha)^2}\right\}d\alpha\right]=\frac{\pi}{4\sqrt{2}}\,.$$

Hence, if $f(a-0)$ is determinate,

$$\frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^a\frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}f(a-0)\,d\alpha=\tfrac{1}{2}f(a-0).$$

Now let

$$f(\alpha)-f(a-0)=\phi(\alpha),$$

so that $\phi(\alpha)\to 0$ as $\alpha\to a-0$. Then

$$\left|\int_\beta^a\frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\phi(\alpha)\,d\alpha\right|\leqq\left|\int_\beta^{a-\delta}\right|+\left|\int_{a-\delta}^a\right|$$

$$<\frac{\tau(\tau^2+2a^2)^{\frac{1}{2}}}{(\tau^2+a^2)(\tau^2+\delta^2)}\int_\beta^a|\phi(\alpha)|\,d\alpha$$

$$+\,\omega\int_{a-\delta}^a\frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,d\alpha,$$

where ω is the upper limit of $|\phi(\alpha)|$ in the interval $(a-\delta, a)$.

The first term is less than $K\tau/(\tau^2+\delta^2)$,

and the second is less than

$$\frac{\omega\sqrt{3}}{a}\int_{a-\delta}^a\frac{\tau\,d\alpha}{\tau^2+(a-\alpha)^2}<\frac{\pi\omega\sqrt{3}}{2a}\,.$$

If $\delta = \tau^s$, where $0 < s < \frac{1}{2}$, each of these expressions tends to zero with τ. Hence

$$\frac{2^{\frac{3}{2}}}{\pi} \lim_{\tau \to 0} \int_\beta^a \frac{\tau(\tau^2 + a^2 + \alpha^2)^{\frac{1}{2}}}{\{\tau^2 + (a+\alpha)^2\}\{\tau^2 + (a-\alpha)^2\}} f(\alpha)\, d\alpha = \tfrac{1}{2} f(a-0).$$

The integral from a to γ can be treated in a precisely similar manner, and so we arrive at the equation (10) of § 31. Thus the result proved by Sommerfeld for integral values of ν is extended to all values of ν whose real part is greater than -1.

§ 34. When $e^{-(\tau x)^2}$ is used as the factor of convergence the work is easier since, by a well-known formula,

$$\int_0^\infty e^{-(\tau x)^2} x\, J^\nu(ax)\, J^\nu(\alpha x)\, dx = \frac{i^\nu}{2\tau^2} e^{-\frac{a^2+\alpha^2}{4\tau^2}} J^\nu\left(\frac{-ia\alpha}{2\tau^2}\right).$$

* Sommerfeld, *loc. cit.* p. 31; Nielsen, *Cylinderfunktionen*, p. 184.

CORRECTIONS

p. 2, *last* 2 *lines*. For $t \to 0$ read $\tau \to 0$ (3 times).
p. 4, *line* 5 *up*. Insert factor x in index of exponential.
p. 10, *line* 12. For $-C$ and $+S$ read C and $-S$.
—— *line* 16. For $-, +, -, \ldots$ read $+, -, +, \ldots$.
p. 13, *line* 11 *up*. For e^{-x} read e^{-x^2}.
p. 15, *line* 6. For a read c.
—— *line* 5 *up*. For e^{-2x} read e^{-zx}.
p. 19, *line* 10. For § 6 read § 10.
—— *line* 13. For 'sum the series on the right' read 'sum the series

$$0-\frac{a_1}{m^2}+0+\frac{3!a_3}{m^4}+\ldots \quad\text{and}\quad \frac{a_0}{m}+0-\frac{2!a_2}{m^3}+0+\ldots\text{'}.$$

— *line* 15. For $\left(\frac{v}{m}\right)^2$ read $\left(\frac{v}{m}\right)^3$.
—— *line* 16. For a_1 read a_0.
p. 21, *line* 4. For the 2nd X read x.
p. 24, *line* 13 *up*. For $t \to 0$ read $\tau \to 0$.
—— *line* 11 *up*. For $r \to 0$ read $\tau \to 0$.
p. 26, *line* 11 *up*. For $\phi^{n(+1)}$ read $\phi^{(n+1)}$.
p. 28, *line* 8. Read Hx^k.
p. 30, *line* 13. For $\tau_0 \leqslant \tau$ read $\tau_0 \leqslant t$.
p. 31, *last line*. For $\int_0^\pi$ read $\int_0^{\frac{1}{2}\pi}$. For $\sin^2 x$ read $\cos^2 x$. Add $\alpha \neq$ even integer; cf. 1904, 4, p. 56, formula (4).
p. 32, *line* 8 *up*. For $\frac{1}{2}\pi$ read $\frac{1}{2}$.
p. 41, *line* 3. For $f(0)$ read $f(+0)$.
—— *line* 10 *up*. For 'in it' read 'on it'.
p. 43, *line* 6. For $x = \alpha$ read $\alpha = a$.
—— *lines* 9 *and* 18. For $\beta < \alpha < \gamma$ read $\beta < a < \gamma$.
—— *line* 10. For 2nd $f(\alpha)$ read $f(a)$.

COMMENTS

The paper continues 1904, 3 and 1904, 4, Part II.

In § 6, the radius of convergence of $\sum n!a_n y^n$ should be σ, where $\sigma > 1/\delta$ (not $\sigma > \delta$), and ρ should be chosen so that $1/\sigma < \rho < \delta$.

In Theorem II, the conditions of § 6 (3) are assumed. Thus $\psi(\tau)$ is defined and regular for $R(\tau) > 0$, since $e^{-\tau x}\phi(x) \to 0$ as $x \to \infty$, for each $\tau > 0$.

In § 15, the condition '$f(x)$ is continuous' should be replaced (for example) by '$f(x)$ is continuous and satisfies Dirichlet's conditions'.

In § 28, conditions (i), (ii), and (iii) are a version of Fejér's conditions, for functions which have an 'absolutely convergent improper Riemann integral'. In § 29, the same conditions are used to obtain the corresponding version of Fejér's theorem for Fourier integrals of 'finite type'; see Pollard.† It follows from Pollard's paper that Hardy's result is equivalent to Fejér's theorem for Fourier series of the corresponding class.

† *Proc. Cambridge Phil. Soc.* 23 (1926), 373–82.

A NOTE ON THE CONTINUITY OR DISCONTINUITY OF A FUNCTION DEFINED BY AN INFINITE PRODUCT

By G. H. Hardy.

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1. Abel's well known theorem on the continuity of power series naturally suggests the question as to whether a similar theorem holds for infinite products. Does the convergence of the product

(1) $$P = \prod_{0}^{\infty}(1+a_n)$$

involve the absolute convergence of the product

(2) $$P_1(x) = \prod_{0}^{\infty}(1+a_n x^n)$$

for all values of x whose modulus is less than unity, and the truth of the equation

$$P_1(x) \to P$$

as $x \to 1$? The path along which $x \to 1$ is here supposed to be any such path as is permitted in Stolz's extension of Abel's theorem, that is to say, any path which lies inside the unit circle, has a tangent at every point, and does not touch the circle. Such a path we shall describe for brevity as a *standard* path.*

This question immediately suggests another: does the convergence of (1) involve that of

(3) $$P_2(x) = \prod_{0}^{\infty}(1+a_n x)$$

for all values of x, and the truth of the equation

(4) $$P_2(x) \to P$$

when $x \to 1$ in any manner?

2. If the product (1) is absolutely convergent, that is to say, if the series Σa_n is absolutely convergent, it can be shown at once that all these questions must be answered in the affirmative.

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* Paths which have no tangents may be dismissed from consideration; no interest attaches to them, and the only result of admitting them is a little unnecessary complication of our definitions.

The question stated above was, if I remember rightly, first suggested to me personally by Prof. V. Ramaswami Aiyar of Gooty, India, in a letter which I received from him a year or two ago.

The simplest proof of this depends upon what Mr. Bromwich has called Tannery's theorem—viz., that, if

$$|g_n(x)| < M_n,$$

where M_n is independent of x, and ΣM_n is convergent, throughout any region D of values of x, then

$$\Pi\{1+g_n(x)\} \tag{5}$$

is uniformly convergent throughout D. If every $g_n(x)$ is continuous, the product is, of course, also continuous.

We shall denote by D_1 any region bounded by a standard curve beginning and ending at the point $x=1$; and by D_2 any region bounded by a closed curve and including the point $x=1$. Then

$$|a_n x^n| \leqslant |a_n|$$

throughout D_1; and

$$|a_n x| \leqslant R|a_n|$$

throughout D_2, R being the greatest distance of any point of D_2 from the origin; and it follows at once that $P_1(x)$ and $P_2(x)$ are uniformly convergent throughout D_1 and D_2 respectively, the boundaries of the regions included.

A theorem similar to Tannery's, rather more general, but rather less simple and natural, was given by Arzelà.* This theorem asserts that, if

(i.) $g_n(x)$ tends uniformly to zero, as $n \to \infty$, for all values of x in D;

(ii.) $\sum_{\nu=0}^{n} |g_\nu(x)| < K$ for all values of n and x;†

then the uniform convergence of $\Sigma g_n(x)$ is a *sufficient* condition for that of $\Pi\{1+g_n(x)\}$.‡

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* *Mem. di Bologna*, ser. 4, t. IV. (1883), p. 427; Stolz und Gmeiner, *Einleitung in die Funktionentheorie*, bd. II, p. 431.

† If Tannery's condition $|g_n(x)| < M_n$, where ΣM_n is convergent, is satisfied, it is evident that Arzelà's two conditions are satisfied. The converse is not true. Suppose, for example, that

$$\begin{array}{ll}
g_0(x) = 1 \quad (0 \leqslant x < \tfrac{1}{2}), & g_0(x) = 0 \quad (\textit{otherwise}); \\
g_1(x) = \tfrac{1}{2} \quad (\tfrac{1}{2} \leqslant x < \tfrac{3}{4}), & g_1(x) = 0 \quad (\textit{otherwise}); \\
g_2(x) = \tfrac{1}{3} \quad (\tfrac{3}{4} \leqslant x < \tfrac{7}{8}), & g_2(x) = 0 \quad (\textit{otherwise}); \\
\ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
g_n(x) = \dfrac{1}{n+1} \quad (1-2^{-n} \leqslant x < 1-2^{-n-1}), & g_n(x) = 0 \quad (\textit{otherwise}).
\end{array}$$

Then it is clear that $\sum_{\nu=0}^{n} |g_\nu(x)| < 1$, and that $g_n(x) \to 0$ uniformly; but Tannery's condition is not satisfied, since $\Sigma(1/n)$ is divergent. There is no difficulty in constructing a similar example in which every g_n is continuous.

‡ If x and $g_n(x)$ are restricted to be real, the condition is also *necessary*.

It is plain that our results concerning the products $\Pi(1+a_n x^n)$, $\Pi(1+a_n x)$ can be deduced at once from Arzelà's as well as from Tannery's theorem.

3. We may now pass on to consider the more interesting case in which the product (1) is only *conditionally* convergent. So far as I am aware, the only general tests of any importance that have ever been given for the conditional convergence of a product are Cauchy's test—viz.,

> *the product* $\Pi(1+a_n)$ *is convergent if* Σa_n *is convergent and* Σa_n^2 *absolutely convergent ;*

and Pringsheim's extension of Cauchy's test—viz.,

> *the product is convergent if* Σa_n, Σa_n^2, ..., Σa_n^{k-1} *are convergent and* Σa_n^k *absolutely convergent.**

A product which is convergent in virtue of Cauchy's or Pringsheim's tests we shall call a *regularly convergent* product.

4. THEOREM A.—*If the product* (1) *is regularly convergent, all the questions of* § 1 *may be answered in the affirmative.*

This result I shall deduce from the following general theorem :—

THEOREM B.—*If the series*†

$$\Sigma g_n(x),\ \Sigma g_n^2(x),\ \ldots,\ \Sigma g_n^{k-1}(x),\ \Sigma |g_n^k(x)|$$

are uniformly convergent throughout any region D in the plane of x, then the product

$$\Pi\{1+g_n(x)\}$$

is uniformly convergent throughout D.

This theorem is very easy to prove. We can choose n_0 so that, for $n \geqslant n_0$, $|g_n| < \delta < 1$, for all values of x in question. We can then ignore the first n_0 factors, so that nothing is lost by supposing $|g_n| < \delta$

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* Pringsheim, *Math. Annalen*, bd. XXII., p. 482 ; Stolz und Gmeiner, *l.c.*, p. 436. The latter test may be stated in the more general form "the product is convergent if

$$\Sigma\left(a_n - \tfrac{1}{2}a_n^2 + \ldots \pm \frac{1}{k-1}\, a_n^{k-1}\right)$$

is convergent and Σa_n^k absolutely convergent," but the extension seems of but little interest.

If a_n is real, Σa_n^2 can, of course, only converge absolutely or diverge to $+\infty$; the product converges or diverges to 0 accordingly. In this case Pringsheim's extension cannot be needed.

† More generally, if $\Sigma\left(g_n - \tfrac{1}{2}g_n^2 + \ldots \pm \dfrac{1}{k-1}\, g_n^{k-1}\right)$, $\Sigma |g_n^k|$ are uniformly convergent.

for all values of n and x. Then

$$\log(1+g_n) = g_n - \tfrac{1}{2}g_n^2 + \ldots + \frac{(-1)^k}{k-1} g_n^{k-1} + (-1)^{k+1}\phi_n,$$

where $$\phi_n = \frac{g_n^k}{k} - \frac{g_n^{k+1}}{k+1} + \ldots,$$

so that $$|\phi_n| < \frac{|g_n|^k}{k}(1+\delta+\delta^2+\ldots) < \frac{|g_n^k|}{k(1-\delta)}.$$

Hence $\Sigma\phi_n$ is uniformly convergent, and so therefore is $\Sigma\log(1+g_n)$.

5. From Theorem B the truth of Theorem A follows almost immediately.

(i.) Let $g_n = a_n x^n$, and let D be the region D_1 of § 2. Then

$$\Sigma a_n,\ \Sigma a_n^2,\ \ldots,\ \Sigma a_n^{k-1},\ \Sigma|a_n^k|$$

are convergent, and therefore, by Stolz's extension of Abel's theorem,

$$\Sigma a_n x^n,\ \Sigma a_n^2 x^{2n},\ \ldots,\ \Sigma a_n^{k-1} x^{(k-1)n},\ \Sigma|a_n^k||x^{kn}|$$

are uniformly convergent throughout D_1.

(ii.) Let $g_n = a_n x$, and let D be the region D_2 of § 2. Then

$$\Sigma a_n x,\ \ \Sigma a_n^2 x^2,\ \ \ldots,\ \ \Sigma a_n^{k-1} x^{k-1},\ \ \Sigma|a_n^k|\,|x^k|$$

are uniformly convergent throughout D_2.

It is easy to deduce, from Theorem B and from the known extensions of Abel's theorem, more general results concerning products of the types

$$\Pi\{1+a_n f_n(x)\};$$

but the cases in which $f_n(x) = x^n$ or x seem so much the most interesting that it is hardly worth while to set any others out at length.

6. There still remains the case in which the product (1) is convergent but not regularly convergent, or, as we may say, *irregularly convergent.*

As regards such irregular convergence one may distinguish two possibilities.

(*a*) It is possible that the product $\Pi(1+a_n)$ may be convergent, although the series Σa_n is not convergent, and indeed even if the series diverges to $+\infty$. The following examples of this are interesting, and we shall have occasion to make use of them later on.

(i.) Consider the product*

(6) $$\prod_{0}^{\infty}\left\{\left(1+\frac{1}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{1}{\sqrt{n+\frac{1}{2}}}\right)\right\},$$

for which $$a_{2\nu}=\frac{1}{\sqrt{\nu-\frac{1}{2}}},\quad a_{2\nu+1}=-\frac{1}{\sqrt{\nu+\frac{1}{2}}}.$$

Here $(1+a_{2\nu})(1+a_{2\nu+1})=1$, so that the product is convergent, and has the value 1, although $\Sigma(a_{2\nu}+a_{2\nu+1})$ or

$$\Sigma\frac{1}{\nu-\frac{1}{4}}$$

diverges to $+\infty$.

(ii.) The product $$\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$$

is convergent, unless θ is a multiple of π,† since the series

$$\Sigma\frac{e^{n\theta i}}{\sqrt{n}},\quad \Sigma\frac{e^{2n\theta i}}{n},\quad \Sigma\frac{1}{n\sqrt{n}}$$

are convergent.

It follows that

(7) $$\Pi\left|1+\frac{e^{n\theta i}}{\sqrt{n}}\right|^2=\Pi\left(1+\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right)$$

is convergent. But $$\Sigma\left(\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right)$$

plainly diverges to $+\infty$.

The product may also converge when some of the later members of the sequence of series $$\Sigma a_n,\ \Sigma a_n^2,\ \ldots,\ \Sigma a_n^{k-1}$$

oscillate or diverge. All such cases afford illustrations of our first possibility with respect to irregular convergence.

(*b*) The second possibility with respect to irregular convergence is that it should not be possible to find a value of k for which

$$\Sigma\,|\,a_n\,|^k$$

is convergent. Suppose, for example, that

$$a_n=\frac{e^{n\theta i}}{\log n},$$

where θ/π is *irrational*. Then Σa_n^k is convergent for *all* values of k, but

* This product is used by Pringsheim (*Math. Annalen*, bd. XXXIII., p. 154) for another purpose.

† The product diverges to infinity if θ is a multiple of 2π, to 0 if θ is an odd multiple of π.

never absolutely. And, so far as the tests at our disposal at present go, the question of the convergence of the product

$$\Pi\left(1+\frac{e^{n\theta i}}{\log n}\right) \tag{8}$$

remains open.

7. I shall return to the second possibility in a moment. But first I wish to show, by means of examples drawn from the first class of irregularly convergent products, that the *mere* convergence of the product (1) is *not* sufficient to ensure an affirmative answer to the questions of § 1.

(i.) *The convergence of* (1) *does not necessarily involve the convergence of* $\Pi(1+a_n x)$ *for any value of* x *other than* $x=0$ *and* $x=1$. We saw above that the product

$$\prod_0^\infty \left\{\left(1+\frac{1}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{1}{\sqrt{n+\frac{1}{2}}}\right)\right\}$$

is convergent. But

$$\prod_0^\infty \left\{\left(1+\frac{x}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{x}{\sqrt{n+\frac{1}{2}}}\right)\right\}$$

is convergent only if $\quad \displaystyle\prod_{\nu=0}^\infty \left(1+\frac{a_\nu}{\nu-\frac{1}{4}}\right)$

is convergent, where $\quad a_\nu = \frac{1}{4}-(x-\frac{1}{2})^2;$

and this is so only if $a_\nu = 0$, *i.e.*, if $x=0$ or $x=1$.

(ii.) The convergence of (1) does, of course, imply the absolute convergence of $\Pi(1+a_n x^n)$ for any value of x numerically less than unity. *But it does not imply the truth of the equation*

$$\Pi(1+a_n x^n) \to \Pi(1+a_n)$$

as $x\to 1$, *even by real values and when* a_n *is real.*

We saw in § 6 that the product

$$\prod_1^\infty \left(1+\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right) \tag{7}$$

is convergent, provided θ is not a multiple of π. Let us denote its value by ϖ. Then I shall prove that if

$$a_n = \frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}$$

then

$$\prod_1^\infty (1+a_n x^n) \to 2\varpi \tag{8}$$

as $x \to 1$ by real values.

In order to prove this we observe that since $\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$ is *regularly* convergent

$$\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}x^n\right) \to \Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$$

as $x \to 1$. It follows that the same relation holds between the products formed by taking the modulus of every factor, and therefore that

$$\Pi\left(1+\frac{2x^n \cos n\theta}{\sqrt{n}}+\frac{x^{2n}}{n}\right) \to \varpi.$$

Our conclusion will therefore be established if we prove that

$$\prod_1^\infty \frac{\left(1+\dfrac{2x^n \cos n\theta}{\sqrt{n}}+\dfrac{x^n}{n}\right)}{\left(1+\dfrac{2x^n \cos n\theta}{\sqrt{n}}+\dfrac{x^{2n}}{n}\right)} = \prod_1^\infty (1+\beta_n),$$

where
$$\beta_n = \frac{x^n(1-x^n)}{n+2x^n \cos n\theta\sqrt{n}+x^{2n}},$$

tends to the limit 2 as $x \to 1$. Now β_n is positive and less than K/n, and so the series $\Sigma\beta_n^2$ is uniformly convergent for $0 \leqslant x \leqslant 1$. Hence also the series

$$\Sigma\{\log(1+\beta_n)-\beta_n\}$$

is uniformly convergent, and so

$$\log \Pi(1+\beta_n)-\Sigma\beta_n \to 0.$$

We therefore require only to show that

$$\Sigma\beta_n \to \log 2.$$

But, if
$$\gamma_n = x^n(1-x^n)/n,$$

we have
$$\gamma_n-\beta_n = \frac{x^n(1-x^n)}{n}\,\frac{2x^n \cos n\theta\sqrt{n}+x^{2n}}{n+2x^n \cos n\theta\sqrt{n}+x^{2n}};$$

so that
$$|\gamma_n-\beta_n| < Kn^{-\frac{3}{2}}.$$

Hence $\Sigma(\gamma_n-\beta_n)$ is absolutely and uniformly convergent, and so

$$\Sigma\gamma_n-\Sigma\beta_n \to 0.$$

But
$$\Sigma \gamma_n = \Sigma \frac{x^n}{n} - \Sigma \frac{x^{2n}}{n} = \log\left(\frac{1}{1-x}\right) - \log\left(\frac{1}{1-x^2}\right)$$
$$= \log(1+x) \to \log 2;$$
and so our conclusion follows, viz., that
$$\Pi(1+a_n x^n) \to 2\Pi(1+a_n)$$
as $x \to 1$.

8. In conclusion, I wish to say something about the second possibility mentioned in § 6. This possibility seems to me very interesting. But I know of no example of such a product, and I am unable to construct one. Indeed, I cannot determine whether the product (8) is ever convergent or not; and the considerations which follow show, I think, that the question is not one which can be settled without considerable difficulty.

Let us consider first a simpler product, viz.,
$$\prod_1^\infty \left(1+\frac{e^{n\theta i}}{n^a}\right) \quad (0 < a < 1).$$
Let k be the least integer such that $ka > 1$. The series Σa_n^k is absolutely convergent, the series
$$\Sigma a_n, \quad \Sigma a_n^2, \quad \ldots, \quad \Sigma a_n^{k-1}$$
are convergent unless one of θ, 2θ, ..., $(k-1)\theta$ is a multiple of 2π. Thus the product is regularly convergent unless θ/π has one of a limited number of rational values; if, *e.g.*, $a = \frac{1}{2}$, $k = 3$, it is regularly convergent unless θ/π is an integer.

Now, let us consider the product (8), for which
$$a_n = e^{n\theta i}/\log n;$$
and let us suppose that θ/π is a rational fraction p/q. The series
$$\Sigma a_n^k = \Sigma \frac{e^{kn\theta i}}{(\log n)^k}$$
is conditionally convergent except for such values of k as make $k\theta/\pi$ an even integer.

First suppose p even. Then q is odd, and Σa_n^k is convergent, except for
$$k = q, 2q, 3q, \ldots,$$
while
$$a_n^q = (\log n)^{-q}.$$
Also
$$\log(1+a_n) = a_n - \tfrac{1}{2}a_n^2 + \ldots - \frac{1}{q-1} a_n^{q-1} + \frac{1}{q} a_n^q (1+\epsilon_n),$$
where $\epsilon_n \to 0$ with n. It follows at once that *the product* (8) *diverges* (*its*

associated product of moduli diverging to $+\infty$) *whenever* θ/π *is a rational fraction with an even numerator.*

Secondly, suppose p odd. Then Σa_n^k is convergent, except for

$$k = 2q,\ 4q,\ 6q,\ \ldots;$$

and an argument similar to that used above shows that *the product* (8) *diverges to* 0 *whenever* θ/π *is a rational fraction with an odd numerator.*

Thus the product (8) is certainly never convergent when θ/π is rational. Is it ever convergent? That, it seems to me, is a very interesting question; but I must confess myself entirely unable to answer it. I can only suggest the problem: *to find a product* $\Pi(1+a_n)$, *such that* Σa_n^k *is always convergent, but never absolutely, and whose convergence, divergence, or oscillation is capable of proof.*

It is hardly necessary to point out that the argument given above applies to any product $\Pi(1+\alpha_n e^{n\theta i})$, where α_n is a positive function of n which tends steadily to 0 as $n \to \infty$, and which is such that

$$\Sigma \alpha_n^{-k}$$

is divergent for all values of k.

CORRECTIONS

p. 42, *2nd footnote.* For $|g_n^k$ read $|g_n^k|$.
pp. 47–8. For 'product (8)' read 'product (8) of § 6' (5 times).
p. 48, *line 2 up.* For α_n^{-k} read α_n^k.

COMMENTS

When $\sum |a_n| < \infty$, § 2, the product $\prod (1+a_n x^n)$ converges absolutely and uniformly in $|x| \leqslant 1$. Hence in this case the boundary of D_1 may be tangential to the unit circle.

Littlewood† considered the question raised in §§ 6 and 8, and proved that: $\prod (1+a_n e^{n\theta i})$ *converges whenever* a_n *decreases to zero and* $\theta/\pi \in S$, *where* S *is a class of irrational numbers which includes all irrational algebraic numbers.*

† *Proc. London Math. Soc.* (2), 8 (1910), 195–9.

THE APPLICATION TO DIRICHLET'S SERIES OF BOREL'S EXPONENTIAL METHOD OF SUMMATION

By G. H. Hardy.

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1. The series

$$\Sigma a_n e^{-\lambda_n s}, \tag{1}$$

where
$$\lambda_{n+1} > \lambda_n, \quad \lambda_n \to \infty,$$

is called a *generalised Dirichlet's series.** If $\lambda_n = n$, it is a power series in $x = e^{-s}$: if $\lambda_n = \log n$, it is an *ordinary Dirichlet's series*

$$\Sigma a_n n^{-s}. \tag{2}$$

The series (1) possesses two lines of convergence

$$R(s) = s_0, \quad R(s) = \bar{s}, \tag{3}$$

where $s_0 \leqslant \bar{s}$: the series is convergent to the right of the first and absolutely convergent to the right of the second. Thus the series

$$\Sigma(-1)^{n-1} n^{-s}, \tag{4}$$

which represents the function

$$(1-2^{1-s})\,\zeta(s) \tag{5}$$

is convergent for $R(s) > 0$, and absolutely convergent for $R(s) > 1$. On the other hand, for the series Σn^{-s}, we have $s_0 = \bar{s} = 1$. It is evident that in the case of any ordinary Dirichlet's series $0 \leqslant \bar{s} - s_0 \leqslant 1$. We may, of course, have $s_0 = \bar{s} = -\infty$, the series then converging abso-

* The literature which concerns Dirichlet's series in general, considered as functions of a complex variable s, is rather scattered. The theory was first attacked seriously by Cahen: "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues," *Annales Sc. de l'École Normale Supérieure* (sér. 3), t. XI, p. 75. See the dissertation of W. Schnee (Berlin, 1908), and his article in the *Rendiconti del Circolo Mat. di Palermo*, t. XXVII, where numerous references are given to papers by Landau and others who have contributed to the theory; a later dissertation by H. Bohr (Copenhagen, 1910); and Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*. The general theory has been rather overwhelmed in the mass of its applications to the problem of the distribution of primes and to the general analytical theory of numbers.

1910, 1 Proceedings of the London Mathematical Society (2), 8, 277–94.

lutely all over the plane, as in the case of the series $\Sigma a^n n^{-s}$ ($|a|<1$) or, we may have $s_0 = \bar{s} = \infty$, the series being never convergent (as in the last example when $|a|>1$). In the general case the inequality $\bar{s}-s_0 \leqslant 1$ need not hold: we may even have $s_0 = -\infty$, $\bar{s} = \infty$, as in the case of the series

$$\Sigma(-1)^{n-1} n^{-a} (\log n)^{-s} \quad (0 < a < 1),$$

which converges conditionally for all values of s.

When $\lambda_n = n$, s_0 and $\bar{s}$ are necessarily equal, and the function represented by the series has necessarily at least one singular point on its line of convergence. No such result holds in the general case.*

The series Σn^{-s} has the one singular point $s=1$, which lies on the line of convergence: and Landau† has shown that, when $a_n > 0$, the real point on the line of convergence is always a singular point of the function represented by the series. On the other hand, the function (5) is an integral function of s. The line of convergence may also be a line of essential singularities, across which the function cannot be continued: Landau‡ has given as an example the series

$$\Sigma 2^{-s 2^n}.$$

2. These circumstances make the problem of the summation of a Dirichlet's series, even in the narrower sense, more difficult and at the same time more interesting than the corresponding problem for power-series. In this connection some very interesting results have been stated in three recent notes in the *Comptes Rendus*, by MM. Bohr and Riesz.

M. Bohr§ considers the application to Dirichlet's series of Cesàro's method of summation by mean values.‖ He begins by proving the following general theorem :—

If Σa_n *is summable or finite* (Ck),¶ *and* $f_n \to 0$ *as* $n \to \infty$, *and the series*

$$\Sigma\,|\Delta f_n|, \quad \Sigma n\,|\Delta^2 f_n|, \quad \ldots, \quad \Sigma n^k\,|\Delta^{k+1} f_n|$$

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* It is easy to see, however, that $s_0 = \bar{s}$ in all cases in which the increase of λ_n is sufficiently rapid to ensure the convergence of $\Sigma e^{-\lambda_n s}$ for all positive values of s.

† *Math. Annalen*, Bd. LXI, p. 537.

‡ *Sitzungsberichte der Akademie zu München*, Bd. XXXVI, p. 191.

§ *Comptes Rendus*, Jan. 11, 1909.

‖ Bromwich, *Infinite Series*, pp. 310 *et seq.*

¶ *Summable* (Ck) means summable by Cesàro's k-th mean: see *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 257. By saying that Σa_n is *finite* (Ck), we mean that, in the notation of the paper just referred to, S_n^k/A_n^k oscillates finitely as $n\to\infty$.

are convergent, then the series $\Sigma a_n f_n$ *is summable* (Ck), *and its sum is equal to that of the series*

$$\Sigma\, S_n^k \Delta^{k+1} f_n,^*$$

which is absolutely convergent.

This theorem is substantially the same as one which I proved in a paper† which I communicated to this Society in 1907, and which M. Bohr had not seen at the time of the publication of his note. It includes, as my theorem did not, the case in which Σa_n is only *finite* (Ck). On the other hand the conditions are less simple than they may be made, in that [as Dr. Bromwich (*Math. Annalen*, Bd. 65, p. 361) has proved] the convergence of $\Sigma n^k \,|\, \Delta^{k+1} f_n \,|$ involves that of all the other series mentioned in the final condition.

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* The notation is that of the paper referred to in the preceding footnote.

† Referred to in the two preceding footnotes.

[This paper unfortunately contains several inaccuracies, which have been pointed out to me by M. Bohr himself (see Bohr, *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen, 1910) and by Mr. J. E. Littlewood. The fact is that, finding the whole difficulty of the investigation to lie in the algebraical work (pp. 258–261), I was careless in writing pp. 261–3, which do not involve any point of very serious difficulty.

In the first place, it should have been explicitly stated that, throughout the proof of Theorem I, the condition $f_n \to 0$ is assumed to be satisfied: this is, of course, implied in calling f_n a "convergence factor" (p. 264). §§ 8, 9, 10 should be modified as follows. We consider first the terms in T_n^k for which $i = 0$. These give

$${}_0T_n^k = \sum_{j=0}^{n} \binom{n-j+k}{k} S_j^k \Delta^{k+1} f_j.$$

In this expression f_m is to be replaced by zero when $m > n$. But we may imagine this convention abandoned without affecting the limit of ${}_0T_n^k / A_n^k$. For this introduces a set of new terms in number depending only on k; for these terms we have $n-j < K$, and so

$$\binom{n-j+k}{k} < K,$$

and also

$$|\, S_j^k / A_n^k \,| < K.$$

And, as each of them involves a factor f_m which tends to zero as $n \to \infty$, their sum also tends to zero as $n \to \infty$.

Again,

$$\binom{n-j+k}{k} - A_n^k = \frac{1}{k!}\{(n-j+1)(n-j+2)\ldots(n-j+k) - (n+1)(n+2)\ldots(n+k)\}$$

is negative and numerically less than Kn^k $(0 \leqslant j \leqslant n)$. Also since

$$(n-j+1)\ldots(n-j+k) - (n+1)(n+2)\ldots(n+k) = -j\,\frac{d}{dx}\{(x+1)(x+2)\ldots(x+k)\},$$

From this theorem M. Bohr deduces the existence of a *line of summability* (Ck) for the series (2), viz.,

$$R(s) = s_k = \overline{\lim_{n\to\infty}} \{(S_n^k\, k!\, n^{-k})/\log n\}.$$

The series is summable (Ck) if $R(s) > s_k$, and not so summable if $R(s) < s_k$. Also

$$0 \leqslant s_k - s_{k+1} \leqslant 1,$$

and the function represented by the series is regular for $R(s) > s_k$. Thus for the series (4), $s_k = -k$, and the series is summable, by one or other of Cesàro's means, all over the plane.

For the general series (1) there are wider possibilities as to summability. Thus, if $\lambda_n = n$, $s_k = s_0 = \bar{s}$ for all values of k. On the other hand, the series

$$\Sigma(-1)^{n-1} n^a (\log n)^{-s} \quad (k-1 < a < k)$$

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where $n-j < x < n$, it follows that $\binom{n-j+k}{k} - A_n^k$ is numerically less than

$$K\nu n^{k-1} \quad (0 \leqslant j \leqslant \nu < n).$$

Hence
$$\frac{{}_0T_n^k}{A_n^k} = \epsilon_n + \sum_0^n S_j^k \Delta^{k+1} f_j + R_0,$$

where
$$|R_0| < \frac{K\nu}{n} \sum_{j=0}^{\nu} j^k \,|\Delta^{k+1} f\,| + K \sum_{\nu+1}^{n} j^k \,|\Delta^{k+1} f_j\,| :$$

and by choosing first ν and then n sufficiently large we see that $R_0 \to 0$.

It follows that
$${}_0T_n^k / A_n^k \to \sum_0^\infty S_j^k \Delta^{k+1} f_j.$$

Next we consider

$${}_iT_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} S_j^k \Delta^{k+1-i} f_{j+i}.$$

Since
$$\binom{n-j-i+k}{k-i} = (n-j+1)(n-j+2)\ldots(n-j+k-i)/(k-i)! < Kn^{k-i},$$

it follows that
$$\left|\frac{{}_iT_n^k}{A_n^k}\right| < \frac{K}{n^i} \sum_{j=0}^{n} |S_j^k \Delta^{k+1-i} f_{j+i}| < \frac{K}{n^i} \sum_{j=0}^{n} j^k\,|\Delta^{k+1-i} f_{j+i}|$$
$$< K \sum_{j=0}^{n} \left(\frac{j}{n}\right)^i j^{k-i}\,|\Delta^{k+1-i} f_{j+i}|.$$

By dividing the range of summation into the two parts $(0, \nu)$, $(\nu+1, n)$, and choosing first ν and then n sufficiently large, we see that

$${}_iT_n^k / A_n^k \to 0.$$

This completes the proof of Theorem A, the statement of which requires no correction. The theorem, as M. Bohr points out, remains true when $f_n \to L$ $(L \neq 0)$, provided that the expression for the sum of the series is modified by the addition of a term SL, where S is the sum of the series Σa_n. This follows at once from Theorem A on replacing f_n by $f_n - L$.

Theorem B requires modification only in the replacing of $\Sigma S_j^k \Delta^{k+1} f_j$ by $\Sigma S_j'^k \Delta^{k+1} f_j$, where

is summable (Ck) all over the plane, but never summable $(C, k-1)$, so that $s_k = -\infty$, $s_{k-1} = \infty$: the sum represents an integral function of s.

3. I propose now to consider the application to the ordinary Dirichlet's series of Borel's method of summation. It is easy to see, by the consideration of two simple examples, that it either may or may not be possible, by the use of Borel's method, to continue a function partially represented by a Dirichlet's series outside the region of convergence of the series.

Suppose first that the series is Σn^{-s}, which converges for $R(s) > 1$, and represents the function $\zeta(s)$, whose only singularity is a simple pole for $s = 1$. Then Borel's integral is

$$\int_0^\infty e^{-x} u(x)\, dx, \tag{6}$$

S'^k is defined as in the text. Then $|S_j'^k| < \epsilon A^k$ for $j \geqslant m_0$, and so

$$\left| \sum_m^{m'} S_j'^k \Delta^{k+1} f_j \right| < \epsilon \sum_m^{m'} A_j^k \,|\Delta^{k+1} f_j| < \epsilon K,$$

for $m' > m \geqslant m_0$; whence the truth of the theorem follows.

The corollaries remain valid: but the first of them requires a few words of proof. The sum of the series $a_0' f_0 + a_1' f_1 + \ldots$ is $\Sigma S_j'^k \Delta^{k+1} f_j$, and is therefore continuous; and the sum of $a_0 f_0 + a_1 f_1 + \ldots$ differs from this by Sf_0, and is therefore also continuous.

It is to be observed that the series

$$\Sigma S_j^k \Delta^{k+1} f_j$$

is, in general, neither uniformly convergent nor continuous. Suppose, *e.g.*, that

$$a_0 = 1,\ a_1 = -1,\ a_2 = 1,\ \ldots,$$

so that $k = 1$, $S = \frac{1}{2}$, $a_0' = 1 - \frac{1}{2} = \frac{1}{2}$; and that f_n is a function, such as x^n, that has the limit 0 for $0 \leqslant x < 1$, and the limit 1 for $x = 1$, the interval of values of x under consideration being $0 \leqslant x \leqslant 1$. Then for $x < 1$, we have

$$\Sigma a_n f_n = \Sigma S_j^1 \Delta^2 f_j = \tfrac{1}{2} \Sigma (j+1) \Delta^2 f_j + \Sigma S_j'^1 \Delta^2 f_j = \tfrac{1}{2} f_0 + \Sigma S'^1 \Delta^2 f_j;$$

the last series being a uniformly convergent series whose limit as $x \to 1$ is plainly zero. Thus $\Sigma a_n f_n$ and $\Sigma S_j^1 \Delta^2 f_j$ have the limit $\frac{1}{2}$. But the last series is discontinuous. For $x = 1$ its sum is zero, but then

$$\Sigma a_n f_n = LS + \Sigma S_j^1 \Delta^2 f_j = \tfrac{1}{2}.$$

These facts, I may point out, are clearly recognised in my earlier paper in Vol. 2, Ser. 2, of these *Proceedings* (pp. 247 *et seq.*).

The genesis of the inaccuracies that I have explained is to be found in a momentary confusion, more natural than excusable, between the relations

$$S_n^k / A_n^k \to S, \quad S_n^k \to S.$$

It is remarkable that so careless a blunder should not have led to more serious error in my results.—*Added, February,* 1910.]

where* $$u(x) = \sum_1^\infty \frac{x^{n-1}}{n^s(n-1)!}.$$

Since† $$u(x) \sim x^{-s}e^x,$$

it follows that (6) is convergent (and then absolutely) if, and only if, $R(s) > 1$, when the original series is convergent.

Secondly, consider the series (4), which represents an integral function of s. Here‡

$$u(x) = \sum_1^\infty \frac{(-x)^{n-1}}{n^s(n-1)!} \sim -\frac{1}{\Gamma(s)} x^{-1} (\log x)^{s-1}.$$

Hence (6) is absolutely convergent for all values of s. It is easy to see that the same is true of

(7) $$\int_0^\infty e^{-x} u^{(\lambda)}(x)\, dx,$$

and so the series is absolutely summable all over the plane.§

4. I shall now prove that, *if the series* Σa_n *is summable, so is* $\Sigma a_n n^{-s}$, *where* $R(s) > 0$.

We are given that the integral (6) is convergent when

$$u(x) = \sum_1^\infty \frac{a_n x^{n-1}}{(n-1)!}.$$

It follows‖ that the integral

(8) $$\int_0^\infty e^{-x} u(xy)\, dx$$

is uniformly convergent for $0 \leqslant y \leqslant 1$. We have therefore, if $R(s) > 0$,

(9) $$\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} dy \int_0^\infty e^{-x} u(xy)\, dx = \int_0^\infty e^{-x} dx \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} u(xy)\, dy.$$

* We are applying Borel's method to $a_1 + a_2 + a_3 + \ldots$ and not to $0 + a_1 + a_2 + \ldots$. The equation would in the latter case be

$$u(x) = \sum_1^\infty \frac{x^n}{n^s n!}.$$

The summability of $a_1 + a_2 + \ldots$ implies that of $0 + a_1 + a_2 + \ldots$, whereas the converse is not true: see Bromwich, *Infinite Series*, p. 273, and a paper by the present writer in the *Quarterly Journal*, there referred to.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 402.

‡ *Ibid.* The formula fails if s is zero or a negative integer; then $u(x)$ tends exponentially to zero as $x \to \infty$.

§ Borel, *Leçons sur les séries divergentes*, p. 99.

‖ Bromwich, *Infinite Series*, pp. 433 *et seq.*

This follows from the following general theorem, which is in substance due to De la Vallée-Poussin.* He considers only the case in which the variables x, y are both real (which is all that is wanted here, the contour C being the line $0 \leqslant y \leqslant 1$). We shall require the more general form of the theorem in § 9.

If $f(x, y)$ is a continuous function of the real variable x and the real or complex variable y, and $\int_0^\infty f(x, y)\,dx$ is uniformly convergent for all values of y lying on a finite contour C, and $\int_C |\phi(y)|\,|dy|$ is convergent, then

$$\int_C \phi(y)\,dy \int_0^\infty f(x, y)\,dx = \int_0^\infty dx \int_C \phi(y) f(x, y)\,dy.$$

The following proof is an adaptation of the proof given by Dr. Bromwich (*Infinite Series*, p. 448) of an analogous theorem for series.

Since $\int_0^\infty f\,dx$ is a continuous function of y, the integral on the left hand is certainly convergent. Given ϵ, we can choose X, so that $\left|\int_{x_1}^\infty f\,dx\right| \leqslant \epsilon$ for $x_1 \geqslant X$. Also, in virtue of the continuity of f,

$$\int_C \phi\,dy \int_0^{x_1} f\,dx = \int_0^{x_1} dx \int_C f\phi\,dy.$$

Hence $$\left|\int_C \phi\,dy \int_0^\infty f\,dx - \int_0^{x_1} dx \int_C f\phi\,dy\right| = \left|\int_C \phi\,dy \int_{x_1}^\infty f\,dx\right| \leqslant \epsilon \int_C |\phi|\,|dy|,$$

for $x \geqslant X$, and so

$$\int_0^\infty dx \int_C f\phi\,dy = \lim_{x_1 \to \infty} \int_0^{x_1} dx \int_C f\phi\,dy = \int_C \phi\,dy \int_0^\infty f\,dx.$$

Now $$\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} u(xy)\,dy = \sum_1^\infty \frac{a_n x^{n-1}}{(n-1)!} \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} y^{n-1}\,dy$$
$$= \Gamma(s) \sum_1^\infty \frac{a_n x^{n-1}}{n^s (n-1)!}.\dagger$$

The last series represents the integral function associated with the series $\Sigma a_n n^{-s}$; and so this series is summable. Incidentally we have proved that its sum is

$$\frac{1}{\Gamma(s)} \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \psi(y)\,dy,$$

where $\psi(y)$ is the sum of the series $\Sigma a_n y^n$.

We notice further that *the sum of the series $\Sigma a_n n^{-s}$ represents a function of s regular for all values of s whose real part is positive.* This follows‡ at once from the fact that

$$\int_0^1 \frac{\partial}{\partial s} \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \psi(y)\,dy = \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \log\log\left(\frac{1}{y}\right) \psi(y)\,dy$$

* *Annales de la Société Scientifique de Bruxelles*, t. XVI.

† The justification of the term-by-term integration presents no difficulty.

‡ Bromwich, *Infinite Series*, p. 438.

is uniformly convergent throughout any continuous region, in the plane of s, which lies entirely to the right of the imaginary axis.

We have thus established the existence of a line $R(s)=\sigma_0$, such that a Dirichlet's series is summable everywhere on its right and nowhere on its left. In other words, *the region of summability is a half plane:* it may, of course, include the whole plane ($\sigma_0=-\infty$), or none of it ($\sigma_0=\infty$).

5. I shall now prove further that, *if the series Σa_n is absolutely summable, so is $\Sigma a_n n^{-s}$, where $R(s)>0$:* *i.e.*, that *the region of absolute summability is also a half plane.*

We are assuming now that

$$\int_0^\infty e^{-x}\,|\,u^{(\lambda)}(x)\,|\,dx \tag{10}$$

is convergent for $\lambda=0, 1, 2, \ldots$, and we have to show that

$$\int_0^\infty e^{-x}\,|\,u_s^{(\lambda)}(x)\,|\,dx, \tag{11}$$

where
$$u_s(x)=\sum_1^\infty \frac{a_n x^{n-1}}{n^s(n-1)!},$$

is convergent. Now

$$u_s(x)=\frac{1}{\Gamma(s)}\int_0^1\left\{\log\left(\frac{1}{y}\right)\right\}^{s-1}u(xy)\,dy,$$

$$|\,u_s(x)\,|\leqslant\frac{1}{|\,\Gamma(s)\,|}\int_0^1\left\{\log\left(\frac{1}{y}\right)\right\}^{R(s)-1}|\,u(xy)\,|\,dy.$$

The integral
$$\int_0^\infty e^{-x}\,|\,u(xy)\,|\,dx$$

is uniformly convergent for $0\leqslant y\leqslant 1$; and this enables us to show, by an argument precisely similar to that employed in the preceding section, that

$$\int_0^\infty e^{-x}dx\int_0^1\left\{\log\left(\frac{1}{y}\right)\right\}^{R(s)-1}|\,u(xy)\,|\,dy,$$

and, *a fortiori*,
$$\int_0^\infty e^{-x}\,|\,u_s(x)\,|\,dx,$$

is convergent. Again,

$$u_s^{(\lambda)}(x)=\frac{1}{\Gamma(s)}\int_0^1\left\{\log\left(\frac{1}{y}\right)\right\}^{s-1}u^{(\lambda)}(xy)\,y^\lambda\,dy,$$

and, using this equation in exactly the same way, we can show that the remainder of the integrals (11) are convergent.

We thus establish the existence of a *line of absolute summability*

$$R(s) = \bar{\sigma}:$$

and clearly $$\sigma_0 \leqslant \bar{\sigma}.$$

6. It is evident that the same reasoning might have been applied to the more general series

$$\Sigma a_n \{\phi(n)\}^{-s},$$

provided only that we could find an expression of $\{\phi(n)\}^{-s}$ in the form

$$\{\phi(n)\}^{-s} = \int_0^\infty e^{-nw} \psi_s(w)\,dw = \int_0^1 \psi_s \left\{\log\left(\frac{1}{y}\right)\right\} y^{n-1}\,dy,$$

where ψ_s is a function such that

$$\int_0^1 \left| \psi_s \left\{\log\left(\frac{1}{y}\right)\right\} \right| dy$$

is convergent.

Suppose, for example, that

$$\phi(n) = \theta(n+a) = (n+a)^{a_0} \{\log(n+a)\}^{a_1} \{\log_2(n+a)\}^{a_2} \ldots,$$

the number of factors being finite, $\log_2 n$, $\log_3 n$, ..., denoting $\log\log n$, $\log\log\log n$, ..., and a, a_0, a_1, ... being real, and a and a_0 positive.

Then it may be deduced from a formula given by Pincherle* that

$$\{\theta(k)\}^{-s} = \int_0^\infty e^{-kw} \psi_s(w)\,dw,$$

where $$\psi_s(w) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{-wt} \{\theta(-t)\}^{-s}\,dt;$$

the path of integration being a straight line, and $e_{p-1} < -\lambda < k$, where p is the number of logarithmic factors of $\theta(k)$, and e_0, e_1, e_2, e_3, ... denote the numbers 1, e, e^e, e^{e^e}, ...

Hence we deduce

$$\{\phi(n)\}^{-s} = \int_0^\infty e^{-(n+a)w} \psi_s(w)\,dw$$

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* *Mem. di Bologna* (4), t. 8, pp. 125 *et seq.*

for all values of n greater than a definite n_0. It may further be proved that as $w \to 0$,

$$\psi_s(w) \sim \frac{w^{a_0 s-1}}{\Gamma(a_0 s)} \left\{ \log\left(\frac{1}{w}\right)\right\}^{-a_1 s} \dots;$$

and, using these facts, we are able to extend the conclusions of §§ 4, 5 to the more general series

$$\Sigma \frac{a_n}{(n+a)^{a_0 s}\{\log(n+a)\}^{a_1 s}\dots}.$$

On the other hand, it is by no means true that the regions of summability or of absolute summability of the general series (1) are necessarily half planes. It is sufficient to consider the series

$$\sum_1^\infty (-1)^{n-1} e^{-ns} = \sum_1^\infty (-1)^{n-1} x^n = \frac{x}{1+x}.$$

This series is summable (absolutely) if $R(x) > -1$. If $s = \xi + i\eta$, this condition is

$$e^{-\xi}\cos\eta > -1;$$

and the region of summability is obtained by cutting out of the plane of s an infinite succession of curvilinear areas whose general shape is easily sketched.

7. The series (4) is a simple example of a series summable by Borel's method all over the plane. For it Borel's method is more effective than any of Cesàro's: later on I shall define a large class of series, all of which resemble (4) in this respect. But it must not be imagined that Borel's method, even as applied to ordinary Dirichlet's series, is *always* as effective as even the simplest of Cesàro's; or that, even when the function represented by the series is regular all over the plane, it can always be continued by exponential summation.

Consider, for example, the series

$$(12) \qquad 1^{-s}+0+0+\dots-8^{-s}+0+\dots+27^{-s}+0+\dots,$$

in which $a_n = (-1)^{k-1}$, when $n = k^3$, and $a_n = 0$ otherwise. I shall prove that *this series is summable when, and only when, it is convergent, absolutely summable when, and only when, it is absolutely convergent.* Thus the function

$$(1-2^{1-3s})\,\zeta(3s)$$

represented by the series when $R(s) > 0$, although an integral function of s, cannot be continued by the use of Borel's method.

It is convenient to consider instead of (12) the series

$$0+1^{-s}+0+\ldots-8^{-s}+0+\ldots+27^{-s}+\ldots,$$

which is certainly summable (absolutely summable) if (12) is summable (absolutely summable). Then

$$u(x)=\sum_1^\infty \frac{(-1)^{n-1}}{n^{3s}}\frac{x^{n^3}}{(n^3)!}.$$

I have considered elsewhere* the asymptotic properties of functions of this type, but it will be necessary now to obtain rather more precise information.

We divide the range of integration into two sets of intervals i_ν, j_ν, i_ν being the interval $(\nu-\delta)^3 \leqslant x \leqslant (\nu+\delta)^3$, where δ is a small fixed positive number. We shall consider first whether the series

$$\sum_{(\nu)}\int_{(i_\nu)} e^{-x}u(x)\,dx \tag{13}$$

is convergent.

If $$v_n = n^{-3s}x^{n^3}/(n^3)!,$$

we may write $u(x)$ in the form

$$\begin{aligned} u(x) &= (-1)^{\nu-1}v_\nu+\left(\sum_1^{\nu-1}+\sum_{\nu+1}^\infty\right)(-1)^{n-1}v_n \\ &= (-1)^{\nu-1}v_\nu+S_1+S_2, \end{aligned}$$

say. It is easy to see that, when x lies in i_ν, v_ν is, to a first approximation, of the order of e^x: a more precise approximation will be obtained later. A straightforward application of Stirling's theorem gives the formula

$$v_n/v_{n+1}=(1+\epsilon_n)\,x^{-3n^2-3n-1}\exp\left\{(9n^2+9n+3)\log n+\tfrac{9}{2}(n+1)\right\},$$

where $\epsilon_n \to 0$ as $n\to\infty$. If $n \geqslant \nu$, we have

$$\begin{aligned} x^{-3n^2-3n-1} &> (\nu+\delta)^{-9n^2-9n-3} \geqslant (n+\delta)^{-9n^2-9n-3} \\ &> n^{-9n^2-9n-3}\exp\left\{-\frac{\delta}{n}(9n^2+9n+3)\right\} \\ &> Kn^{-9n^2-9n-3}e^{-9\delta n}, \end{aligned}$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 335 *et seq.*; *Messenger of Math.*, Vol. 39, pp. 28 *et seq.*

and, accordingly, $$v_n/v_{n+1} > Ke^{(\frac{9}{2}-\delta)n} > e^{4n}:$$

and it follows that

(14) $$|S_2| < Ke^{-4\nu}|v_\nu|.$$

Again, $$S_1 = \left(\sum_1^{n_0} + \sum_{n_0+1}^{\nu-1}\right)(-1)^{n-1}v_n = S_1' + S_1'',$$

say. We choose n_0, once for all, so as to justify the application of Stirling's theorem to the terms for which $n > n_0$. Then, if $n_0 < n < \nu$, we have

$$x^{-3n^3-3n-1} < (\nu-\delta)^{-9n^3-9n-3} \leqslant (n+1-\delta)^{-9n^3-9n-3}$$
$$< n^{-9n^3-9n-3} \exp\left[-\left\{\frac{1-\delta}{n} - \frac{(1-\delta)^2}{2n^2}\right\}(9n^3+9n+3)\right]$$
$$< Kn^{-9n^3-9n-3}e^{-9(1-\delta)n},$$

and so $$v_n/v_{n+1} < Ke^{-(\frac{9}{2}-\delta)n} < e^{-4n}.$$

It follows that

(15) $$|S_1''| < Ke^{-4\nu}|v_\nu|.$$

Also S_1'' is a polynomial in n, of degree n_0^3, while v_ν is (roughly) of order e^x or e^{ν^3}. Hence, when ν is large

(16) $$|S_1'| < Ke^{-4\nu}|v_\nu|.$$

From (14), (15), and (16) it follows that, in i_ν,

(17) $$u(x) = (-1)^{\nu-1}v_\nu(1+\rho_\nu),$$

where $$|\rho_\nu| < Ke^{-4\nu}.$$

Again, $$v_\nu = \frac{\nu^{-3s}x^{\nu^3}}{(\nu^3)!} = \frac{\nu^{-3s}(\nu+t)^{3\nu^3}}{(\nu^3)!} \quad (|t| < \delta)$$
$$= \frac{\nu^{-3s}}{\sqrt{(2\pi)}} \exp\{3\nu^3\log(\nu+t) - 3(\nu^3+\tfrac{1}{2})\log\nu + \nu^3 + r_\nu\},$$

where $$|r_\nu| < K/\nu^3.$$

We easily deduce that

(18) $$v_\nu = \frac{\nu^{-3s-\frac{3}{2}}}{\sqrt{(2\pi)}} e^{x-\frac{3}{2}\nu t^2}(1+\epsilon_\nu),$$

where $$|\epsilon_\nu| < K/\nu.$$

From this it follows that

$$|e^{-x}\{u(x)-(-1)^{\nu-1}v_\nu\}| < K\nu^{-3R(s)-\frac{3}{2}}e^{-4\nu},$$

which is obviously, whatever the value of s, even when multiplied by the

length of the interval i_ν, the general term of an absolutely convergent series. Hence the convergence of (13) depends entirely upon that of the series

(19) $$\Sigma(-1)^{\nu-1}\int_{(i^\nu)} e^{-x} v_\nu \, dx,$$

or of

(20) $$\Sigma(-1)^{\nu-1}\nu^{-3s-\frac{3}{2}}\int_{(\nu-\delta)^3}^{(\nu+\delta)^3} e^{-\frac{9}{2}\nu t^2}(1+\epsilon_\nu)\,dx.$$

Now $$\int_{(\nu-\delta)^3}^{(\nu+\delta)^3} e^{-\frac{9}{2}\nu t^2}dx = 3\int_{-\delta}^{\delta}(\nu+t)^2 e^{-\frac{9}{2}\nu t^2}dt$$

$$= 6\int_0^{\delta}(\nu^2+t^2)\,e^{-\frac{9}{2}\nu t^2}dt$$

$$= 6\nu^2(1+\eta_\nu)\int_0^{\infty} e^{-\frac{9}{2}\nu t^2}dt$$

$$= \nu^{\frac{3}{2}}\sqrt{(2\pi)}(1+\eta_\nu),$$

where $$|\eta_\nu| < K/\nu.*$$

Hence the series (19) may be written in the form

(21) $$\sqrt{(2\pi)}\,\Sigma(-1)^{\nu-1}\nu^{-3s}(1+\eta_\nu);$$

and this series is absolutely convergent if $R(s) > \frac{1}{3}$, and conditionally convergent if $0 < R(s) \leqslant \frac{1}{3}$.

It remains to consider the series

(22) $$\Sigma\int_{(j_\nu)} e^{-x}u(x)\,dx,$$

where j_ν is the interval $(\nu+\delta)^3 \leqslant x \leqslant (\nu+1-\delta)^3$. But it follows at once from the work in my note in the *Messenger of Mathematics* quoted above that, in j_ν,

$$|e^{-x}u(x)| < K_1 x^{K_2 R(s)+K_3} e^{-K_4 x^{\frac{1}{3}}},$$

where K_1, K_2, K_3 and K_4 are constants: and from this it follows that the series (22) is absolutely convergent for all values of s. And this completes the proof of the assertion made at the beginning of this section.

The same conclusions may be extended to the series

$$(1^k)^{-s}+0+\ldots-(2^k)^{-s}+0+\ldots+(3^k)^{-s}+0+\ldots,$$

where k is any integer greater than 2.

---

* A smaller limit may be assigned for $|\eta_\nu|$, but this one is sufficient for our present purpose.

8. The series considered in the last section are interesting as examples of the possibilility of Borel's method proving less powerful than Cesàro's. For it follows from investigations of Cesàro and myself* that the summability $(C1)$ of the series

$$a_1+a_2+a_3+\dots$$

involves that of all the series

$$a_1+0+0+a_2+0+0+0+0+a_3+0+\dots,$$
$$a_1+0+0+0+0+0+0+a_2+0+0+0+0+\dots,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

in which a_ν occurs respectively in the ν^2-th, ν^3-th, ... position. Hence, for example, the series (12) is summable $(C1)$ for $-\frac{1}{3} < R(s) \leqslant 0$: it may be proved to be summable (Ck) for $-\frac{1}{3}k < R(s) \leqslant -\frac{1}{3}(k-1)$; and so summable by one or other of Cesàro's means all over the plane.

9. Let us call

$$\Sigma a_n n^{-s} \tag{23}$$

the Dirichlet's series *associated with* the power series

$$\Sigma a_n x^n. \tag{24}$$

If (24) has a radius of convergence greater than unity, (23) is absolutely convergent for all values of s, and represents an integral function of s. If the radius of convergence of (24) is less than unity, (23) is never convergent. If the radius of convergence of (24) is unity, (23) may converge for none or for some or for all values of s; examples are given by the series

$$\Sigma e^{\sqrt{n}} n^{-s}, \quad \Sigma n^{-s}, \quad \Sigma e^{-\sqrt{n}} n^{-s}.$$

I shall assume only that the radius of convergence of (24) is positive: the series then possesses a circle of convergence and a polygon of summability.† And I shall now prove the following theorem :—

If the point $x = 1$ lies within the polygon of summability of the series (24), *the associated Dirichlet's series is summable by Borel's method for all values of s, and represents an integral function of s.*

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* See *Quarterly Journal*, Vol. 38, pp. 269 *et seq.*: Bromwich, *Infinite Series*, pp. 386 *et seq.*

† Borel, *Leçons sur les séries divergentes*, p. 126; Bromwich, *Infinite Series*, p. 295.

Borel's sum is given by the integral

$$\int_0^\infty e^{-x} u_s(x)\, dx, \tag{25}$$

where

$$u_s(x) = \sum_1^\infty \frac{a_n x^{n-1}}{n^s (n-1)!}. \tag{26}$$

Now

$$n^{-s} = \frac{i\Gamma(1-s)}{2\pi} \int_W (-w)^{s-1} e^{-nw}\, dw,^* \tag{27}$$

where $\qquad (-w)^{s-1} = \exp\{(s-1)\log(-w)\},$

the logarithm being real when w is real and negative, and where W represents a contour beginning and ending at the infinitely distant end of the positive real axis and surrounding the positive real axis by a counter-clockwise circuit. We shall suppose that the point of W, where $R(w)$ has its algebraical minimum, is its point of intersection with the negative real axis, where $w = -\delta$, δ being a positive number as small as we please.

The equation (27) holds for all values of s save $s = 1, 2, 3, \ldots$: values which we shall at present omit from consideration.

From (27) we at once deduce

$$\begin{aligned} u_s(x) &= \frac{i\Gamma(1-s)}{2\pi} \int_W (-w)^{s-1} e^{-w} u(xe^{-w})\, dw \\ &= \frac{i\Gamma(1-s)}{2\pi} \int_T (\log t)^{s-1}\, u(xt)\, dt, \end{aligned} \tag{28}$$

where T is a loop from the origin in the plane of T round the point $t=1$, as shown in the figure. The term by term integration here employed is of a type whose justification presents no difficulty.

We shall now prove that

$$\int_0^\infty e^{-x} dx \int_T (\log t)^{s-1} u(xt)\, dt = \int_T (\log t)^{s-1} dt \int_0^\infty e^{-x} u(xt)\, dx; \tag{29}$$

or, in other words, that the integral (25) is convergent, and may be calculated by an inversion of the order of integration.

The integral

$$\int_0^\infty e^{-x} u(xt)\, dx \tag{30}$$

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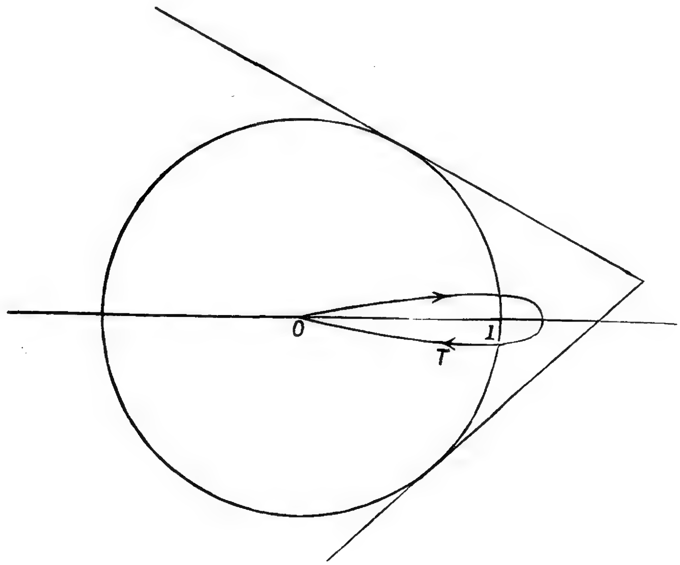
* Whittaker, *Modern Analysis*, p. 182.

is uniformly (and absolutely) convergent throughout any region in the plane of t which lies entirely inside the polygon of summability of the series $\Sigma a_n t^n$. If $t=1$ lies inside the polygon it is evident (see the figure) that we can draw W so that T lies entirely in the polygon, and



then (30) is uniformly convergent for all values of t on T. Our conclusion then follows at once from the auxiliary theorem of § 4.

The proof that the " sum " of the series represents an analytic function regular all over the plane may now be supplied as in § 4. Incidentally we see that the sum is

$$\frac{i\Gamma(1-s)}{2\pi}\int_T (\log t)^{s-1}\,\phi(t)\,dt,$$

where $\phi(t)$ is the function represented by $\Sigma a_n t^n$ and its continuation by Borel's integral.

So far we have assumed that s is not a positive integer: if it is we replace (27) by the equation

$$n^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty w^{s-1}e^{-nw}\,dw,$$

and argue in precisely the same way.

10. The conclusions of the last section may be extended (cf. § 6) to more general series of the type $\Sigma a_n \{\phi(n)\}^{-s}$, such as

$$\sum_{n_0}^{\infty} \frac{a_n}{(n+a)^{\alpha_0 s}\{\log(n+a)\}^{\alpha_1 s}\dots}.$$

It is to be observed that they are quite independent of any assumption that the Dirichlet's series is ever *convergent:* thus they apply to such series as

$$\Sigma(-1)^{n-1}a^n n^{-s}, \quad \Sigma(-1)^{n-1}a^{2n+1}(2n+1)^{-s},$$

where a is any positive number, however large. But certainly the most interesting case of the theorem is:

If $\Sigma a_n x^n$ is convergent for $|x| < 1$, and the function represented by the series is regular for $x = 1$, then the associated Dirichlet's series is summable for all values of s, and represents an integral function of s.

The last assertion (having no reference to summability) is easily established directly. As examples of series which satisfy these conditions, we may mention

$$\Sigma\frac{\cos n\theta}{n^s}, \quad \Sigma\frac{\sin n\theta}{n^s}, \tag{31}$$

$$\Sigma\frac{1}{n^{\beta_0}(\log n)^{\beta_1}\dots}\frac{e^{ni\theta}}{n^s},$$

where

$$\theta \neq 0 \pmod{2\pi}.$$

With the help of the remarks made at the beginning of this section the conclusion may be extended to such series as, *e.g.*,

$$\Sigma\frac{\xi^n e^{ni\theta}}{(n+b)^{\beta_0}\{\log(n+b)\}^{\beta_1}\dots[(n+a)^{\alpha_0}\{\log(n+a)\}^{\alpha_1}\dots]^s}.$$

11. It is instructive to verify the results of §§ 9, 10 in the case of the simple series (31). Writing $2\pi\phi$ for θ, and using Hurwitz's formula*

$$\zeta(s, \phi) = 2\Gamma(1-s)(2\pi)^{s-1}\sum_1^{\infty} n^{-(1-s)}\sin\{(2n\phi+\tfrac{1}{2}s)\pi\}, \tag{32}$$

where

$$0 < \phi < 1, \quad R(s) < 1,$$

and $\zeta(s, \phi)$ is the generalised Riemann ζ-function defined by the series

$$\phi^{-s}+(\phi+1)^{-s}+(\phi+2)^{-s}+\dots,$$

* Lindelöf, *Le calcul des résidus*, p. 107.

and its continuations, we see that

$$C_s(\phi) = \sum_1^\infty \frac{\cos 2n\pi\phi}{n^s} = \frac{(2\pi)^{1-s}}{4\Gamma(1-s)\sin\frac{1}{2}s\pi}\{\zeta(s,\ \phi)+\zeta(s,\ 1-\phi)\},$$

$$S_s(\phi) = \sum_1^\infty \frac{\sin 2n\pi\phi}{n^s} = \frac{(2\pi)^{1-s}}{4\Gamma(1-s)\cos\frac{1}{2}s\pi}\{\zeta(s,\ \phi)-\zeta(s,\ 1-\phi)\}.$$

The functions $\zeta(s, \phi)$, $\zeta(s, 1-\phi)$ are each regular save for a simple pole, with unit residue, at $s = 1$. It is clear that the only possible singularities of C and S are simple poles at the points $s = 0, -2, -4, \ldots$ and $s = -1, -3, -5, \ldots$ respectively. To verify that C and S are in fact regular at these points, we have to show that

$$\zeta(-2k,\ \phi)+\zeta(-2k,\ 1-\phi) = 0, \quad \zeta(-2k-1,\ \phi) = \zeta(-2k-1,\ 1-\phi):$$

and these equations are easily verified by the help of known results in the theory of the Zeta and Bernoullian functions.*

12. If $\Sigma a_n x^n$ is regular for $|x| < 1$, and has a simple pole for $x = 1$, then $\Sigma a_n n^{-s}$ is regular all over the plane except for a simple pole at $s = 1$. If $\Sigma a_n x^n$, at $x = 1$, behaves like

$$(1-x)^a\, \phi(x),$$

where $\phi(x)$ is regular, then $\Sigma a_n n^{-s}$ has simple poles for

$$s = -a, \quad -1-a, \quad -2-a, \quad \ldots,$$

unless a is integral. If a is a positive integer, there are no poles; if a negative integer, the poles are

$$1,\ 2,\ \ldots,\ -a.$$

These results follow at once from the consideration of the equation

$$\Sigma a_n n^{-s} = \frac{i}{2\Gamma(s)\{\sin(a+s)\pi\}}\int_W (-w)^{a+s-1}\chi(e^{-w})\,dw,$$

where $\chi(e^{-w})$ is a function regular for $w = 0$. I do not imagine that they are new: but they are worth stating here in connection with the results of §§ 9–11.

* See Barnes, *Messenger of Mathematics*, Vol. XXIX, pp. 74 *et seq*, 88 *et seq*.

CORRECTIONS

p. 289, *line* 4. For i^ν read i_ν.

p. 291, *line* 9 *up*. For T read t.

p. 293, *line* 6. Read n^{-s}.

—— *line* 11 *up*. Read n^{β_0}.

Corrections to footnote, pp. 279–81.

p. 279, *line* 8. For Theorem I read Theorem A.

p. 280, *line* 4. For f read f_j.

p. 281, *line* 1. For A^k read A_j^k.

—— *line* 7 *up*. For Vol. 2 read Vol. 4. The reference is to 1907, 2.

COMMENTS

The addendum at the foot of pp. 279–81 contains corrections to 1908, 1, and is an important part of that paper; see the Comments on 1908, 1.

In the quotation from Bromwich in § 2, it is to be understood that the hypothesis $f_n \to 0$ is retained.

The formula, § 2, for the (C, k)-abscissa s_k of $\sum a_n/n^s$, should be

$$s_k = \overline{\lim}\{\log|S_n^k k!\, n^{-k}|/\log n\},$$

and is valid when either side of the equation is positive; see Bohr.† If we apply the formula to the series $\sum a_n'/n^{s'}$, where $a_n' = n^\lambda a_n$, $s' = s+\lambda$, $\lambda > 0$, we obtain $s_k = s_k' - \lambda$, when either side is $> -\lambda$.

In § 3, and elsewhere, *absolute summability* is defined in Borel's sense; see the Comments on 1904, 4.

In § 4, it is assumed that the integral (8) is uniformly convergent for $0 \leqslant y \leqslant 1$, if it is convergent for $y = 1$. For $0 < \delta \leqslant y \leqslant 1$ this was proved by Phragmén,‡ but his proof does not apply to the range $0 \leqslant y \leqslant \delta$; see the Comments on 1904, 4. A complete proof is given in 1911, 8; see also D.S., Theorems 129 and 130. A similar assumption in § 5 may be justified by the same argument.

In § 9, it is stated that the integral (30) is 'uniformly (and absolutely) convergent throughout any region in the plane of t which lies entirely inside the polygon of summability'. This was stated by Bromwich (1st edn., p. 296), for uniform convergence, but he said that it follows from Phragmén's theorem, which he had misquoted. Hardy indicates a proof in 1911, 8, and gives a complete proof in D.S., Theorem 133.

The properties of the series (12), developed in §§ 7–8, are used in 1916, 8, § 3. In terms of Riesz's typical means, the first group of statements in § 8 may be expressed by saying that summability $(R, n, 1)$ implies summability $(R, n^2, 1)$, $(R, n^3, 1)$,.... This follows from case (b) of the Cesàro–Hardy theorem in 1907, 5. Case (a) of the theorem shows that the inferences are reversible. The statements about the series (12) may be expressed by saying that the series $\sum(-1)^{n-1}n^{-3s}$ is summable $(R, n^3, 1)$, i.e. $(R, n, 1)$, for $3R(s) > -1$, and summable (R, n^3, k) for $3R(s) > -k$. The equivalence of (R, n^3, k) and (R, n, k), $k = 2, 3, \ldots$, is proved in 1916, 5.

† *Bidrag* . . ., pp. 85–93, English translation, pp. 74–81.

‡ *Comptes rendus* 132 (1901), 1396–9.

THEOREMS RELATING TO THE SUMMABILITY AND CONVERGENCE OF SLOWLY OSCILLATING SERIES

By G. H. Hardy.

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1. When, in this paper, I say that a series is summable, I mean "summable by mean values." The best known and most useful of methods of summation by mean values is that due to Cesàro. If

$$\begin{aligned} s_n &= a_1 + a_2 + \ldots + a_n, \\ s_n^{(1)} &= s_1 + s_2 + \ldots + s_n, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ s_n^{(r)} &= s_1^{(r-1)} + s_2^{(r-1)} + \ldots + s_n^{(r-1)}, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

and
$$(r!\, s_n^{(r)})/n^r \to s,$$

as $n \to \infty$, the series Σa_n is said to be "summable by Cesàro's mean of the r-th order." For the sake of brevity I shall say that such a series is "*summable* (Cr)." A series which is summable (Cr) is certainly summable (Ck) for all values of k greater than r.*

It is easy to see that if Σa_n is summable (Cr), then

$$a_n/n^r \to 0 :\dagger$$

and this result shows that the scope of Cesàro's method is somewhat restricted. For example, the series

$$a - a^2 + a^3 - \ldots \quad (a > 1)$$

cannot be summable (Cr) for any value of r.

2. Thus Cesàro's method is inapplicable if, to put it roughly, $|a_n|$ is *too large* when n is large. It does not, however, appear to have been observed that it is also inapplicable if $|a_n|$ is *too small* when n is large.

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* Bromwich, *Infinite Series*, p. 312.

† *Ibid.*, p. 318.

That this is so appears from the theorem :—

If $na_n \to 0$ *as* $n \to \infty$, *then the series* Σa_n *cannot be summable by any of Cesàro's means unless it is convergent.*

This theorem is an immediate corollary from well known theorems of Frobenius, Hölder, Cesàro, and Tauber. If Σa_n is summable (Cr), and s is its sum, then

$$\Sigma a_n x^n \to s,$$

as $x \to 1$.* But the last equation, together with $na_n \to 0$, involves the convergence of Σa_n.†

3. It is, however, possible to prove a more general theorem, viz.:

If $|na_n| < K$, *then the series* Σa_n *cannot be summable by any of Cesàro's means unless it is convergent.*

This theorem I shall now proceed to prove: the extension is of some importance in that it enables us to deal with a particularly interesting case, viz., that of the series

$$\Sigma n^{-1-ai}$$

(where a is real).

Suppose first that $r = 1$. Then

$$(1) \qquad \frac{s_1+s_2+\ldots+s_n}{n} = \frac{na_1+(n-1)\,a_2+\ldots+a_n}{n}$$

$$= \frac{n+1}{n}\,s_n - \frac{a_1+2a_2+\ldots+na_n}{n}.$$

If we put

$$(2) \qquad na_n = b_n,$$

and denote by t_n, $t_n^{(1)}$, ... the functions of n formed from the b's as are the s's from the a's, we may write the equation (1) in the form

$$(3) \qquad s_n^{(1)}/n = (n+1)\,s_n/n - t_n/n.$$

Hence, if Σa_n is summable $(C1)$, the necessary and sufficient condition that it should be convergent is that

$$(4) \qquad t_n/n \to 0.‡$$

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* Bromwich, *Infinite Series*, p. 313.

† *Ibid.*, p. 251.

‡ This is a known result: see Bromwich, *l.c.*; Vivanti, *Theorie der Analytischen Funktionen*, p. 429; and the papers by Tauber and Pringsheim there quoted.

I shall now prove that the conditions

$$s_n^{(1)}/n \to s, \quad |b_n| < K,$$

necessarily involve (4), and so the convergence of the series.

We have
$$\sum_1^n a_\nu = \sum_1^n \frac{b_\nu}{\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} + \frac{t_n}{n},$$

where
$$\Delta\phi(\nu) \equiv \phi(\nu) - \phi(\nu+1).$$

Hence

(5)
$$\sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} = \frac{ns_n - t_n}{n} = \frac{s_{n-1}^{(1)}}{n};$$

and so the summability ($C1$) of the series involves the convergence (to the same sum) of the series

(6)
$$\sum_1^\infty \frac{t_\nu}{\nu(\nu+1)}.$$

Now suppose that t_ν/ν does not tend to the limit zero. We may, without real loss of generality, restrict ourselves to the case in which a_n is real. For, if $a_n = \alpha_n + i\alpha_n'$, the series Σa_n is convergent or summable if, and only if, $\Sigma \alpha_n$ and $\Sigma \alpha_n'$ are both convergent or summable. If $|na_n| < K$, then α_n and α_n' satisfy the same condition. Finally, if t_ν/ν has not the limit zero, the same must be true of at least one of τ_ν/ν and τ_ν'/ν, where $t_\nu = \tau_\nu + i\tau_\nu'$, and τ_ν and τ_ν' are formed from α_ν and α_ν' just as is t_ν from a_ν.

Thus we suppose a_ν real. Then we can find values of N, as large as we like, and such that

(7)
$$t_N > K_1 N$$

(or $t_N < -K_1 N$), K_1 being a positive constant which we may without loss of generality suppose less than K.

Let N_1 be the least integer such that

$$N_1 \geqslant \left(1 - \frac{K_1}{2K}\right) N = K_2 N,$$

say. Then also $N_1 < K_3 N$, where K_3 is another constant which we may suppose as nearly equal to K_2 as we like, and so certainly less than 1.

If $N_1 \leqslant n \leqslant N$, we have

$$|t_N - t_n| = |b_{n+1} + b_{n+2} + \dots + b_N| < (N - N_1) K \leqslant \tfrac{1}{2} K_1 N,$$

and so, by (7),
$$t_n \geqslant t_N - |t_N - t_n| > \tfrac{1}{2} K_1 N.$$

Hence $$\sum_{N_1}^{N} t_\nu \Delta \frac{1}{\nu} > \tfrac{1}{2}K_1 N \sum_{N_1}^{N} \Delta \frac{1}{\nu} = \tfrac{1}{2}K_1 N \left(\frac{1}{N_1} - \frac{1}{N+1}\right)$$

$$> \tfrac{1}{2}K_1 \left(\frac{1}{K_3} - 1\right) = K_4,$$

say: and this inequality is evidently inconsistent with the convergence of the series (6). Hence the theorem is proved when $r = 1$.

4. I shall now prove that *a series for which* $|na_n| < K$ *cannot be summable* (Cr) *without being summable* $(C, r-1)$.

It is easy to verify that

$$s_n^{(r)} = \binom{r+n-1}{r} a_1 + \binom{r+n-2}{r} a_2 + \ldots + \binom{r}{r} a_n,$$

$$s_n^{(r-1)} = \binom{r+n-2}{r-1} a_1 + \binom{r+n-3}{r-1} a_2 + \ldots + \binom{r-1}{r-1} a_n,$$

$$t_n^{(r-1)} = \binom{r+n-2}{r-1} a_1 + \binom{r+n-3}{r-1} 2a_2 + \ldots + \binom{r-1}{r-1} na_n.$$

By means of these equations it is easily proved that

(8) $$rs_n^{(r)} + t_n^{(r-1)} = (n+r)\, s_n^{(r-1)},$$

(9) $$rs_n^{(r)} + t_{n+1}^{(r-1)} = (n+1)\, s_{n+1}^{(r-1)}.$$

The second of these equations is a generalisation of the second of the equations (5), to which it reduces if we put $r=1$ and write $n-1$ for n. The first may be written in the form

$$r\frac{s_n^{(r)}}{n^r} = \left(\frac{n+r}{n}\right)\frac{s_n^{(r-1)}}{n^{r-1}} - \frac{t_n^{(r-1)}}{n^r},$$

which is a generalisation of (3), and shows that *the necessary and sufficient condition that a series, known to be summable by r means, should be summable by r*—1 *means, is that*

$$t_n^{(r-1)}/n^r \to 0$$

as $n \to \infty$.

Again, $$s_n = \sum_1^n \frac{b_\nu}{\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} + \frac{t_n}{n}$$

$$= \sum_1^{n-2} t_\nu^{(1)} \Delta^2 \frac{1}{\nu} + t_{n-1}^{(1)} \Delta \frac{1}{n-1} + \frac{t_n}{n}$$

$$= \ldots \quad \ldots \quad \ldots \quad \ldots,$$

or

$$(10) \qquad \sum_{1}^{n-r} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} = s_n - \sum_{0}^{r-1} t_{n-k}^{(k)} \Delta^k \frac{1}{n-k}.$$

Now, by (5),

$$s_n - \frac{t_n}{n} = \frac{s_{n-1}^{(1)}}{n};$$

and it is easily verified that

$$s_n - \frac{t_n}{n} - t_{n-1}^{(1)} \Delta \frac{1}{n-1} = \frac{2s_{n-2}^{(2)}}{n(n-1)}.$$

These equations suggest that

$$(11) \qquad s_n - \sum_{0}^{r-1} t_{n-k}^{(k)} \Delta^k \frac{1}{n-k} = \frac{r!\,(n-r)!}{n!} s_{n-r}^{(r)}.$$

That this is so may be verified by induction. The equation (11), assumed true for $r=p$, will be true for $r=p+1$ also if only

$$\frac{p!\,(n-p)!}{n!} s_{n-p}^{(p)} - \frac{(p+1)!\,(n-p-1)!}{n!} s_{n-p-1}^{(p+1)} = t_{n-p}^{(p)} \Delta^p \frac{1}{n-p}.$$

But
$$\Delta^p \frac{1}{n-p} = \frac{p!\,(n-p-1)!}{n!},$$

so that the equation which we have to verify is

$$(n-p)\, s_{n-p}^{(p)} - (p+1)\, s_{n-p-1}^{(p+1)} = t_{n-p}^{(p)},$$

and if we write r for $p+1$, and then n for $n-r$, this reduces to (9). Thus (11) is established.

From (10) and (11) it follows that

$$(12) \qquad \sum_{1}^{n-r} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} = \frac{r!\,(n-r)!}{n!} s_{n-r}^{(r)}.$$

Hence *the necessary and sufficient condition that the series Σa_n should be summable by Cesàro's mean of the r-th order is that the series*

$$(13) \qquad \sum_{1}^{\infty} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu}$$

should be convergent.

5. I shall now prove that, for the special class of series that we are considering, the convergence of (13) involves

$$t_n^{(r-1)}/n^r \to 0;$$

and therefore the summability of the series Σa_n by $r-1$ means. As before, we are at liberty to suppose a_n real.

Suppose that the last relation is untrue. Then we can find a value of N, as large as we like, and such that

(14) $$t_N^{(r-1)} > K_1 N^r,$$

or $$t_N^{(r-1)} < -K_1 N^r,$$

say the former. We may, as in § 3, suppose $K_1 < K$; and we choose N_1 as in § 3. Let $N_1 \leqslant n \leqslant N$. Then

$$t_N^{(r-1)} - t_n^{(r-1)} = \sum_1^N \binom{r-1+N-s}{r-1} sa_s - \sum_1^n \binom{r-1+n-s}{r-1} sa_s$$

$$= \sum_1^n \left\{ \binom{r-1+N-s}{r-1} - \binom{r-1+n-s}{r-1} \right\} sa_s$$

$$+ \sum_{n+1}^N \binom{r-1+N-s}{r-1} sa_s.$$

Both the coefficients of sa_s are positive; and so the sum is certainly increased numerically by writing K throughout in place of sa_s. Hence

$$| t_N^{(r-1)} - t_n^{(r-1)} | < K \left[\sum_1^n \left\{ \binom{r-1+N-s}{r-1} - \binom{r-1+n-s}{r-1} \right\} + \sum_{n+1}^N \binom{r-1+N-s}{r-1} \right]$$

$$= K \sum_1^{N-n} \binom{r-1+N-s}{r-1}$$

$$< K(N-n) \binom{r+N-2}{r-1} < K(N-n)\, N(N+1) \dots (N+r-2)$$

$$< \bar{K}(N-N_1)\, N^{r-1}$$

(15) $$< \tfrac{1}{2} K_1 \bar{K} N^r / K = \tfrac{1}{2} \bar{K}_1 N^r,$$

where $\bar{K}$, $\bar{K}_1$ are constants which we may suppose greater than K, K_1 by as little as we please. From (14) and (15) it follows that

$$t_n^{(r-1)} > \tfrac{1}{2} K_1' N^r \quad (N_1 \leqslant n \leqslant N),$$

K_1' being a constant which we may suppose less than K_1 by as little as

we please. Thus

$$\sum_{N_1}^{N} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} > \tfrac{1}{2} K_1' N^r \sum_{N_1}^{N} \Delta^r \frac{1}{\nu}$$

$$= \tfrac{1}{2} K_1' N^r (r-1)! \left\{ \frac{1}{N_1(N_1+1)\dots(N_1+r-1)} - \frac{1}{(N+1)(N+2)\dots(N+r)} \right\}.$$

But
$$\frac{1}{N_1(N_1+1)\dots(N_1+r-1)} > \frac{K_5}{N_1^r} > \frac{K_5}{K_3^r N^r},$$

$$\frac{1}{(N+1)(N+2)\dots(N+r)} < \frac{1}{N^r},$$

K_5 being a constant less than 1 by as little as we please. Hence, finally,

$$\sum_{N_1}^{N} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} > \tfrac{1}{2} K_1' (r-1)! \left\{ \frac{K_5}{K_3^r} - 1 \right\} > K_6,$$

say: and this inequality is evidently incompatible with the convergence of the series (13).

6. It follows that, if $|na_n| < K$, the series Σa_n, if summable (Cr), is summable $(C, r-1)$: and so the series is summable $(C1)$, and so convergent. That is to say, *a series for which* $|na_n| < K$ *cannot be summable by Cesàro's means unless it is convergent.*

The following examples of this theorem are interesting.

(i) The series
$$\Sigma n^{-1-\alpha i} \quad (\alpha \gtrless 0)$$

is known not to be convergent for any value of α. It follows that it is not summable by any of Cesàro's means: *i.e.*, that the region of definition of the Riemann Zeta-function cannot be extended by Cesàro's method of summation.

This result has been stated by Riesz,* who deduces it from a general theorem in analytic function-theory.

The simplest proof of this particular result, however, is obtained by considering the function

$$\Sigma n^{-1-\alpha i} x^n.$$

* *Comptes Rendus*, June 21, 1909.

As $x \to 1$, this function behaves like*

$$\Gamma(-ai)\left(\log \frac{1}{x}\right)^{ai},$$

and so does not tend to a limit. It follows, by well known theorems, that the series cannot be summable by Cesàro's means, nor yet by Borel's exponential method, since summability in any of these ways would involve the existence of such a limit.†

(ii) Fejér has shown that the Fourier's series corresponding to any continuous‡ function is summable (C1). Now suppose that $f(x)$ is not only continuous, but also monotonic. Then, by the second theorem of mean value,

$$\int_0^{2\pi} f(x)\cos nx\,dx = f(0)\int_0^{\xi}\cos nx\,dx + f(2\pi)\int_{\xi}^{2\pi}\cos nx\,dx,$$

which is numerically less than K/n. Hence the Fourier's series satisfies the condition $|na_n| < K$, and is therefore *convergent*. This result may be immediately extended to any case in which $f(x)$ is continuous and of limited total fluctuation.

In short, the theorem proved in the last section enables us to present Dirichlet's classical result as a mere corollary of Fejér's theorem.

7. The theorem of § 5 naturally suggests the following question: cannot Tauber's theorem itself be generalised by replacing the condition $na_n \to 0$ by the more general condition $|na_n| < K$; *i.e., is it not true that the conditions* (i) *that* $\Sigma a_n x^n$ *tends to a limit* s *as* $x \to 1$, *and* (ii) *that* $|na_n| < K$, *involve the convergence of the series* Σa_n*?*

I have not been able to prove this theorem, and I am inclined to think that it is not true. At the same time, I have been quite unsuccessful in my attempts to construct an example to the contrary.

8. The fact that such finitely oscillating series§ as

$$\Sigma n^{-1-ai},\quad \Sigma n^{-1}(\log n)^{-1-ai},\quad \ldots,$$

cannot be summed by Cesàro's means suggests a variety of interesting questions. The first is whether we can devise other analogous, purely

* *Proc. London Math. Soc.*, Vol. 3, p. 385.

† Bromwich, *Infinite Series*, pp. 291, 313.

‡ This hypothesis has been very widely generalised: see Hobson, *The Theory of Functions of a Real Variable*, pp. 707 *et seq.*

§ For proofs that these series oscillate finitely see a paper by Dr. Bromwich, *Proc. London Math. Soc.*, Vol. 6, pp. 327 *et seq.*, where further references are given.

arithmetical, methods of summation which will be more effective in such cases. This question has in part been answered by Riesz,* who has devised methods of summation which are generalisations of all of Cesàro's. In what follows we shall be concerned solely with the generalisation of Cesàro's first method.

Let μ_n be the general term of a divergent series of positive terms which, from a certain value of n onwards, tend steadily to zero, so that, in the notation of Du Bois Reymond's *Infinitärcalcül*,† we may write

$$\mu_n \prec 1.‡$$

And let
$$\lambda_n = \mu_1 + \mu_2 + \ldots + \mu_n,$$

so that $\lambda_{n+1}/\lambda_n \to 1$. Then we shall say that the series is *summable* $(R1, \mu_n)$ if
$$(\mu_2 s_1 + \mu_3 s_2 + \ldots + \mu_{n+1} s_n)/\lambda_{n+1}$$

tends to a limit s as $n \to \infty$, and we shall call s the *sum* of the series.

An equivalent way of stating the definition is to demand that

$$s_n - \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_{n+1}} \to s$$

as $n \to \infty$.

If we put $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 1$, ..., $\lambda_n = n$ we are led to Cesàro's definition. The case of most interest after Cesàro's is that in which

$$\mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = \tfrac{1}{2}, \quad \ldots, \quad \mu_{n+1} = \frac{1}{n}, \quad \ldots,$$

when $\lambda_{n+1} \sim 1/n$. Our sum is then defined as

$$\lim_{n\to\infty} \left(\frac{s_1}{1} + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \Big/ \log n.§$$

It is known‖ that, if Σb_n, Σc_n are two divergent series of positive terms, such that
$$c_n/c_{n+1} \geqslant b_n/b_{n+1}$$

* *Comptes Rendus*, July 5, 1909.

† For an exposition of the ideas and notation of the *Infinitärcalcül*, see my "Orders of Infinity" (*Camb. Math. Tracts*, No. 12, 1910).

‡ This restriction on the definition is not essential: we might suppose $\mu_n = \log n$, n, ... But a theorem quoted later in this section shows that if the definition is effective for such a form of μ_n it will also be effective when $\mu_n = 1$, when the definition reduces to Cesàro's. Thus nothing can be gained by considering such cases.

§ I had framed this definition before seeing Riesz's note.

‖ *Quarterly Journal*, Vol. 38, p. 271; Bromwich, *Infinite Series*, p. 386. Both parts of the theorem there proved (and not merely the first part, of which alone use is made here) are really due to Cesàro [*Rom. Acc. Lincei Rend.* (IV), 4, 1888, p. 452].

for all values of n, then the equation

$$\frac{b_1 s_1 + b_2 s_2 + \ldots + b_n s_n}{b_1 + b_2 + \ldots + b_n} \to s$$

implies the equation
$$\frac{c_1 s_1 + c_2 s_2 + \ldots + c_n s_n}{c_1 + c_2 + \ldots + c_n} \to s.$$

For practical purposes this is equivalent to the assertion: *if* Σa_n *is summable* $(R1, \mu_n)$ *it is summable* $(R1, m_n)$ *for any form of* m_n, *such that*

$$m_n \prec \mu_n.$$

For example, a series *summable* $(R1, 1)$ is certainly *summable* $(R1, 1/n)$: *i.e.*, a series summable $(C1)$ is summable by the special form of Riesz's method alluded to above. *The efficacy of the method rises as the rapidity of decrease of* μ_n *rises*—provided of course μ_n does not decrease so rapidly that $\Sigma \mu_n$ ceases to diverge. Thus, by taking

$$\mu_n = 1,\ \frac{1}{n},\ \frac{1}{n \log n},\ \frac{1}{n \log n \log \log n},\ \ldots,$$

we obtain a series of methods of summation of increasing efficacy.

9. We have seen that Σn^{-1-ai} is not summable $(C1)$. It is, however, as Riesz has pointed out, summable $(R1, 1/n)$; as is almost immediately deducible from the formula

$$\sum_1^n \nu^{-1-ai} = -\frac{n^{-ai}}{ai} + \zeta(1+ai) + O\left(\frac{1}{n}\right),$$

where, in Landau's notation, $O\{\phi(n)\}$ denotes a function of n whose modulus is less than $K\phi(n)$ for all values of n from a certain value onwards. Another simple example of a series summable $(R1, 1/n)$ is

$$1-1+0+1+0+0+0-1+0+\ldots,$$

where the non-zero terms occur in the 1st, 2nd, 4th, 8th, ... places. This series is not summable $(C1)$ or (Cr) or by Borel's method, since the series

$$x - x^2 + x^4 - x^8 + \ldots$$

oscillates as $x \to 1$.*

* *Quarterly Journal*, Vol. 38, pp. 276 *et seq.*; Bromwich, *Infinite Series*, p. 389.

10. The summability $(R1, \mu_n)$ of Σa_n does not, as we have seen, involve the existence of a limit for $\Sigma a_n x^n$ as $x \to 1$. But there is an interesting theorem with respect to summability $(R1, \mu_n)$ which is a generalisation of Frobenius's theorem concerning summability $(C1)$; and corresponding theorems which are generalisations of Tauber's theorem and the theorem of § 3 (so far as it applies to summation by Cesàro's *first* mean).

These theorems I shall now proceed to state and to prove.

(i) *Analogue of Frobenius's Theorem.—If Σa_n is summable $(R1, \mu_n)$ to the sum s, and*

$$s_n e^{-\lambda_n x} \to 0,$$

as $n \to \infty$, for any positive value of x, then $\Sigma a_n e^{-\lambda_n x}$ is convergent for $x > 0$, and

$$\Sigma a_n e^{-\lambda_n x} \to s$$

*as $x \to 0$.**

(ii) *Analogue of Tauber's Theorem.—If $\lambda_n a_n / \mu_n \to 0$, and the series $\Sigma a_n e^{-\lambda_n x}$ (then certainly convergent for $x > 0$) tends to the limit s as $x \to 0$, then Σa_n is convergent to the sum s.*

(iii) *Analogue of the Theorem of § 3.—If $|\lambda_n a_n / \mu_n| < K$, the series Σa_n cannot be summable $(R1, \mu_n)$ without being convergent.*

The second of these theorems has been proved by Landau :† I shall now proceed to prove the others.

11. We have

$$\sum_1^n a_\nu e^{-\lambda_\nu x} = \sum_1^{n-1} s_\nu \Delta e^{-\lambda_\nu x} + s_n e^{-\lambda_n x}.$$

* In Frobenius's theorem there is no condition similar to the second of these conditions. In fact the existence of a limit for $(s_1 + s_2 + \ldots + s_n)/n$ *implies* that $s_n e^{-nx} \to 0$ for any positive x. This is not so when the divergence of $\Sigma \mu_n$ is very slow.

Suppose Σa_n summable $(R1, \mu_n)$. Then it is easy to see that

$$s_n = (\epsilon_{n+1} \lambda_{n+1} - \epsilon_n \lambda_n)/\mu_{n+1},$$

where $\epsilon_n \to 0$, and that the condition that $s_n e^{-\lambda_n x} \to 0$ will certainly be satisfied if (to put the matter roughly) λ_n/μ_n increases more slowly than any positive power of e^{λ_n}. This condition is satisfied if $\mu^n = 1$, $\lambda_n = n$: but not if $\mu_n = 1/n$, $\lambda_n \sim \log n$.

† *Monatshefte für Mathematik und Physik*, Jahrgang XVIII, p. 12.

The last term tends to zero. Also

(16) $$\Delta e^{-\lambda_\nu x} = e^{-\lambda_\nu x} - e^{-\lambda_{\nu+1} x}$$

$$= x \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-xt}\,dt < x\mu_{\nu+1} e^{-\lambda_\nu x}.$$

Now $\Sigma \mu_{\nu+1} \lambda_\nu^{-\xi}$ is convergent for $\xi > 1$:* *a fortiori* $\Sigma \mu_{\nu+1} e^{-\lambda_\nu x}$ is convergent for any positive x. Also, for any positive value of r, $|s_\nu e^{-\lambda_\nu r}| < K$, where K depends only on r, and

$$|s_\nu \Delta e^{-\lambda_\nu x}| < Kx\mu_{\nu+1} e^{-\lambda_\nu (x-r)},$$

so that $\Sigma s_\nu \Delta e^{-\lambda_\nu x}$ is absolutely convergent for $x > r$: as r is arbitrarily small, this condition may be replaced by $x > 0$. It follows that

(17) $$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty s_\nu \Delta e^{-\lambda_\nu x}.$$

Hence $$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty \mu_{\nu+1} s_\nu \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right)$$

$$= \sum_1^\infty (\mu_2 s_1 + \ldots + \mu_{\nu+1} s_\nu)\, \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right),$$

provided only that

$$(\mu_2 s_1 + \ldots + \mu_{n+1} s_n)\left(\frac{1}{\mu_{n+1}} \Delta e^{-\lambda_n x}\right) \to 0:$$

and as $\mu_2 s_1 + \ldots + \mu_{n+1} s_n \sim \lambda_{n+1} s$ and $\lambda_{n+1}/\lambda_n \to 1$, this follows at once from (16). Thus we have

(18) $$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty (s + \epsilon_\nu)\, \lambda_{\nu+1} \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right),$$

where $\epsilon_n \to 0$ as $n \to \infty$.

If we had applied the same analysis to the particular case in which

$$a_1 = 1, \quad a_2 = a_3 = \ldots = 0, \quad s_n = 1,$$

we should have arrived at the equation

(19) $$e^{-\lambda_1 x} + \frac{\mu_1}{\mu_2} \Delta e^{-\lambda_1 x} = \sum_1^\infty \lambda_{\nu+1} \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right).$$

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* Abel, *Œuvres*, t. II, p. 198; Bromwich, *Infinite Series*, p. 32.

Before proceeding further, we observe that

$$\frac{\mu_{\nu+2}\Delta e^{-\lambda_\nu x}}{\mu_{\nu+1}\Delta e^{-\lambda_{\nu+1}x}} = \frac{\mu_{\nu+2}(e^{\mu_{\nu+1}x}-1)}{\mu_{\nu+1}(1-e^{-\mu_{\nu+2}x})} \geqslant 1,$$

so that

$$\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right) \geqslant 0 \tag{20}$$

for all values of ν and x.

From (18) and (19) it follows that

$$\Sigma a_\nu e^{-\lambda_\nu x} - se^{-\lambda_1 x} - \frac{\mu_1 s}{\mu_2}\Delta e^{-\lambda_1 x} = \sum_1^\infty \epsilon_\nu \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right),$$

$$\begin{aligned}|\Sigma a_\nu e^{-\lambda_\nu x} - s| < |s|(1-e^{-\lambda_1 x}) + \frac{\mu_1}{\mu_2}|s|\Delta e^{-\lambda_1 x} \\ + \sum_1^N \left|\epsilon_\nu \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right)\right| \\ + \delta_N \sum_1^\infty \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right),\end{aligned}$$

where δ_N is the greatest value of $|\epsilon_n|$ for $n > N$. The last term is less than $\delta_N\{1+(\mu_1/\mu_2)\}$, and so N can be chosen so as to make it as small as we please. After N is fixed we can choose x so as to make the remaining terms as small as we please: and so

$$\Sigma a_\nu e^{-\lambda_\nu x} \to s$$

as $x \to 0$.

12. We have now to prove the extension of the theorem of § 3. From the equation

$$\frac{\mu_2 s_1 + \mu_3 s_2 + \ldots + \mu_{n+1} s_n}{\lambda_{n+1}} = s_n - \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_{n+1}},$$

it follows that the necessary and sufficient condition that a series, known to be summable ($R1$), shall be convergent, is that

$$\frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_{n+1}} \to 0. \tag{21}$$

Let $$b_n = \lambda_n a_n, \quad t_n = b_1 + b_2 + \ldots + b_n.$$

Then $$s_n = \sum_1^n \frac{b_\nu}{\lambda_\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\lambda_\nu} + \frac{t_n}{\lambda_n},$$

$$\sum_1^{n-1} t_\nu \Delta \frac{1}{\lambda_\nu} = s_n - \frac{t_n}{\lambda_n} = \frac{\mu_2 s_1 + \ldots + \mu_{n+1} s_n}{\lambda_{n+1}} - \frac{\mu_{n+1} t_n}{\lambda_n \lambda_{n+1}}.$$

Now, *ex hypothesi*, $|b_n| < K\mu_n$, and so $|t_n| < K\lambda_n$: hence

$$\mu_{n+1} t_n / \lambda_n \lambda_{n+1} \to 0.$$

It follows that

(22) $$\sum_1^\infty t_\nu \Delta \frac{1}{\lambda_\nu}$$

is convergent: we have now to show that this necessarily implies (21), and so the convergence of Σa_n. As in §§ 3 and 5, we may suppose a_n real.

If (21) is not satisfied we can find values of N, as large as we please, and such that $$t_N > K_1 \lambda_{N+1}$$

(or $t_N < -K_1 \lambda_{N+1}$): we can, without loss of generality, suppose $K_1 < K$. Let N_1 be the least integer, such that

$$\lambda_{N_1+1} \geqslant \left(1 - \frac{K_1}{2K}\right) \lambda_{N+1} = K_2 \lambda_{N+1},$$

say. Then, also $\lambda_{N_1+1} < K_3 \lambda_{N+1}$, where K_3 is another constant which we can suppose as nearly equal to K_2 as we please.

If $N_1 \leqslant n \leqslant N$, we have

$$\begin{aligned} |t_N - t_n| = |b_{n+1} + b_{n+2} + \ldots + b_N| &\leqslant K(\mu_{n+1} + \mu_{n+2} + \ldots + \mu_N) \\ &\leqslant K(\lambda_N - \lambda_{N_1}) \\ &< \bar{K}(\lambda_{N+1} - \lambda_{N_1+1}) \\ &< \tfrac{1}{2}\bar{K}_1 \lambda_{N+1}, \end{aligned}$$

$\bar{K}$ and $\bar{K}_1$ being constants which we may suppose as nearly equal as we please to K and K_1 respectively. Hence

$$t_n \geqslant t_N - |t_N - t_n| > \tfrac{1}{2} K_1' \lambda_{N+1},$$

where K_1' also is as nearly equal to K_1 as we please.

Thus
$$\sum_{N_1}^{N} t_\nu \Delta \frac{1}{\lambda_\nu} > \tfrac{1}{2} K_1' \lambda_{N+1} \left(\frac{1}{\lambda_{N_1}} - \frac{1}{\lambda_{N+1}}\right)$$
$$> \tfrac{1}{2} K_1' \left(\frac{1}{K_3} - 1\right) = K_4,$$

say: and this inequality is plainly inconsistent with the convergence of the series (22). Hence the theorem is established.*

13. Suppose, in particular, that

$$\mu_1 = 0, \quad \mu_n = 1/(n-1) \quad (n > 1).$$

Then the series is summable if

$$\left(\frac{s_1}{1} + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \Big/ \log n \to s,$$

and the condition $|\lambda_n a_n / \mu_n| < K$ takes the form $|n \log n a_n| < K$. If this condition is satisfied the series cannot be summable unless it is convergent.

The analogue of Frobenius's theorem in this case takes the form: *if the series Σa_n is summable, and $s_n n^{-x} \to 0$ for all positive values of x, then $\Sigma a_n n^{-x}$ is convergent, and $\Sigma a_n n^{-x} \to s$ as $x \to 0$.*

It follows from the theorem cited in § 8 that by making $\Sigma \mu_n$ diverge *more and more slowly*, we obtain more and more powerful methods of summation. The theorem just proved, however, assigns a limit to the efficacy of the method for any particular choice of μ_n.

Thus the series
$$\Sigma n^{-1-ai}$$

is not summable when $\mu_n = 1$: but it is summable if $\mu_n = 1/n$ and, *a fortiori*, if $\mu_n \prec 1/n$. The series

$$\Sigma \frac{1}{n(ln)^{1+ai}}$$

is not summable when $\mu_n = 1/n$, since now $|n ln a_n| = 1$, but it is, as we shall see in a moment, summable when $\mu_n = 1/nln$, and, *a fortiori*, if $\mu_n \prec 1/nln$: and so on. In general, *the slower the oscillations of a series,*

* In giving our definition we restricted ourselves to the case in which $\mu_n \prec 1$. All that we have actually assumed in §§ 11, 12 is that $\lambda_{n+1}/\lambda_n \to 1$, a condition certainly satisfied when $\mu_n \prec 1$, but satisfied also if, *e.g.*, $\mu_n = n, n^2, \ldots$ (not if $\mu_n = e^n$). But, as we stated before, the case of real interest is that in which $\mu_n \prec 1$.

the harder it is to sum: this, at first sight, appears paradoxical. But the paradox disappears when we reflect that, if there are no oscillations, *i.e.*, if all the terms are positive, *no* method of summation by mean values can be effective except when the series is convergent.

14. Consider, in particular, the series

$$\Sigma \phi'(n)\, e^{ai\phi(n)}, \tag{23}$$

where $\phi(n)$ is a function which tends steadily to infinity, while $\phi'(n)$ tends steadily to zero: so that, assuming $\phi(n)$ to be in every sense *à croissance régulière*, we have

$$\phi(n) \prec n.$$

Suppose further that $\Sigma\{\phi'(n)\}^2$ is convergent, as will certainly be the case if

$$\phi(n) \prec n^{\frac{1}{2}-\delta}, \quad \phi(n) \prec n^{\frac{1}{2}}(ln)^{-\frac{1}{2}-\delta}, \quad \ldots, \quad (\delta > 0).$$

Then it is known* that

$$\sum_1^n \phi'(\nu)\, e^{ai\phi(\nu)} = \frac{1}{ai} e^{ai\phi(n)} + C + \epsilon_n, \tag{24}$$

where C is a constant, and $\epsilon_n \to 0$ as $n \to \infty$. Hence the condition that the series (23) should be summable $(R1, \mu_n)$ is that

$$\{\mu_2 e^{ai\phi(1)} + \mu_3 e^{ai\phi(2)} + \ldots + \mu_{n+1} e^{ai\phi(n)}\}/\lambda_{n+1}$$

should tend to a limit as $n \to \infty$.

Suppose that $$\mu_{n+1} = \phi'(n).$$

Then a renewed application of the formula (24) shows at once that this condition is satisfied, and that the limit is 0. Hence we can certainly sum the series by taking

$$\mu_{n+1} \preccurlyeq \phi'(n).$$

If, *e.g.*, $\phi(n) = \log n$, we can sum the series by taking $\mu_{n+1} \preccurlyeq 1/n$, and so on.

It is easy to extend this conclusion to more general series of the type

$$\Sigma \chi(n)\, e^{ai\phi(n)},$$

where $\phi(n) \prec n$ and $\Sigma \chi(n)\, \phi'(n)$ is convergent. In many cases of this kind it may be shown that

$$s_n \sim \frac{\chi(n)}{ai\phi'(n)} e^{ai\phi(n)}.$$

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* Bromwich, *Proc. London Math. Soc.*, Vol. 6, p. 330.

If we take
$$\mu_{n+1} = \phi'(n),$$
and suppose
$$\chi(n) \prec \phi'(n)\, e^{x\phi(n)}$$
for any positive value of x, we see that
$$\Sigma \chi(n)\, e^{(ai-x)\phi(n)}$$
has a limit as $x \to 0$: and, with appropriate restrictions on $\chi(n)$, this conclusion may be extended to more general series of the type
$$\Sigma \chi(n)\, e^{ai\phi(n)} \{\psi(n)\}^{-x},$$
where
$$\psi(n) \preccurlyeq e^{\phi(n)}.$$

15. Euler defined the sum of the oscillatory series Σa_n as the limit, if it exists, of $\Sigma a_n x^n$ as $x \to 1$. It is natural to generalise this definition by saying that the sum of the series is

(25) $$\lim_{x \to 0} \Sigma a_n e^{-\lambda_n x},$$

if it exists. The theorems of the preceding sections give us a number of criteria as to the existence or non-existence of this limit.

As λ_n is chosen so as to tend to infinity less and less rapidly, the greater becomes the probability that the series in (25) will not converge for all positive values of x—in which case there can, of course, be no question of the existence of the limit (25). This is obvious: the more interesting fact, which emerges from our previous analysis, is this—that, the *less* the increase of λ_n, then, if the series in (25) converges for all positive values of x, the *greater* the probability of the existence of the limit. Thus
$$\lim \Sigma n^{-1-ai} e^{-nx}$$
does not exist, but
$$\lim \Sigma n^{-1-ai} n^{-x}$$
does: and
$$\lim \Sigma n^{-1} (\log n)^{-1-ai} n^{-x}$$
does not, but
$$\lim \Sigma n^{-1} (\log n)^{-1-ai} (\log n)^{-x}$$
does: and so on. As the definition becomes less effective in one direction, it becomes, by way of compensation, more effective in another.

As an illustration of this general principle, I shall prove the following theorem:*—

If
$$\Sigma a_n n^{-x}$$

* This theorem is independent of any assumption as to the summability of Σa_n by mean values.

is convergent for $x > 0$, *and if the series* $\Sigma a_n x^n$ *tends to a limit as* $x \to 1$, *then the series* $\Sigma a_n n^{-x}$ *will tend to the same limit as* $x \to 0$.

That is to say, of the two series

$$\Sigma a_n x^n, \quad \Sigma a_n n^{-x},$$

the convergence of the second for $x > 0$ involves that of the first for $0 < x < 1$, but the converse is not true: but, if the second is convergent, then the existence of a limit for the first as $x \to 1$ involves that of a limit for the second as $x \to 0$, while again the converse is not true.

16. I shall base my proof on the following theorem, which I have proved elsewhere:*—

If
$$\phi(t) = \sum_1^\infty a_n e^{-nt},$$
and $R(x) > 0$, *then*
$$\Sigma a_n n^{-x} = \frac{1}{\Gamma(x)} \int_0^\infty \phi(t)\, t^{x-1}\, dt,$$
whenever the series on the left-hand side is convergent.

Since $x\Gamma(x) \to 1$ as $x \to 0$, and it is given that $\phi(t)$ tends to a limit, which we may denote by $\phi(0)$, as $t \to 0$, it is clear that what we have to show is that

$$\int_0^\infty x t^{x-1} \phi(t)\, dt \to \phi(0). \tag{26}$$

Here $\phi(t)$ is a function continuous for $t = 0$, and, as is easily seen, less than a constant multiple of e^{-t} when t is large.

Let
$$\phi(t) - \phi(0) = \psi(t),$$
so that $\psi(t) \to 0$ with t. Then we have

$$\int_0^\infty x t^{x-1} \phi(t)\, dt = \phi(0) + \int_0^1 x t^{x-1} \psi(t)\, dt + \int_1^\infty x t^{x-1} \phi(t)\, dt.$$

The last integral is numerically less than

$$Kx \int_1^\infty e^{-t}\, dt < Kx,$$

* *Messenger of Mathematics*, Vol. XXXIX, p. 136.

and has the limit 0 as $x \to 0$. Again, if we choose δ so that $|\psi(t)| < \epsilon$, for $0 \leqslant t \leqslant \delta$, we have

$$\left| \int_0^1 xt^{x-1}\psi(t)\,dt \right| < \epsilon \int_0^\delta xt^{x-1}\,dt + x\int_\delta^1 t^{x-1}\,|\psi(t)|\,dt.$$

The first term is less than ϵ; and when δ is fixed we can choose x_0 so that the second is less than ϵ for $0 \leqslant x \leqslant x_0$. Hence

$$\int_0^1 xt^{x-1}\psi(t)\,dt \to 0; \tag{27}$$

and the truth of (26) follows.

This result is capable of considerable generalisation. Thus in the note in the *Messenger of Mathematics*, quoted above, I proved that:—

If $\phi(n)$ is any positive function of n which tends steadily to infinity with n, and $R(x) > 0$, then

$$\int_0^\infty \{\Sigma a_n e^{-t\phi(n)}\}\, t^{x-1}\,dt = \Gamma(x)\,\Sigma \frac{a_n}{\{\phi(n)\}^x},$$

whenever the last series is convergent: and from this it may be deduced that:

If $\Sigma a_n e^{-x\phi(n)} \to s$ as $x \to 0$, then

$$\Sigma \frac{a_n}{\{\phi(n)\}^x} \to s$$

as $x \to 0$.

17. All the theorems that have been discussed in this paper suggest numerous generalisations.

(i) Riesz has given a definition of summability (Rk) related to summability ($R1$) as Cesàro's more general method is to its simplest case. It is suggested that we extend Theorems (i) and (iii) to cover this case: thereby obtaining analogues of Hölder's extension of Frobenius's theorem and of the result of § 5.

(ii) The theorems of the last section suggest more general theorems as to the cases in which the conditions

$$\Sigma a_n e^{-\lambda_n x} \to s, \qquad \Lambda_n \prec \lambda_n$$

imply

$$\Sigma a_n e^{-\Lambda_n x} \to s.$$

(iii) Our theorems are all capable of generalisations bringing in com-

plex values of x, and resembling in this respect Stolz's generalisation of Abel's original theorem on the continuity of power series.

(iv) All our theorems have analogues concerning integrals of the type

$$\int_1^\infty \phi(t)\,\{\psi(t)\}^{-x}\,dt.$$

It is not my object, however, to work out a whole hierarchy of theorems of a similar character: my desire is rather to take a definite text—"the arithmetic methods of summation of more or less slowly oscillating series"—and to prove just those theorems necessary to throw a general light upon the subject. There will be time to work out proofs of the more elaborate theorems when the need arises for their application to some special class of functions.

[*Added April*, 1910.—It is perhaps worth while pointing out that when, in § 3, $na_a \to 0$ (and not merely $|na_n| < K$), then a much simpler direct proof of the theorem can be given. For, if $b_n = na_n \to 0$, it follows at once that

$$t_n/n = (b_1+b_2+\ldots+b_n)/n \to 0,$$

so that the condition (4) of § 3 is obviously satisfied. A similar simplification can be effected in § 12, when $\lambda_n a_n/\mu^n \to 0$ (instead of merely $|\lambda_n a_n/\mu_n| < K$). For then $b_n/\mu_n \to 0$, and so

$$\frac{t_n}{\lambda_n} = \frac{b_1+b_2+\ldots+b_n}{\mu_1+\mu_2+\ldots+\mu_n} \to 0,$$

so that the condition (21) of § 12 is certainly satisfied.

The proof of §§ 4, 5 can also be simplified in the same way in this case. But the theorems obtained in this way would not be general enough to enable us to deal with any of the most interesting cases considered in the paper.]

CORRECTIONS

p. 309, *line* 5 *up*. For $1/n$ read $\log n$.

p. 311, 1*st footnote, last line*. For μ^n read μ_n.

p. 320, *line* 13. For na_a read na_n.

—— *line* 8 *up*. For μ^n read μ_n.

COMMENTS

The idea of replacing Tauber's† condition $a_n = o(1/n)$ by $a_n = O(1/n)$ occurs in 1908, 2, but in a different context (see the Comments on 1908, 2). The problem of replacing Cesàro summability by Abel summability in Hardy's O-Tauberian theorem (see § 7) was solved by Littlewood.‡ The O-condition in Hardy's theorem, §§ 3–6, was replaced by the one-sided condition $a_n > -K/n$ by Landau.§ Littlewood's theorem is extended similarly in 1914, 4. Tauberian theory is further developed in 1913, 1, 2, 3, and 10; 1914, 4 and 11; 1916, 8; 1920, 7; 1921, 6 (in Vol. II); 1924, 7; 1926, 5; 1930, 4; 1931, 8; 1936, 1; 1943, 4. Except for 1913, 3, which is a continuation of the present paper, these are all by Hardy and Littlewood.

In Landau's proof (loc. cit.) of his one-sided analogue of Hardy's theorem, his basic lemma (which is equivalent to the final result) may be expressed in the form: *if* $P_n \geqslant 0$ *and* $P_n \to g\ (H, 2)$, *then* $P_n \to g\ (H, 1)$. He recognized that this was analogous to a lemma that he had used in his proof of the 'prime number theorem', that: *if* $f(x)$ *is continuous and differentiable*, $xf'(x)$ *increases and* $f(x)/x \to 1$ *as* $x \to \infty$, *then* $f'(x) \to 1$; *Handbuch* . . .,‖ pp. 260–1 (cf. Lemma 1 of 1914, 4).

Summability $(R1, \mu_n)$, § 8, without the restriction $\mu_n \prec 1$, is Riesz's†† *discontinuous* method $(R^*, \lambda, 1)$, which is equivalent to his *continuous* method $(R, \lambda, 1)$, defined by the limit

$$\lim_{\omega\to\infty} \sum_{\lambda_n<\omega} (1-\lambda_n/\omega)a_n.$$

Hardy's definition is obtained by taking $\omega = \lambda_{m+1}$. The remarks in § 8, concerning the relative strength of methods, are further developed in 1916, 5.

An argument used in §§ 6 and 9 is a corollary of the analogue for Borel summability of Abel's limit theorem (1904, 4, § 7, and 1911, 8) and of the consistency (regularity) of Borel summability (1904, 3). This principle was independently formulated by Doetsch‡‡ as a theorem. See also Knopp§§ for some related results.

In Theorem (i) of § 10, the condition $s_n e^{-\lambda_n x} \to 0$ as $n \to \infty$, for every $x > 0$, is equivalent to the convergence of $\sum a_n e^{-\lambda_n x}$ for every $x > 0$; see H.R., Theorem 7. Thus the theorem may be stated in the form: *If* $\sum a_n$ *is summable* $(R1, \mu_n)$ *to* s *and* $\sum a_n e^{-\lambda_n x}$ *converges for every* $x > 0$, *then*

$$\sum a_n e^{-\lambda_n x} \to s \quad \text{as } x \to 0.$$

A more general result, stated in 1913, 3, § 2, is included in Theorem 23 of H.R.; see also 1911, 2, § 25.

In Theorems (i)–(iii) of § 10, the restriction $\mu_n \to 0$ (or $\lambda_{n+1}/\lambda_n \to 1$; cf. § 12, footnote) is unnecessary. Landau proved Theorem (ii) without such a restriction, and

† *Monatsch. für Math. u. Phys.* 8 (1897), 273–7.

‡ *Proc. London Math. Soc.* (2), 9 (1911), 434–48.

§ *Prace Mat.-Fiz.* 21 (1910), 97–177.

‖ Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 1. Teubner, Leipzig, 1909.

†† For references, see the Comments on 1911, 1.

‡‡ *Math. Annalen* 104 (1931), 403–14.

§§ *Rend. del Circolo Mat. di Palermo* (2), 1 (1952), 129–38.

there is no restriction in Theorem 23 of H.R., which includes Theorem (i). In the proof of Theorem (iii), § 12, the condition $\lambda_{n+1}/\lambda_n \to 1$ is used (a) to prove that

$$\mu_{n+1} t_n/\lambda_n \lambda_{n+1} \to 0,$$

(b) to prove that, for arbitrarily large N,

$$K(\lambda_N - \lambda_{N_1}) < \tfrac{1}{2}\bar{K}_1 \lambda_{N+1},$$

where $\bar{K}_1$ is close to K_1, $K_1 < K$, and

$$\lambda_{N_1} < (1-\tfrac{1}{2}K_1/K)\lambda_{N+1} \leqslant \lambda_{N_1+1}.$$

Now (a) may be avoided, if we use the formula

$$\sum_1^n t_r \Delta(1/\lambda_r) = (\mu_2 s_1 + \ldots + \mu_{n+1} s_n)/\lambda_{n+1},$$

the sequence in (a) being simply the nth term of the series on the left. But, if for some arbitrarily large N, $\lambda_{N+1}-\lambda_N$ is small and $\lambda_N - \lambda_{N-1}$ is large (so that $N_1 = N-1$), we may have

$$(\lambda_N - \lambda_{N_1})/\lambda_{N+1} \sim (\lambda_N - \lambda_{N-1})/\lambda_N \sim 1$$

as N runs through a subsequence of integers. Thus, if λ_n is unrestricted, (b) cannot be carried out, even if the factor $\frac{1}{2}$ is replaced by a number close to 1. On the other hand, Ananda-Rau|||| proved that the theorem is true with unrestricted λ_n. The result is obtained by letting the role of the interval $((1-\frac{1}{2}K_1/K)\lambda_{N+1}, \lambda_{N+1})$ be played by the interval $(\lambda_{N+1}, (1+\frac{1}{2}K_1/K)\lambda_{N+1})$. The result is needed to fill a gap in 1913, 3, which was noticed by Ananda-Rau; see the Comments on 1913, 3.

|||| *Proc. London Math. Soc.* (2), 17 (1918), 334–6.

THEOREMS CONNECTED WITH MACLAURIN'S TEST FOR THE CONVERGENCE OF SERIES

By G. H. HARDY.

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1. In a paper recently published in these *Proceedings*,* Dr. Bromwich gave a very interesting extension of the theorem (usually but inaccurately ascribed to Cauchy) that, when $f(x)$ is a positive decreasing function of x, the series

(1) $$\sum^{\infty} f(n)$$

and the integral

(2) $$\int^{\infty} f(x)\,dx$$

converge or diverge together. Dr. Bromwich proved that, *if*

(i) *$f(x)$ is positive and tends steadily to zero ;*

(ii) *$\phi(x)$ is positive and tends steadily to infinity,*

(iii) *$\phi'(x)$ tends steadily to zero,*

(iv) *the integral $\int^{\infty} f(x)\,\phi'(x)\,dx$ is convergent, then*

$$\int^{X} F(x)\,dx - \sum^{[X]} F(n),$$

where $F(x) = f(x)e^{i\phi(x)}$ and $[X]$ denotes the integral part of X, tends to a finite limit as $X \to \infty$.

Roughly, it may be said that what Dr. Bromwich proved amounted to this, that the relations between (1) and (2), as regards convergence or divergence, established by Maclaurin and Cauchy when $f(x)$ is positive and decreasing, still subsist when $f(x)$ is multiplied by an oscillatory factor of the type

$$\begin{matrix}\cos\\ \sin\end{matrix}\,\phi(x),$$

provided $\phi(x)$ tends to infinity more slowly than x.

* Vol. 6, pp. 327 *et seq.*

2. Dr. Bromwich's theorem may be generalised and its proof simplified as follows :—

THEOREM 1.—*If* (i) $F(x)$ *possesses a continuous derivative* $F'(x)$, (ii) $F(x) \to 0$ *as* $x \to \infty$, (iii) *the integral*

$$\int^{\infty} |F'(x)|\,dx$$

is convergent, then
$$\int_0^X F(x)\,dx - \sum_1^{[X]} F(n)$$

tends to a finite limit, viz.,
$$J_1 = -\int_0^{\infty} (x-[x])\,F'(x)\,dx,$$
as $X \to \infty$.*

Let $$j_\nu = \int_{\nu-1}^{\nu} F(x)\,dx - F(\nu) = \int_{\nu-1}^{\nu} \{F(x)-F(\nu)\}\frac{d}{dx}(x-\nu+1)\,dx$$
$$= -\int_{\nu-1}^{\nu} (x-[x])\,F'(x)\,dx.$$

Then
$$|j_\nu| \leqslant \int_{\nu-1}^{\nu} |F'(x)|\,dx.$$

Hence Σj_ν is convergent (absolutely) and the result follows immediately, since
$$\int_{[X]}^{X} F(x)\,dx \to 0.$$

3. In the case considered by Dr. Bromwich, we have
$$F'(x) = \{f'(x)+if(x)\,\phi'(x)\}\,e^{i\phi(x)},$$
and the convergence of $\int^{\infty} |F'(x)|\,dx$ follows immediately from that of
$$\int^{\infty} f'(x)\,dx, \qquad \int^{\infty} f(x)\,\phi'(x)\,dx.$$

Dr. Bromwich points out to me that the last inequality of § 2 may be established with equal simplicity by the method of argument adopted by himself (*loc. cit.*, p. 329). In fact
$$j_\nu = \int_{\nu-1}^{\nu} \{F(x)-F(\nu)\}\,dx = -\int_{\nu-1}^{\nu} dx \int_x^{\nu} F'(t)\,dt,$$
and so
$$|j_\nu| \leqslant \int_{\nu-1}^{\nu} |F'(t)|\,dt.$$

So far as applications are concerned, there is practically nothing to choose

* It is not difficult to establish a similar theorem relating to double series and integrals.

between Dr. Bromwich's result and mine. For the sake of completeness I give an example in which my result applies and his does not. It is obtained by taking

$$F(x) = x^{-a} \exp\{ix^b \cos(\log x)\},$$

where $0 < b < a < 1$.

4. Either theorem enables us, as Dr. Bromwich shows, to determine very simply the behaviour, from the point of view of convergence, of many interesting series, such as $\Sigma(\log n)^p n^{-1-ai}$. But neither theorem is sufficiently general to deal with many other simple and interesting cases that naturally present themselves.

Consider, for example, the series

$$\Sigma \frac{e^{in^a}}{n^b},$$

where $0 < a < 1$, $b > 0$. The integral

$$\int^{\infty} \frac{e^{ix^a}}{x^b}\, dx$$

is convergent if $a+b > 1$, and it is natural to suppose that the same is the case with the series: and this is in fact true. But the conditions of Theorem 1 are satisfied only if $b > a$.

It is therefore desirable to investigate more general theorems.

5. It will easily be verified that, if we transform the expression j_ν of § 2 by a further integration by parts, we obtain*

$$\int_{\nu-1}^{\nu} \{f(x)+\tfrac{1}{2}f'(x)\}\, dx - f(\nu) = \tfrac{1}{2}\int_{\nu-1}^{\nu} (y^2-y) f''(x)\, dx,$$

or, say,

$$j_\nu^{(2)} = \tfrac{1}{2}\int_{\nu-1}^{\nu} (y^2-y) f''(x)\, dx,$$

where

$$y = x-[x].$$

And generally, if we denote by $\phi_k(x)$ the Bernoullian polynomial of degree k, so that†

$$\phi_1 = x, \quad \phi_2 = x(x-1), \quad \phi_3 = x(x-1)(x-\tfrac{1}{2}), \quad \phi_4 = x^2(x-1)^2, \quad \ldots,$$

* In what follows I revert to the notation $f(x)$, $f(n)$ instead of $F(x)$, $F(n)$.

† Bromwich, *Infinite Series*, p. 236.

we have

$$(3)\quad j_\nu^{(2k)} = \int_{\nu-1}^{\nu}\left\{f(x)+\tfrac{1}{2}f'(x)+\frac{B_1}{2!}f''(x)-\frac{B_2}{4!}f^{(4)}(x)+\dots\right.$$
$$\left.+\frac{(-1)^{k-1}B_k}{2k!}f^{(2k)}(x)\right\}dx-f(\nu)$$
$$=-\frac{1}{(2k+1)!}\int_{\nu-1}^{\nu}\phi_{2k+1}(y)\,f^{(2k+1)}(x)\,dx$$
$$=\frac{1}{(2k+2)!}\int_{\nu-1}^{\nu}\phi_{2k+2}(y)\,f^{(2k+2)}(x)\,dx.$$

These formulæ are, in fact, merely a slight transformation of one of the standard forms of the "Euler-Maclaurin Sum Formula."*

From (3) we at once deduce

THEOREM 2.—*If either of the integrals*

$$\int^{\infty}|f^{(2k+1)}(x)|\,dx,\qquad \int^{\infty}|f^{(2k+2)}(x)|\,dx$$

is convergent, then

$$\int_0^{[X]}\left\{f(x)+\tfrac{1}{2}f'(x)+\frac{B_1}{2!}f''(x)-\dots\right\}dx-\sum_1^{[X]}f(\nu)$$

tends, as $X\to\infty$, to a finite limit, viz., one of the integrals

$$J_{2k+1}=-\frac{1}{(2k+1)!}\int_0^{\infty}\phi_{2k+1}(x-[x])\,f^{(2k+1)}(x)\,dx,$$
$$J_{2k+2}=\frac{1}{(2k+2)!}\int_0^{\infty}\phi_{2k+2}(x-[x])\,f^{(2k+2)}(x)\,dx.$$

If also $f(x)$ and its first $2k$ derivatives tend to zero, as $x\to\infty$, then

$$\int_0^{X}f(x)\,dx-\sum_1^{[X]}f(\nu)$$

tends, as $X\to\infty$, to the limit

$$\tfrac{1}{2}f(0)+\frac{B_1}{2!}f'(0)-\frac{B_2}{4!}f^{(3)}(0)+\dots+\frac{(-1)^{k-1}B_k}{2k!}f^{(2k-1)}(0)+J_l,$$

where $l=2k+1$ or $l=2k+2$, as the case may be.

We have, in fact, merely to observe that the maximum of $|\phi_{2k+1}(y)|$ or $|\phi_{2k+2}(y)|$ depends on k only and not on $[X]$.

* Bromwich, *Infinite Series*, p. 239.

I have not thought it worth while to encumber this theorem (or the others which follow) with explicit statements as regards the continuity of such derivatives of $f(x)$ as figure in them. All such derivatives are, of course, supposed continuous: only conditions relating to their behaviour as $x \to \infty$ are in any way relevant to the problem. Sometimes, however, these conditions may be broken for $x = 0$ or some other value of x. Then the *formulæ* may need modification, but the application of the theorem to questions of convergence is not affected.

6. If
$$f(x) = x^{-b} e^{ix^a},$$
where $0 < a < 1$, it is easy to see that
$$f^{(s)}(x) \sim (ia)^s x^{s(a-1)-b} e^{ix^a},$$
which yields an absolutely convergent integral if $s(a-1)-b < -1$ or
$$s > (1-b)/(1-a).$$

It is always possible to choose s so as to satisfy this condition: hence *the series and integral*
$$\int^{\infty} x^{-b} e^{ix^a} dx, \qquad \Sigma n^{-b} e^{in^a} \quad (0 < a < 1)$$
in all cases converge or oscillate together, i.e., converge if, and only if, $a+b > 1$.

If $b > a$ *is implied by* $a+b > 1$, *i.e.*, if $a \leqslant \frac{1}{2}$, but not if $a > \frac{1}{2}$, this result may be inferred from Theorem 1.

7. It is naturally suggested by the preceding results that it should be possible to prove theorems of a similar character, but relating to series and integrals that are only *summable* and not convergent. The integral
$$\int_0^{\infty} f(x)\, dx$$
is said to be summable (Cr), to sum S, if

$$\frac{r!}{x^r} \left(\int_0^x dt\right)^{r+1} f(t) \to S, \tag{4}$$
as $x \to \infty$.*

This definition is strictly analogous to Cesàro's definition of the sum

* Some properties of integrals which are summable $(C1)$ are proved by Hardy, *Quarterly Journal*, Vol. XXXV, p. 22; Moore, *Trans. Amer. Math. Soc.*, Vol. VIII, p. 299; Bromwich, *Math. Annalen*, Bd. LXV, S. 350. The above definition is an immediate generalisation of that adopted by these writers.

s of an oscillatory series Σa_ν, which may be put in the form

(5) $$\frac{r!}{n^r}\left(\sum_{\nu=1}^{n}\right)^{r+1} a_\nu \rightarrow s.$$

Now, by a well known formula,*

$$\left(\int_0^x dt\right)^{r+1} f(t) = \frac{1}{r!}\int_0^x (x-t)^r f(t)\,dt.$$

Hence our definition is equivalent to

(6) $$\int_0^x \left(1-\frac{t}{x}\right)^r f(t)\,dt \rightarrow S$$

or

(7) $$\int_0^1 x f(xu)(1-u)^r\,du \rightarrow S.$$

And, when either of these forms is adopted, there is no longer any reason why r should be restricted to be an integer; we may suppose that r has any real value greater than -1.

Suppose, *e.g.*, that
$$f(x) = x^p e^{aix},$$
where a is real and not zero. Then the integral on the left-hand side of (7) is
$$x^{p+1}\int_0^1 u^p(1-u)^r e^{aixu}\,du.$$

Now it is known that, when $|x|$ is large,

$$\int_0^1 u^p(1-u)^r e^{aixu}\,du = \frac{\Gamma(p+1)}{(-aix)^{p+1}}(1+\epsilon)+\frac{\Gamma(r+1)}{(aix)^{r+1}}e^{aix}(1+\epsilon'),$$

where ϵ, ϵ' are small of the order $1/x$, and the many valued functions in the denominators have their values fixed by conventions explained precisely in the papers referred to in the foot-note below.†

It follows at once that
$$\int_0^\infty x^p e^{aix}\,dx$$
is summable (Cr), and has the sum
$$\Gamma(p+1)/(-ai)^{p+}$$
if, and only if, $r > p$.

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* See, *e.g.*, Jordan, *Cours d'Analyse*, t. III, p. 59.

† Jacobsthal, *Math. Annalen*, Bd. LVI, S. 129; Barnes, *Trans. Camb. Phil. Soc.*, Vol. XX, p. 253; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, p. 401.

8. Now M. Marcel Riesz has replaced* Cesàro's definition by

$$\sum_{\nu=1}^{n}\left(1-\frac{\nu}{n}\right)^{r} a_{\nu} \rightarrow s, \tag{8}$$

a definition precisely analogous in form to that adopted in equation (7) above for the value of the summable integral. This formal analogy renders it convenient to adopt this definition in the analysis which follows.

Before proceeding further, I may remark that I shall consider explicitly only the case in which r is a positive integer. This is the really interesting case. But the definition (8), like (7), applies equally well for non-integral values of r, and, in a good deal of the succeeding analysis, the restriction that r is integral is in no way necessary.

9. M. Riesz has indicated an important generalisation of the definition expressed by (8). Let $\lambda(x)$ be any function of x which tends steadily to infinity with x. Then we may define the sum s of the series Σa_n by the relation

$$\sum_{\nu=1}^{n}\left\{1-\frac{\lambda(\nu)}{\lambda(n)}\right\}^{r} a_{\nu} \rightarrow s, \tag{9}$$

which reduces to (8) if $\lambda(x)=x$. If this limit exists, we shall say that the series Σa_n is *summable* (Rr) *with sum-function* $\lambda(n)$. When $r=1$ this definition reduces to

$$s_n-\frac{1}{\lambda(n)}\sum_{\nu=1}^{n}\lambda(\nu)\,a_{\nu} \rightarrow s$$

or to

$$\frac{\mu_2 s_1+\mu_3 s_2+\ldots+\mu_n s_{n-1}}{\lambda(n)} \rightarrow s,$$

where $\mu_\nu=\lambda(\nu)-\lambda(\nu-1)$; which is the definition that I considered in a recent paper in these *Proceedings*.†

The corresponding definition for the integral is obviously

$$\int_0^x\left\{1-\frac{\lambda(t)}{\lambda(x)}\right\}^{r} f(t)\,dt \rightarrow S. \tag{10}$$

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* *Comptes Rendus*, July 5th, 1909. M. Riesz has established (as is easy enough when r is integral) the substantial equivalence of the two definitions.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301.

If this limit exists, I shall say that the integral

$$\int_0^\infty f(x)\,dx$$

is *summable* (*Rr*), *with sum-function* $\lambda(x)$, *to sum* S.

If we put
$$\lambda(t) = T, \qquad \lambda(x) = X,$$

and denote by $\bar{\lambda}$ the function inverse to λ, we see that (10) is equivalent to

$$\int_0^X \left(1-\frac{T}{X}\right)^r f\{\bar{\lambda}(T)\}\,\bar{\lambda}'(T)\,dT.$$

That is to say,
$$\int^\infty f(x)\,dx$$

is summable (*Rr*), with sum-function $\lambda(x)$, if, and only if,

$$\int^\infty f\{\bar{\lambda}(x)\}\,\bar{\lambda}'(x)\,dx$$

is summable (*Rr*), with sum-function x. *The adoption of Riesz's general definition* instead of Cesàro's is, in the case of integrals, substantially equivalent to a change in the independent variable.*

Thus it is easy to prove that

(11)
$$\int^\infty (\log x)^p x^{-1+ai}\,dx$$

is not summable (*Cr*) for any value of r. The substitution $x = e^y$, however, leads to the integral

$$\int^\infty y^p e^{aiy}\,dy,$$

which is summable (*Cr*) for $r > p$. Hence (11) is summable (*Rr*), with sum-function $\log x$, for $r > p$. We shall see in a moment that the same is true of the series

$$\Sigma (\log n)^p n^{-1+ai}.$$

10. We are now in a position to establish the following theorem :—

THEOREM 3.—*If $f(x)$ is subject to the conditions of Theorem* 1, *viz., that $f(x) \to 0$ and*

$$\int^\infty |f'(x)|\,dx$$

* I call the definition (10) Riesz's because he has doubtless used it as well as the definition (9). It would indeed be a plausible conjecture that M. Riesz was led to the definitions (8) and (9) through some consideration of integrals.

is convergent, and $\lambda(x)$ *is any function of* x *which has a continuous derivative and tends steadily to infinity with* x, *then*

$$\Sigma_n = \int_0^n \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r f(t)\,dt - \sum_1^n \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^r f(\nu)$$

tends, as $n \to \infty$, *to the finite limit* J_1 *of Theorem* 1, *so that the summability of the series* $\Sigma f(\nu)$ *follows from that of the integral, the definitions of summability being Riesz's definitions of the r-th order, with sum-functions* $\lambda(x)$, $\lambda(n)$.

Let
$$\psi(t) = \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r f(t).$$

Then
$$\Sigma_n = \sum_1^n j_\nu,$$

where $j_\nu = \int_{\nu-1}^{\nu} \psi(t)\,dt - \psi(\nu) = -\int_{\nu-1}^{\nu} (t-[t])\,\psi'(t)\,dt = u_\nu + v_\nu + w_\nu$,

u_ν, v_ν, w_ν being defined by the equations

$$u_\nu = -\int_{\nu-1}^{\nu} (t-[t]) f'(t)\,dt, \tag{12}$$

$$v_\nu = \int_{\nu-1}^{\nu} (t-[t]) \left(1 - \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r\right) f'(t)\,dt, \tag{13}$$

$$w_\nu = \frac{r}{\lambda(n)} \int_{\nu-1}^{\nu} \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} \lambda'(t) f(t)\,dt. \tag{14}$$

In the first place,
$$|u_\nu| < \int_{\nu-1}^{\nu} |f'(t)|\,dt,$$

so that Σu_ν is convergent (absolutely). In the second place,

$$|w_\nu| < \frac{r}{\lambda(n)} \int_{\nu-1}^{\nu} \lambda'(t)\,|f(t)|\,dt.$$

But it is easy to prove that $\phi(t) \to 0$ involves

$$\frac{1}{\lambda(n)} \int_0^n \phi(t)\,\lambda'(t)\,dt \to 0:*$$

---

* This is an immediate generalisation from the well known case in which $\lambda(n) = n$. Choose N so that $|\phi| < \epsilon$ for $x \geqslant N$. Then

$$\frac{1}{\lambda(n)} \left|\int_0^n \phi\lambda'\,dt\right| < \frac{1}{\lambda(n)} \int_0^N |\phi\lambda'|\,dt + \frac{\epsilon}{\lambda(n)} \int_N^n \lambda'(t)\,dt < \frac{1}{\lambda(n)} \int_0^N |\phi\lambda'|\,dt + \epsilon < 2\epsilon,$$

if n is large enough.

and so
$$\sum_1^n w_\nu \to 0.$$

Finally,
$$\left|\sum_1^n v_\nu\right| < \int_0^n \left(1 - \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r\right) |f'(t)|\, dt.$$

Let
$$\int_t^\infty |f'(u)|\, du = \Phi(t).$$

Then, integrating by parts,
$$\left|\sum_1^n v_\nu\right| < -\Phi(n) + \left(1 - \left\{1 - \frac{\lambda(0)}{\lambda(n)}\right\}^r\right) \Phi(0)$$
$$- \frac{r}{\lambda(n)} \int_0^n \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} \lambda'(t)\, \Phi(t)\, dt.$$

The first two terms evidently tend to zero as $n \to \infty$. The last is numerically less than
$$\frac{r}{\lambda(n)} \int_0^n \lambda'(t)\, |\Phi(t)|\, dt,$$

and so also tends to zero.

It follows that $\Sigma_n \to \sum_1^\infty u_\nu = -\int_0^\infty (t - [t])\, f'(t)\, dt,$

and the proof of Theorem 3 is accordingly completed.

11. A simple example of the theorem just proved is afforded by the case in which
$$f(x) = (\log x)^p\, x^{-1+ai}.$$

Here
$$f'(x) \sim (ai - 1)(\log x)^p\, x^{-2+ai},$$

and the conditions of the theorem are certainly satisfied. It follows from § 9 that the series
$$\Sigma (\log n)^p\, n^{-1+ai}$$

is summable (Rr), with sum function $\log n$, if, and only if, $r > p$.

12. The question now arises as to whether we can prove a still more general theorem related to Theorem 3 as Theorem 2 is related to Theorem 1.

I shall content myself with proving this in two special cases, viz., (1) when

$$\lambda(x) = x$$

(so that we are using definitions substantially equivalent to Cesàro's), and (2) when $\lambda(x)$ is arbitrary and $k = 0$. The absolutely general case appears to require rather elaborate preliminaries.

13. The formula (3) of § 5, applied to the function

$$\psi(t) = \left(1-\frac{t}{n}\right)^r f(t),$$

gives

$$(15)\quad \int_{\nu-1}^{\nu} \left\{\psi(t)+\tfrac{1}{2}\psi'(t)+\frac{B_1}{2!}\psi''(t)-\ldots+\frac{(-1)^{k-1}B_k}{2k!}\psi^{(2k)}(t)\right\} dt-\psi(\nu)$$

$$= \frac{1}{(2k+2)!}\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt.$$

Suppose now that, as in Theorem 2,

$$\int^{\infty} |\,f^{(2k+2)}(t)\,|\,dt,$$

and is convergent, and also that $f^{(2k+1)}(t) \to 0$. Then we can prove that

$$\sum_{1}^{n}\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt \to \int_0^{\infty} \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt.$$

We have, in fact,

$$\psi^{(2k+2)}(t) = f^{(2k+2)}(t)-\left\{1-\left(1-\frac{t}{n}\right)^r\right\}f^{(2k+2)}(t)$$

$$+\sum_{s=1}^{\sigma}(-1)^s s!\binom{2k+2}{s}\binom{r}{s}n^{-s}\left(1-\frac{t}{n}\right)^{r-s}f^{(2k+2-s)}(t),$$

where σ is the lesser of the numbers $2k+2$ and r.

We may therefore write

$$\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt = u_\nu+v_\nu+\sum_{s=1}^{\sigma} w_\nu^{(s)},$$

where $$u_\nu = \int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t]) f^{(2k+2)}(t)\,dt,$$

$$v_\nu = -\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t]) \left\{1-\left(1-\frac{t}{n}\right)^r\right\} f^{(2k+2)}(t)\,dt,$$

$$w_\nu^{(s)} = C_s n^{-s} \int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t]) \left(1-\frac{t}{n}\right)^{r-s} f^{(2k+2-s)}(t)\,dt,$$

C_s being a number dependent only on k, r, s, and not upon ν or n.

Since $$|u_\nu| < M \int_{\nu-1}^{\nu} |f^{(2k+2)}(t)|\,dt,$$

where M is the maximum of $\phi_{2k+2}(y)$ for $0 < y < 1$, it follows that Σu_ν is convergent (absolutely).

Again, $$\left|\sum_1^n v_\nu\right| < M \int_0^n \left\{1-\left(1-\frac{t}{n}\right)^r\right\} |f^{(2k+2)}(t)|\,dt.$$

Let $$\Phi(t) = \int_t^\infty |f^{(2k+2)}(u)|\,du.$$

Then $$\int_0^n \left\{1-\left(1-\frac{t}{n}\right)^r\right\} |f^{(2k+2)}(t)|\,dt = -\Phi(n) + \frac{r}{n}\int_0^n \left(1-\frac{t}{n}\right)^{r-1} \Phi(t)\,dt.$$

The first term on the right-hand side has the limit zero. The second is less than

$$\frac{r}{n}\int_0^n \Phi(t)\,dt,$$

and so also tends to zero. It follows that

(16) $$\sum_1^n v_\nu \to 0.$$

Finally, $$\left|\sum_1^n w_\nu^{(s)}\right| < K n^{-s} \int_0^n |f^{(2k+2-s)}(t)|\,dt,$$

where K is independent of n. I shall prove in the next section that the expression on the right-hand side tends to zero as $n \to \infty$, for $s = 1, 2, \ldots, \sigma$. It will then follow that

(17) $$\sum_1^n w_\nu^{(s)} \to 0 \quad (s = 1, 2, \ldots, \sigma),$$

and from (16) and (17) will follow the truth of the assertion made at the beginning of this section.

14. Since $f^{(2k+1)}(t) \to 0$, it follows at once that

$$\frac{1}{t}\int_0^t |f^{(2k+1)}(u)|\,du \to 0,$$

as $t \to \infty$. Hence

$$|f^{(2k)}(t)| \leqslant |f^{(2k)}(0)| + \int_0^t |f^{(2k+1)}(u)|\,du = t\epsilon_t,$$

where $\epsilon_t \to 0$. Hence

$$\frac{1}{t^2}\int_0^t |f^{(2k)}(u)|\,du \to 0,$$

and from this we deduce as above that

$$|f^{(2k-1)}(t)| = t^2\epsilon_t,$$

where $\epsilon_t \to 0$. It is evident that a mere repetition of this argument is sufficient to prove what we require.

It is therefore proved that the sum from $\nu = 1$ to $\nu = n$ of the right-hand side of the equation (15) tends, as $n \to \infty$, to the limit

$$J_{2k+2} = \frac{1}{(2k+2)!}\int_0^\infty \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt.$$

15. We have now to consider the left-hand side of (15), which is

$$\Sigma_n + \tfrac{1}{2}\{\psi(n)-\psi(0)\} + \frac{B_1}{2!}\{\psi'(n)-\psi'(0)\} - \frac{B_2}{4!}\{\psi^{(3)}(n)-\psi^{(3)}(0)\} + \dots$$

$$+ \frac{(-1)^{k-1}B_k}{2k!}\{\psi^{(2k-1)}(n)-\psi^{(2k-1)}(0)\},$$

where
$$\Sigma_n = \int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu).$$

Now
$$\psi^{(s)}(t) = \sum_{\kappa=0}^{\lambda} (-1)^\kappa \kappa! \binom{s}{\kappa}\binom{r}{\kappa} n^{-\kappa}\left(1-\frac{t}{n}\right)^{r-\kappa} f^{(s-\kappa)}(t),$$

where λ is the lesser of the numbers r, s. From this it follows at once that

(18) $$\psi^{(\ )}(0) = f^{(s)}(0) + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$. Also

(19) $$\psi^{(s)}(n) = 0 \quad (r > s),$$

since, if $r > s$, every term in $\psi^{(s)}(n)$ contains a factor which vanishes for $t = n$. On the other hand, if $r \leqslant s$, there is one term in $\psi^{(s)}(n)$ which does not vanish for $t = n$, viz., that for which $\kappa = r$, and

(20) $$\psi^{(s)}(n) = (-1)^r r! \binom{s}{r} n^{-r} f^{(s-r)}(n) \quad (r \leqslant s).$$

From (18), (19), and (20) it follows that the left-hand side of (15) may be expressed in the form

$$\Sigma_n + \epsilon_n - \tfrac{1}{2}f(0) - \frac{B_1}{2!}f'(0) + \frac{B_2}{4!}f^{(3)}(0) - \ldots - \frac{(-1)^{k-1}B_k}{2k!}f^{(2k-1)}(0) + \chi(n),$$

where $$\chi(n) = (-1)^r r! \, n^{-r} \Sigma \frac{(-1)^{p-1}B_p}{2p!}\binom{2p-1}{r} f^{(2p-1-r)}(n),$$

the summation extending to all the values of p (if any) which satisfy the inequalities
$$r \leqslant 2p-1 \leqslant 2k-1.$$

We therefore assume, in addition to the conditions already imposed upon f, that
$$x^{-r} f^{(2p-1-r)}(x) \to 0,$$

for $r \leqslant 2p-1 \leqslant 2k-1$. Thus, if $r = 1$, we must have
$$f(x)/x \to 0, \quad f''(x)/x \to 0, \quad \ldots, \quad f^{(2k-2)}(x)/x \to 0;$$

if $r = 2k-1$, we must have
$$f(x)/x^{2k-1} \to 0;$$

while, if $r \geqslant 2k$, no conditions of this type are needed. If, *e.g.*, $k = 1$, the only case in which one of these conditions is required is when $r = 1$, the condition then being
$$f(x)/x \to 0.$$

16. We have thus proved—

THEOREM 4.—*If*

(i) $\int^{\infty} |f^{(2k+2)}(x)| \, dx$ *is convergent*

(ii) $f^{2k+1}(x) \to 0,$

(iii) $f^{(2p-1-r)}(x)/x^r \to 0$

for such values of p, if any, as satisfy the inequalities

$$r \leqslant 2p-1 \leqslant 2k-1,$$

then
$$\int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu)$$

tends, as $n \rightarrow \infty$, to the limit

$$\tfrac{1}{2}f(0)+\frac{B_1}{2!}f'(0)-\frac{B_2}{4!}f^{(3)}(0)+\ldots$$

$$+\frac{(-1)^{k-1}B_k}{2k!}f^{(2k-1)}(0)+\frac{1}{(2k+2)!}\int_0^\infty \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt,$$

so that the summability (Rr), with sum-function n, of the series $\Sigma f(n)$ follows from that of the integral $\int f(x)\,dx$.

In the simplest case, that of $k=0$, the theorem takes the form

THEOREM 5.—*If* (i) $\int^\infty |f''(x)|\,dx$ *is convergent,* (ii) $f'(x) \rightarrow 0$, *then*

$$\int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu) \rightarrow \tfrac{1}{2}f(0)+\frac{1}{2!}\int_0^\infty \phi_2(t-[t])\,f''(t)\,dt,$$

if $r>0$; and if $r=0$, the result is still true, provided only

$$f(x) \rightarrow 0.$$

17. Let us consider, for example, the case in which

$$f(x) = x^{-b}e^{ix^a} \quad (0<a<1).$$

Then
$$f^{(s)}(x) \sim (ia)^s\, x^{s(a-1)-b}\, e^{ix^a},$$

which tends to zero, if $s(a-1)-b<0$, and possesses an absolutely convergent integral, if $s(a-1)-b<-1$. This last condition will be satisfied for $s=2k+2$, if

$$2k+2 > (1-b)/(1-a).$$

This is certainly so for sufficiently large values of k.

The condition that $f^{(2k+1)}(x) \to 0$ requires

$$2k+1 > -b/(1-a);$$

and, as $1/(1-a) > 1$, this is a consequence of the former condition.

The condition $$f^{(2p-1-r)}(x)/x^r \to 0,$$

reduces to $$(2p-1-r)(a-1)-b-r < 0.$$

This is *least* likely to be satisfied when p has its smallest possible value, viz., that given by $r = 2p-1$. It then reduces to

$$b+r > 0.$$

But the condition $b+r > 0$ is certainly a *necessary* condition for the summability of

$$\Sigma n^{-b} e^{in^a},$$

by Cesàro's r-th mean.* Hence there is no loss of generality involved in this set of conditions.

We can now apply Theorem 4, and we see that the problem of the summability of the series is reduced to the corresponding problem for the integral

$$\int x^{-b} e^{ix^a} dx.$$

We have therefore only to determine whether

$$\int_0^n \left(1-\frac{t}{n}\right)^r \frac{e^{it^a}}{t^b} dt$$

tends to a limit as $n \to \infty$. Putting

$$t = nu^{1/a} = (Nu)^{1/a},$$

we obtain $$\frac{1}{a} N^{(1-b)/a} \int_0^1 (1-u^{1/a})^r u^{[(1-b)/a]-1} e^{iNu} du.$$

Now it may easily be proved, by a slight modification of the argument used in establishing the asymptotic formula quoted in § 7, that

$$\int_0^1 (1-u^{1/a})^r u^{[(1-b)/a]-1} e^{iNu} du = \frac{\Gamma\{(1-b)/a\}}{(-iN)^{(1-b)/a}} (1+\epsilon) + \frac{\Gamma(r+1)}{a^r (iN)^{r+1}} e^{iN} (1+\epsilon'),$$

* Bromwich, *Infinite Series*, p. 318.

where ϵ and ϵ' are of order $1/N$ when N is large. It follows that the integral

$$\int_0^\infty \frac{e^{it^a}}{t^b}\,dt$$

is summable (Cr), and has the sum

$$\frac{1}{a}\Gamma\left(\frac{1-b}{a}\right)\Big/(-i)^{(1-b)/a},$$

if, and only if, $r+1>(1-b)/a$; *i.e.*, if

$$b+(r+1)\,a>1.^*$$

And so the same is true of the series: that is to say, *the series and integral*

$$\sum^\infty n^{-b}e^{in^a}, \qquad \int^\infty x^{-b}e^{ix^a}\,dx \quad (0<a<1)$$

are both convergent if $a+b>1$, *both summable* $(R1)$ *if* $2a+b>1$, *both summable* $(R2)$ *if* $3a+b>1$, *and so on, with sum function* n.

18. The last question which arises is as to whether there is an analogue of Theorem 4, when Riesz's more general definition is adopted. The answer is, of course, in the affirmative; but, for the reason stated in § 12, I content myself with considering the simplest case, the analogue of Theorem 5.

THEOREM 6.—*If* (i) $\int^\infty |f''(x)|\,dx$ *is convergent,* (ii) $f'(x)\to 0$, (iii) $\lambda(x)$ *is a steadily increasing function of* x, *tending to infinity with* x, *and possessing a continuous derivative,* (iv) $\{\lambda'(x)/\lambda(x)\}f(x)\to 0$, *then*

$$\int_0^n\left\{1-\frac{\lambda(t)}{\lambda(x)}\right\}^r f(t)\,dt-\sum_1^n\left\{1-\frac{\lambda(\nu)}{\lambda(n)}\right\}^r f(\nu)$$

tends, as $n\to\infty$, *to the limit*

$$\tfrac{1}{2}f(0)+\tfrac{1}{2}\int_0^\infty \phi_2(t-[t])\,f''(t)\,dt,$$

if $r>0$; *and the conclusion remains valid for* $r=0$, *provided* $f(x)\to 0$.

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* Of course we suppose $b<1$ in order that the integral should converge at the lower limit: this is quite irrelevant to the main issue. If $b>1$, the integral is absolutely convergent as regards the upper limit.

We have, in fact, if Σ_n denotes the expression whose limit we are considering,

$$\Sigma_n + \tfrac{1}{2}\{\psi(n) - \psi(0)\} = \frac{1}{2!}\int_0^n \phi_2(t-[t])\psi''(t)\,dt,$$

where
$$\psi(t) = \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r f(t).$$

Clearly $\psi(n) = 0$, unless $r = 0$, in which case we impose the additional condition that
$$\psi(x) = f(x) \to 0.$$

Again,

$$\psi''(t) = \left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^r f''(t) - \frac{2r\lambda'(t)}{\lambda(n)}\left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} f'(t)$$
$$+ \frac{r(r-1)\{\lambda'(t)\}^2}{\{\lambda(n)\}^2}\left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^{r-2} f(t)$$
$$- \frac{r\lambda''(t)}{\lambda(n)}\left\{1 - \frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} f(t).$$

The reader who has followed the arguments of the preceding paragraphs with care (especially those of §§ 10, 13) will easily convince himself that everything reduces to proving that $f'(x) \to 0$ involves

$$(a) \quad \frac{1}{\lambda(x)}\int_0^x \lambda'(t) f'(t)\,dt \to 0,$$

$$(b) \quad \frac{1}{\{\lambda(x)\}^2}\int_0^x \{\lambda'(t)\}^2 f(t)\,dt \to 0,$$

$$(c) \quad \frac{1}{\lambda(x)}\int_0^x \lambda''(t) f(t)\,dt \to 0.$$

Of these relations the first has already been proved. Again,

$$\frac{1}{\lambda(x)}\int_0^x \lambda''(t) f(t)\,dt = \frac{\lambda'(x)}{\lambda(x)} f(x) - \frac{\lambda'(0)}{\lambda(x)} f(0) - \frac{1}{\lambda(x)}\int_0^x \lambda'(t) f'(t)\,dt,$$

and so also tends to zero if the condition

$$\frac{\lambda'(x)}{\lambda(x)} f(x) \to 0$$

is satisfied, as we have supposed. Also, if this condition is satisfied, we can find X, so that

$$|f(x)| < \epsilon\lambda(x)/\lambda'(x) \quad (x \geqslant X).$$

Then $$\frac{1}{\{\lambda(x)\}^2}\left|\int_0^x \{\lambda'(t)\}^2 f(t)\,dt\right| < \frac{1}{\{\lambda(x)\}^2}\int_0^X \{\lambda'(t)\}^2\,|f(t)|\,dt$$

$$+\frac{\epsilon}{\{\lambda(x)\}^2}\int_X^x \lambda(t)\,\lambda'(t)\,dt.$$

The second term is less than $\frac{1}{2}\epsilon$; and by choosing x large enough, when X is fixed, we can ensure that the first term is also less than $\frac{1}{2}\epsilon$. The relations (a), (b), (c) are accordingly established and Theorem 6 is proved.

CORRECTIONS

p. 131, *line* 2 *up*. For $)^{p+}$ read $)^{p+1}$.
—— *footnote, last line*. For Vol. 3 read Vol. 2.
p. 138, *line* 8 *up*, and *p.* 139, *line* 7. For 'left-hand side of (15)' read 'left-hand side of (15) summed from 1 to n'.
p. 142, *line* 4 *up*. For $\lambda(x)$ read $\lambda(n)$.

COMMENTS

Riesz defined his *discontinuous* typical means (R^*, λ, r) in 1909,† and his *continuous* typical means (R, λ, r) later the same year.‡ The method defined by (9), § 9 is (R^*, λ, r), and that defined by (8), § 8 is (R^*, n, r), The footnote to § 8 refers to an assertion by Riesz (in paper (1)) that the (R^*, n, r) means are 'entièrement equivalentes aux *moyennes arithmétiques*', i.e. to the (C, r) means. Riesz found later that his assertion was incorrect,§ and showed that there are series which are summable (R^*, n, r), $r = 2$ or 3, but not summable (C, r). He also established the equivalence of (R, n, r) and (C, r) for $r \geqslant 0$,‖ and the equivalence of (R^*, n, r) and (C, r) for $-1 < r \leqslant 1$; see Riesz (4). It is now known†† that (R^*, n, r) and (C, r) are equivalent for $-1 < r < 2$, but not equivalent for $r \geqslant 2$. The (R, n, r) method has no useful meaning for $r < 0$; see Agnew‡‡ and the Comments on 1911, 2.

It follows that the summability (C, r) of

$$\sum n^{-b} e^{in^a} \quad (0 < a < 1)$$

for $b+(r+1)a > 1$ (r an integer) is not completely established by Theorem 4, if $r \geqslant 2$. In D.S., p. 147, Hardy remarks that 'it is not difficult to modify the argument so as to take account of non-integral ω, and prove that the series is summable (R, n, r)'. He also gives (Theorem 84) a proof, not depending on the equivalence theorem, that the Cesàro means of order r ($r > -1$) are of the form

$$n^{-r-b+(r+1)(1-a)} e^{in^a}(K+o(1))+C+o(1),\ K \neq 0.$$

† Riesz (1), *Comptes rendus* 149 (1909), 18–21.
‡ Riesz (2), ibid. 149 (1909), 909–12.
§ Riesz (3), ibid. 152 (1911), 1651–4, footnote; for details, see Riesz (4), *Proc. London Math. Soc.* (2), 22 (1924), 412–19; D.S., p. 114.
‖ Riesz (3). Some further details are given in Hobson's *Theory of functions of a real variable*, Vol. 2 (1926), Ch. I; see also Gergen, *Duke Math. J.* 3 (1937), 133–48; D.S., Theorem 58 (for r an integer); Ingham, *Pub. Ramanujan Institute* 1 (1968–9), 107–13.
†† Peyerimhoff, *Proc. American Math. Soc.* 7 (1956), 335–47, for $r = 4, 5, \ldots$; Kuttner, *J. London Math. Soc.* 37 (1962), 354–64, in the general case.
‡‡ *Trans. American Math. Soc.* 35 (1933), 532–48.

A GENERAL VIEW OF THE THEORY OF SUMMABLE SERIES.

By G. H. Hardy and S. Chapman.

CONTENTS OF THE PAPER.

1911, 2 (with S. Chapman) *Quarterly Journal of Mathematics*, 42, 181–215.

I.

Introduction and bibliography.

§ 1. THE results contained in this paper are, for the most part, applications of a general principle formulated by one of us in two papers published in 1904,* "a general principle which may often guide us in our choice of a convention as to the value to be attributed to an otherwise meaningless expression." The principle is that, "when a number of limit operations, performed in a definite order on a function of several variables, lead to a definite result, but do not do so when performed in another definite order, we are to agree that the result of the formal carrying out of the second sequence of operations *means* the result of the first sequence."

These remarks have reference (in so far as our present applications of them are concerned) to the problem of the summation of oscillatory series. Since the time when the papers in which they occur were written, the arithmetic theory of summability has developed rapidly: new definitions have been framed and old ones generalised, and important and unlooked-for applications of the theory have been found.† In the present paper we propose to discuss some very general classes of definitions, which number among their members the definitions which are most familiar and which have been most widely applied. It is possible in this way to obtain a clearer insight into the nature of these definitions, and to coordinate them as particular cases of the general principle already quoted. It will also be found that the point of view here adopted suggests a considerable variety of further questions, and leads simply and naturally to interesting generalisations of a number of well-known theorems.

§ 2. There is another question which we shall discuss along with this one, viz., the application of methods of summation to *convergent* series.

The object of the ordinary methods of summation is to correlate a definite number, a "sum," with a series which has no sum in the ordinary sense. All such methods, if they are to be useful, must obey a *condition of consistency*—when

* G. H. Hardy, "On differentiation and integration of divergent series," *Cambridge Philosophical Transactions*, vol. XIX., p. 297; "Researches in the theory of divergent series and divergent integrals," *Quarterly Journal*, vol. XXXV., p. 22. See also Bromwich, *Infinite Series*, pp. 267 *et seq.*

† See the bibliography in § 3.

applied to convergent series they must yield the number which is its sum according to the ordinary definition. It is easy to imagine methods—such methods will be found in this paper (*e.g.*, § 26)—which satisfy this condition but are *wholly ineffective* for the summation of oscillating series: methods, that is to say, which are capable of summing *convergent series only*. But it is possible to go further,* and to imagine methods which will not sum even all convergent series, but only particular classes of them; such methods also will be considered in this paper, and possess properties of considerable interest. They also must, in so far as they are applicable, be subject to a condition of consistency. But their interest is quite different in character from that of the methods useful for oscillating series; there is naturally no particular object in attributing a "sum" to a series which already has one in the ordinary sense. We apply methods of summation to convergent series because of the light which they throw on the *nature* of the convergence of such series, and on the relation of their convergence to the non-convergence of oscillatory but summable series.

§ 3. It will be convenient to append a list of memoirs dealing with methods for the summation of series. This list does not profess to be complete: we have included in it only (*a*) memoirs later in date than those quoted in § 1 and (*b*) memoirs to which we shall refer in the course of the paper.

(1) H. Bohr: "Sur la série de Dirichlet," *Comptes Rendus*, 11 January, 1909.

(2) —— "Bidrag til de Dirichlet'ske Raekkers Theori," *Dissertation*, Copenhagen, 1910.

(3) E. Borel: "Mémoire sur les séries divergentes," *Annales de l'Ecole Normale Supérieure*, t. XVI., p. 1.

(4) T. J. I'a Bromwich: "Various extensions of Abel's lemma," *Proc. Lond. Math. Soc.*, vol. VI., p. 58.

(5) —— "The relation between the convergence of series and of integrals," *ibid.*, vol. VI., p. 326.

(6) —— "On the limits of certain infinite series and integrals, *Math. Ann.*, bd. LXV., s. 350.

(7) —— (and G. H. Hardy): "Some extensions to multiple series of Abel's theorem on the continuity of power series," *Proc. Lond. Math. Soc.*, vol. II., p. 161.

(8) S. Chapman: "Non-integral orders of summability of series and integrals," *Proc. Lond. Math. Soc.* (unpublished).

(9) L. Fejér: "Untersuchungen über Fouriersche Reihen," *Math. Ann.*, bd. LVIII., s. 51.

* S. Chapman (8)—see the bibliography in § 3.

(10) L. Fejér: "Ueber die Laplacesche Reihe," *ibid.*, bd. LXVII., s. 76.

(11) —— "Sur le développement d'une fonction arbitraire suivant les fonctions de Laplace," *Comptes Rendus*, 3 February, 1908.

(12) —— "Lebesque'sche Konstanten und divergente Fourierreihen," *Crelle's Journal*, bd. CXXXVIII., s. 22.

(13) M. Fekete: "Sur les séries de Dirichlet," *Comptes Rendus*, 25 April, 1910.

(14) G. H. Hardy: "Further researches in the theory of divergent series and integrals," *Camb. Phil. Trans.*, vol. XXI., p. 1.

(15) —— "Some theorems concerning infinite series, *Math. Ann*, bd. LXIV., s. 77.

(16) —— "On certain oscillating series," *Quarterly Journal*, vol. XXXVIII., p. 269.

(17) —— "Some theorems connected with Abel's theorem on the continuity of series," *Proc. Lond. Math. Soc.*, vol. IV., p. 247.

(18) —— "Generalisation of a theorem in the theory of divergent series," *ibid.*, vol. VI., p. 247.

(19) —— "The application to Dirichlet's series of Borel's exponential method of summation," *ibid.*, vol. VIII., p. 277.

(20) —— "Theorems relating to the summability and convergence of slowly oscillating series," *ibid.*, vol. VIII., p. 301.

(21) —— "Theorems connected with Maclaurin's test for the convergence of series," *ibid.* (unpublished).

(22) —— "Notes on some points in the integral calculus," *Messenger of Mathematics*, vol. XL., p. 108.

(23) K. Knopp: "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Dissertation*, Berlin, 1907.

(24) E. Landau: "Ueber die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes," *Monatshefte für Math.*, bd. XVIII., s. 8.

(25) J. E. Littlewood: "On the generalisation of Tauber's theorem," *Proc. Lond. Math. Soc.* (unpublished).

(26) C. N. Moore: "On the introduction of convergence factors into summable series and summable integrals," *Trans. Amer. Math. Soc.*, vol. VIII., p. 299.

(27) —— "The summability of the developments in Bessel functions, with applications," *ibid.* vol. X., p. 391.

(28) A. Pringsheim: "Ueber das Verhalten von Potenzreihen auf dem Convergenzkreise," *Münchener Sitzungsberichte*, bd. XXX., s. 43.

(29) —— "Ueber die Divergenz gewisse Potenzreihen au der Convergenzgrenze," *ibid.*, bd. XXXI., s. 505.

(30) M. Riesz: "Sur les séries trigonométriques, *Comptes Rendus*, 7 October, 1907.

(31) —— "Sur les séries de Dirichlet," *ibid.*, 21 June, 1909.

(32) —— "Sur la sommation des séries de Dirichlet," *ibid.*, 5 July, 1909.

(33) —— "Sur les séries de Dirichlet et les séries entières, *ibid.*, 22 November, 1909.

(34) E. Le Roy: "Sur les séries divergentes et les fonctions definies par un développement de Taylor," *Annales de la Faculté des Sciences de Toulouse*, sér. 2, t. 2, p. 317.

(35) W. Schnee: "Die Identität des Cesàroschen und Hölderschen Grenzwertes," *Math. Ann*, bd. LXVII., s. 110.

(36) — Tauber: "Ein Satz aus der Theorie der unendlichen Reihen," *Monatshefte für Math.*, bd. VIII., s. 273.

II.

General principles.

§ 4. Let $$f(n, p, \nu)$$
be a function of the three variables n, p, ν, defined for all real values of n (or all values numerically greater than some positive number N), all positive values of p (or all values greater than some positive number P), and all positive integral values of ν (including 0). Suppose further that $f(n, p, \nu)$ has the following properties:—

(i) $f = 0$ for $\nu > |n|$, so that
$$\lim_{\nu \to \infty} f = 0;$$

(ii)
$$\lim_{n \to +\infty} f = \phi(p, \nu)$$
and
$$\lim_{n \to -\infty} f = \psi(p, \nu)$$
exist for all values of p and ν in question;

(iii)
$$\lim_{p \to \infty} f = 1;$$

(iv)
$$\lim_{\nu \to \infty} \lim_{n \to +\infty} f = 0,$$
$$\lim_{\nu \to \infty} \lim_{n \to -\infty} f = \infty;$$

(v)
$$\lim_{p \to \infty} \lim_{n \to +\infty} f = 1,$$
$$\lim_{p \to \infty} \lim_{n \to -\infty} f = 1.$$

We now define $S(n, p)$ by the equation
$$S(n, p) = \sum_{\nu=0}^{[n]} u_\nu f(n, p, \nu),$$
where $[n]$ denotes the greatest integer less than or equal to $|n|$, so that $[n]$ is always positive. Then
$$\lim_{p \to \infty} S(n, p) = \sum_0^{[n]} u_\nu,$$
and therefore

(i)
$$\lim_{n \to \infty} \lim_{p \to \infty} S(n, p)$$

(i)′ $$= \lim_{n\to-\infty} \lim_{p\to\infty} S(n, p) = s,$$

if and only if the series Σu_ν *is convergent, with the sum s.*

The general principle of § 1 suggests that, when Σu_ν is *not* convergent, we should define its "sum" s by one or other of the equations

(2) $$\lim_{p\to\infty} \lim_{n\to+\infty} S(n, p) = s,$$

(2)′ $$\lim_{p\to\infty} \lim_{n\to-\infty} S(n, p) = s$$

if either of these repeated limits exists. And, moreover, as explained in § 2, the general principle may also be applied to convergent series so long as the "sum" so obtained is not different from the ordinary sum

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} u_\nu.$$

We shall call $s_n = \sum_0^n u_\nu$ the n^{th} *partial sum* of the series Σu_ν, $f(n, p, \nu)$ the *summing function*, and $S(n, p)$ the n^{th} *partial sum-function.*

§ 5. It is possible that a function such as $S(n, p)$ should possess a *double* limit in Pringsheim's sense—a limit obtained by making n and p tend simultaneously but independently to infinity. Here we must distinguish between *two* double limits, one in each of the two upper quadrants of the plane (n, p). We shall confine ourselves for the moment, in the interests of clearness, to the positive quadrant.

It is well known that if $S(n, p)$ has a limit when $p \to \infty$ (as is the case here), then the existence of the double limit involves that of the repeated limit

$$\lim_{n\to\infty} \lim_{p\to\infty} S(n, p),$$

i.e., the convergence of the series Σu_ν. Only in this case, then, can the double limit exist.* But although, save in the

* It will generally be the case that

$$\lim_{n\to\infty} S(n, p)$$

exists for any fixed p and is equal to

$$\sum_0^\infty u_\nu \phi(\nu, p).$$

If this be so, the existence of the double limit would also necessarily involve the existence of the repeated limit (2).

case of convergence, the double limit cannot exist, there is, of course, no reason why $S(n, p)$ should not tend to a limit when n and p tend to infinity simultaneously, but *in some special manner*, *i.e.*, so that some functional relation

$$(3) \qquad F(n, p) = 0$$

holds between n and p. If any such limit exists, it may be taken as a definition of the "sum" of Σu_ν. We shall then say that n and p tend to infinity along the path (F), and refer to the sum s as *corresponding to that path.* We have thus to consider not only the definition by means of the repeated limit (2), but also an infinity of definitions by means of double limits corresponding to particular paths.

Thus far we have been confining our remarks to the positive quadrant. In this quadrant the coefficients of the terms u_ν in the expression $S(n, p)$ are less than unity; they are, in a sense, *convergence factors* introduced to smooth out the oscillations of $\sum\limits_{\nu=0}^{n} u_\nu$ and to make it likely that $S(n, p)$ will tend to a limit related in a useful manner to the series Σu_ν. The main object of the limiting processes which are confined to this quadrant is to find this limit or "sum"; a secondary object is to examine the nature of the oscillation or non-convergence of the series.

When n and p lie in the *left* upper quadrant, however, the coefficients of the terms u_ν will in general be *greater* than unity: they are not convergence-factors but *divergence-factors.* They are not inserted in $S(n, p)$ with the object of finding the sum of the series, for that we may assume already known. The question of interest is to determine, to put it roughly, *how divergent* the factors may be taken ere $S(n, p)$ ceases to tend to a limit—a limit which may be a repeated limit (2′), a double limit, or a limit taken along some path in the negative quadrant with an equation such as (3). In this way we shall be able, to some extent, to classify the possible modes of convergence of the series.

III.

Examples of the general method.

§ 6. It will probably make these general considerations clearer if we illustrate them at once by means of some particular examples.

A. Methods of Cesàro-Riesz.

Let $\lambda(n)$ be a function of n which possesses a continuous and positive derivative $\lambda'(n)$ for all positive values of n, and which tends to ∞ with n: and let

(1) $$\lambda(n) = -\lambda|n| \quad (n<0),$$

so that $\lambda(n)$ is an odd function of n.*

Let

(2) $$f(n, p, \nu) = \left\{1 - \frac{\lambda(\nu)}{\lambda|n|}\right\}^{\lambda(n)/p}$$

for $0 \leqq \nu \leqq |n|$, and

(2') $$f(n, p, \nu) = 0$$

for $\nu > |n|$. Then it is obvious that conditions (i) and (iii) of § 4 are satisfied. Also

$$\lim_{n\to+\infty} f = e^{-\lambda(\nu)/p}, \qquad \lim_{n\to-\infty} f = e^{\lambda(\nu)/p},$$

so that conditions (ii), (iv), and (v) are also satisfied.

The repeated limit (1) of § 4 exists if, and only if, Σu_ν is convergent. Let us consider in what circumstances the repeated limit (2) of § 4 exists.

§ 7. We require the following theorem, related to what Dr. Bromwich† calls Tannery's theorem, in the same way as Abel's and Dirichlet's tests for the uniform convergence of series are related to Weierstrass's.

THEOREM. Let (i) Σa_ν *be convergent to sum* s,

(ii) $$0 \leqq b_\nu(n) \leqq 1,$$

(iii) $b_\nu(n)$ *increase or decrease steadily as* ν *goes from* 0 *to* n,

(iv) $$\lim_{n\to\infty} b_\nu(n) = 1.$$

Then will $$F(n) = \sum_0^n a_\nu b_\nu \to s$$

as $n \to \infty$.

We have

$$F(n) = \left(\sum_0^{N-1} + \sum_N^n\right) a_\nu b_\nu = F_1(n) + F_2(n),$$

* λ will be discontinuous at the origin unless $\lambda(0) = 0$.

† *Infinite Series*, pp. 113, 123.

say. Choose N so that

$$\left|\sum_{m_1}^{m_2} a_\nu\right| < \epsilon \quad (m_2 \geqq m_1 \geqq N).$$

Then
$$|F_2(n)| < \epsilon.^*$$

When N is fixed we can choose n_0 so that

$$\left|F_1(n) - \sum_0^{N-1} a_\nu\right| < \epsilon$$

for $n \geqq n_0$; and we see at once that

$$|F(n)| \leqq \left|F_1(n) - \sum_0^{N-1} a_\nu\right| + \left|\sum_N^\infty a_\nu\right| + |F_2(n)| < 3\epsilon$$

for $n \geqq n_0$. Thus the theorem is proved.†

§ 8. Suppose in particular that

$$a_\nu = e^{-\lambda(\nu)/p}\, u_\nu,$$

$$b_\nu = e^{\lambda(\nu)/p}\left\{1 - \frac{\lambda(\nu)}{\lambda|n|}\right\}^{\lambda(n)/p},$$

so that
$$S(n, p) = \sum_0^{[n]} u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda|n|}\right\}^{\lambda(n)/p} = \sum_0^{[n]} a_\nu b_\nu.$$

Then, writing λ for $\lambda(\nu)$ and λ_0 for $\lambda(n)$, we have

$$b_\nu = \exp\left(-\frac{\lambda^2}{2p\lambda_0} - \frac{\lambda^3}{3p\lambda_0^2} - \dots\right),$$

and it is clear that b_ν satisfies the conditions of the theorem. Thus

$$\lim_{n \to \infty} S(n, p) = \sum_0^\infty u_\nu e^{-\lambda(\nu)/p},$$

---

* Bromwich, *Infinite Series*, p. 54.

† There is no difficulty in extending the theorem to the case in which $b_\nu(n)$ is complex: we replace conditions (ii) and (iii) by the conditions

(ii)′ $$|b_0(n)| < K,$$

(iii)′ $$\sum_{m_1}^{m_2} |b_\nu(n) - b_{\nu+1}(n)| < K.$$

It is easy to frame analogous theorems for products.

provided only that the series on the righl-hand side is convergent. And hence *the repeated limit* (2) *of* § 4 *exists, and equals s, whenever the Dirichlet's series*

(3) $$\Sigma u_\nu e^{-x\lambda(\nu)}$$

is convergent for $x > 0$ *and tends to s as* $x \to 0$. We shall express this shortly by saying that *Abel's limit exists for the Dirichlet's series* (3).

If $\lambda(n) \to \infty$ with sufficient rapidity, the series (3) will have the property that its lines of convergence and absolute convergence coincide. In this case (which occurs, *e.g.*, if $\lambda(n) = n$) the proof can be simplified by using Tannery's theorem in place of that of § 7.

Similarly we prove that, *if the repeated limit* (2′) *of* § 4 *is to exist, we require not only that the series* Σu_ν *should be convergent, but also that the series* (3) *should be convergent for some negative values of* x.

Thus, in particular, when $\lambda(n) = n$, the existence of one of the three repeated limits (2), (1), (2′) corresponds to one of the three hypotheses:—

(2) Abel's limit

$$\lim_{x \to 1} \Sigma u_\nu x^\nu \quad (0 < x < 1)$$

exists (as for such series as $1 - 1 + 1 - \ldots$, $1 - 2 + 3 - \ldots$, etc.);

(1) Σu_ν is convergent;

(2)′ $\Sigma u_\nu x^\nu$ has a radius of convergence greater than unity.

§ **9**. So much for the three repeated limits: we must now consider limits along paths F (§ 5). Suppose, for example, that the path F is defined by

(4) $$\lambda(n) = \alpha p,$$

where α is a constant. The corresponding definition of the sum is as

$$\lim_{n \to \infty} \sum_0^{[n]} u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda[n]}\right\}^\alpha.$$

Thus the limits corresponding to paths (4) are precisely the limits adopted by Riesz* to define the sum of a series. When $\lambda(n) = n$ and α is a positive integer, these definitions are

* See his memoirs, (31)–(33).

known to agree with Cesàro's. In this case, if the limit exists, we shall say that Σu_ν is *summable* (R, λ, α). Generally, when n and p are restricted to move along a path F, we write

$$\lambda(n)/p = \omega(n);$$

and, if the corresponding limit exists, we say that Σu_ν is *summable* (R, λ, ω). If $\lambda(n) = n$ we may write (C, ω) for (R, n, ω).

We shall describe the whole system of methods of summation as *the Cesàro-Riesz system of type* λ, and we shall call $\omega(n)$ the *order* of the particular method considered.

It is sometimes convenient, for purposes of graphical representation, to regard $\lambda(n)$ and p as our independent variables rather than n and p. The curves (4) then become straight lines. We may then speak of *rectilinear* orders of summability and *curvilinear* orders—the rectilinear orders are those of Riesz's methods.

§ 10. We shall consider these methods of summation in greater detail later on (§§ 20–27). We may however mention here one or two particular results that throw some light upon their range of applicability.

(1) The series

$$1^s - 2^s + 3^s - \dots \quad (s > -1)$$

is summable $(R, n, s+\epsilon)$ for any positive ϵ, but not summable (R, n, s). This is still true if λ is substituted for n, provided (to put it roughly) the order of λ, as $n \to \infty$, is less than that of some power of n.

The series is *finite* (R, n, s) or (R, λ, s)—that is to say, $S(n, p)$ oscillates finitely as n and p tend to infinity along the curve $n = sp$ or $\lambda(n) = sp$.

(2) The series

$$e^{(\log 1)^2} - e^{(\log 2)^2} + e^{(\log 3)^2} - \dots$$

$$e^{1^s} - e^{2^s} + e^{3^s} - \dots,$$

where $0 < s < 1$, are not summable (R, λ, α) for any value of α. But they may be summed by methods of curvilinear order.

(3) Any convergent series Σu_ν, such that $\nu u_\nu \to 0$, is summable $(R, n, -1+\epsilon)$ for any positive ϵ.*

The rectilinear paths $\lambda = \alpha p$, where $\alpha < -1$, cover an angle $\frac{1}{4}\pi$ bordering upon the negative axis of λ. Paths

* S. Chapman (8).

which lie within this region are abnormal, and present peculiarities which can hardly be taken account of in a general theory. Thus divergent series of positive terms may be summable (with sum 0) along paths lying in this region.*

(4) The series $\Sigma\nu^{-1-ti}$ is not summable (R, n, α) for any value of α. Abel's limit does not exist for the series $\Sigma\nu^{-1-ti}x^{\nu}$, and it is probable that the series is not summable (R, n, ω) for any form of ω, though of this we have at present no formal proof. On the other hand the series is summable (R, λ, ϵ), when $\lambda = \log|n|$, for any positive ϵ.†

§11. The discussions which precede will have suggested to the reader the following general conclusions.

Given a Cesàro-Riesz scheme of type λ, *the corresponding methods of summation are more or less powerful according as the paths* F, *to which they correspond, approximate more or less closely to the positive axis of* n *(or of* λ*)—or, in other words, according as the increase of*

$$\omega = \lambda/p$$

is more or less rapid. The MOST *powerful method, in any such scheme, is that defined by the use of the repeated limit* (2), *which corresponds to the positive* n*-axis itself.*

A series summable (R, λ, ω), *to sum* s, *is summable* $(R, \lambda, \bar{\omega})$ *if* $\bar{\omega} > \omega$ *for* $n \geqq n_0$*: and in all cases of summability the repeated limit* (2) *exists and is equal to* s.

It will be well to say at once that we have not succeeded in establishing these conclusions in their full generality; and it may be that further conditions are required before they become universally valid. But there can be no doubt that they are generally true in the sense in which "generally" means "in all cases of interest"—and that they apply not only to these "methods of Cesàro-Riesz" but to many other systems of methods of summation. Certain interesting cases in which they are certainly true will be discussed in §§23–24.

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* S. Chapman (8)

† G. H. Hardy (20); M. Riesz (32). In connection with the assertions of §10 see also §§27–29. When we say "$\lambda = \log n$" it is convenient to regard λ as being defined, more precisely, by the equations

$$\lambda(n) = \log|n| \quad (|n| \geqq 1),$$
$$\lambda(n) = 0 \quad (-1 < n < 1)$$

in order to avoid irrelevant complications near $n = 0$.

B. Methods of Le Roy.

§ 12. A particularly interesting definition of the sum of an oscillatory series is that given by Le Roy, who defines the sum of Σu_ν as

$$\lim_{t\to 1} \Sigma \frac{\Gamma(\nu t+1)}{\Gamma(\nu+1)} u_\nu.^*$$

This method is an exceedingly powerful one; far more powerful than Borel's or Cesàro's or Euler's (which amounts practically to defining the sum of Σu_ν as $\lim\limits_{x\to 1} \Sigma u_\nu x^\nu$).† Thus, if $\Sigma u_\nu = \Sigma a_\nu x^\nu$ is a power-series which has a finite radius of convergence, Le Roy's definition gives its sum at all points of the "étoile"‡ of $f(x)$, the analytic function which it represents when convergent; thus it gives the sum $1/(1-x)$ for Σx^ν, save when x is real and greater than (1).

A definition substantially equivalent to Le Roy's is that by means of the limit

$$(2) \qquad \lim_{\delta\to\infty} \Sigma u_\nu e^{-\delta\nu\log\nu},$$

which has, in the case of power series of finite radius of convergence, precisely similar powers.

To prove this it is only necessary to consider the series Σx^ν; the extension to the general power series may then be effected precisely on the lines adopted by himself.§

The series in (2) is convergent for any positive δ: also

$$\Sigma x^\nu e^{-\delta\nu\log\nu} = 1 + \int_C x^z e^{-\delta z\log z} \frac{dz}{e^{2\pi iz}-1},$$

where C is a contour enclosing the points 1, 2, 3, ... (but not the origin) and going off to infinity in a direction roughly parallel to the real axis (fig. 1). This contour may be transformed into the contour C' shown in fig. 1, consisting of two lines making acute angles ϕ_1, ϕ_2 with the positive real axis of z.

It is easy to see that, unless x is real and positive, ϕ_1 and ϕ_2 can be chosen so that

$$\int_{C'} x^z e^{-\delta z\log z} \frac{dz}{e^{2\pi iz}-1} \to \int_{C'} \frac{x^z\,dz}{e^{2\pi iz}-1}$$

---

* Le Roy (34); Bromwich, *Infinite Series*, p. 299.

† Bromwich, *Infinite Series*, p. 266.

‡ The region bounded by lines drawn from the singular points to infinity directly away from the origin.

§ Le Roy, *l.c.*

as $\delta \rightarrow 0$, and that the value of this integral is $x/(1-x)$. Hence our conclusion is established.*

If now, in the investigations of §§ 6—11, we take

$$\lambda(n) = n \log |n|,$$

we see that the repeated limit (2) assumes the form

$$\lim_{p \rightarrow \infty} \Sigma u_\nu e^{-(\nu \log \nu)/p},$$

a limit equivalent to (2) above. Thus *Le Roy's definition*, or rather the substantially equivalent definition by which we have replaced it, *is equivalent to the definition by the repeated limit* (2) *of the Cesàro-Riesz system of type* $n \log |n|$. In this system there are, of course, included an infinity of definitions of less range of applicability. In particular, those corresponding to paths

$$n \log |n| = \alpha p$$

do not differ materially in range from the corresponding definitions of the ordinary Cesàro-Riesz scheme of type n. The wider range of the system as a whole† is entirely accounted for by the wider range of the methods for which $\omega(n) \rightarrow \infty$.

The results of this and the preceding section may be extended to a considerable class of forms of $\lambda(n)$—roughly to the class

$$\lambda(n) = n\,\phi(n),$$

where $\phi|n|$ is a function which tends to infinity with $|n|$, but more slowly than any power of $|n|$.

C. *A new system of definitions.*

§ 13. An interesting particular case of our general definition of §§ 4—5 is that in which $f(n, p, \nu)$ does not explicitly involve n. We may then write

$$\begin{aligned} f(n, p, \nu) &= \phi(p, \nu) \quad && (0 \leqq \nu \leqq n) \\ &= 0 && (\nu > n), \end{aligned}$$

if n is positive, and

$$\begin{aligned} f(n, p, \nu,) &= \psi(p, \nu) \quad && (0 \leqq \nu \leqq |n|) \\ &= 0 && (\nu > |n|), \end{aligned}$$

---

* Cf. G. H. Hardy (the second paper quoted in § 1) for an analogous discussion for integrals.

† The series $1 - x + x^2 - \dots \quad (x > 1)$ is summable by methods of this system, though Abel's limit obviously does not exist for it or for any power-series whose radius is less than unity.

The repeated limit (2′) exists for this system if, and only if, $\Sigma u_\nu x^\nu$ is an integral function.

if n is negative. If

$$\lim_{\nu\to\infty} \phi = 0, \qquad \lim_{\nu\to\infty} \psi = \infty,$$

$$\lim_{p\to\infty} \phi = 1, \qquad \lim_{p\to\infty} \psi = 1,$$

it is clear that the conditions of § 4 are all satisfied.

Thus a natural choice is

$$\phi = e^{-\lambda(\nu)/p}, \qquad \psi = e^{\lambda(\nu)/p}.*$$

Then

(1) $$S(n, p) = \sum_0^{[n]} u_\nu e^{-\{\lambda(\nu)/p\}\,\mathrm{sgn}\, n}.\dagger$$

The existence of the repeated limit (2), as in the case of the Cesàro-Riesz system, corresponds to the existence of Abel's limit for the Dirichlet's series $\Sigma u_\nu e^{-x\lambda(\nu)}$. Similarly the interpretation of the existence of the repeated limit (2′) is the same as in § 8. Thus, for the series

$$1 - 1 + 1 - \ldots,$$

we have, when $\lambda(n) = n$,

$$S(n, p) = \sum_0^{[n]} (-1)^\nu e^{-\nu/p} = \frac{1 + (-1)^{[n]} e^{-([n]+1)/p}}{1 + e^{-1/p}},$$

which tends to the limit $\frac{1}{2}$ if, and only if,

$$n/p \to \infty.$$

§ 14. The great simplicity of this definition makes it worth while to examine a few special cases. First, let us consider the *convergent* series

$$1^{-s} - 2^{-s} + 3^{-s} \ldots \quad (0 < s < 1).$$

Using the familiar formula

$$\frac{1}{(\nu+1)^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(\nu+1)u} u^{s-1}\, du$$

we obtain‡

$$S(n, p) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u} u^{s-1} \frac{1 \pm e^{-([n]+1)(u-\delta)}}{1 + e^{-(u-\delta)}}\, du,$$

---

* Or $e^{-(\lambda/p)^2}$, $e^{(\lambda/p)^2}$, etc.

† Here sgn n is "the sign of n," *i.e.*, $n/|n|$.

‡ Here $n < 0$, and $f = e^{+\nu/p}$.

where $\delta = 1/p$ and the ambiguous sign depends on whether $[n]$ is even or odd. It is very easy to prove that, of the two parts into which this integral falls apart, the first tends to the limit

$$(2) \qquad \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-u}u^{s-1}}{1+e^{-u}}\,du,$$

and that the second will tend to zero, provided

$$\frac{e^{\delta[n]}}{\Gamma(s)}\int_0^\infty e^{-[n]u}u^{s-1}\,du = e^{\delta[n]}[n]^{-s}$$

does so, or if $\quad s\log|n| - (|n|/p) \to \infty.$

This is satisfied, *e.g.*, if

$$p > \frac{|n|}{k\log|n|} \quad (k<s).*$$

We may apply a similar method to the oscillatory series

$$1^s - 2^s + 3^s - \dots \quad (s>0)$$

if we use loop-integrals instead of linear-integrals. We then obtain results resembling some which will be found later (§ 27) in connection with the Cesàro-Riesz system of type n.

We shall find, in fact, that the series is summable if

$$(n/p) - s\log n \to \infty.$$

There is, however, a different method, less simple but more powerful, which leads to wider results.

§ 15. Consider the series

$$e^{\sqrt{1}} - e^{\sqrt{2}} + e^{\sqrt{3}} - \dots.$$

Here $\quad S(n,p) = \sum_0^{[n]} (-1)^\nu e^{\sqrt{\nu}-(\nu/p)}$

$$= 1 + \frac{1}{2\pi i}\left(\int_{\mu-i\infty}^{\mu+i\infty} - \int_{\lambda-i\infty}^{\lambda+i\infty}\right)\frac{\pi}{\sin z\pi}e^{\sqrt{z}-(z/p)}\,dz,$$

where λ, μ are constants such that $0<\lambda<1$, $[n]<\mu<[n]+1$ (*e.g.*, $\lambda=\frac{1}{2}$, $\mu=[n]+\frac{1}{2}$). This follows at once from Cauchy's theorem if we observe that $\operatorname{cosec} z\pi$, where $z=\xi+i\eta$, tends to zero, as $|\eta|\to\infty$, like $2e^{-|\eta|\pi}$. In $e^{\sqrt{z}}$, of course, we take

* When n is negative (*i.e.*, if we are dealing with *convergent* series) then increasing p (as compared with n) *increases* the efficacy of our method. When n is positive, of course, the reverse is true.

the principal value of $\sqrt{z}$. It is easy to see that the second line-integral tends to the limit

$$-\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\pi}{\sin z\pi}e^{\sqrt{z}}\,dz.$$

As regards the other line-integral we have only to observe that, if $z=\mu+i\eta$,

$$\left|\frac{\pi}{\sin z\pi}\right|<Ke^{-|\eta|\pi},$$

$$|e^{-(z/p)}|=e^{-\mu/p}<e^{-[n]/p},$$

$$|e^{\sqrt{z}}|=e^{\sqrt{\sqrt{(\mu^2+\eta^2)}}}<e^{\sqrt{\mu}+\sqrt{|\eta|}}<Ke^{\sqrt{[n]}+\sqrt{|\eta|}},$$

and we see that the integral certainly tends to zero if

$$e^{-[n]/p+\sqrt{[n]}}\int_{-\infty}^{\infty}e^{-|\eta|\pi+\sqrt{|\eta|}}d\eta$$

does so, *i.e.*, if $\qquad ([n]/p)-\sqrt{[n]}\to\infty,$

or if $\qquad (n/p)-\sqrt{n}\to\infty.$

This method may be extended (under very general conditions) to the series

$$\phi(1)-\phi(2)+\phi(3)-\ldots,$$

where $\phi(\nu)$ is a function of n, which tends to infinity more slowly than $e^{\delta\nu}$, however small be δ—the general condition is

$$(n/p)-\log\phi(n)\to\infty.$$

It is very interesting to compare these results with those which we shall obtain later for the methods of Cesàro-Riesz.

We observe finally, with a view to this comparison, that the convergent series of § 14 is summable along any line $n/p=\alpha$ (α positive or negative), while the oscillatory series of § 15 are not summable along any such line.

D. Methods of Borel.

§ 16. Another interesting system of methods of summation is included in those of § 13.

We confine ourselves in this case to positive values of n, and take

$$\phi(p,\nu)=\frac{1}{\nu!}\int_0^p e^{-x}x^{\nu}dx.*$$

* More generally, $\qquad \phi(p,\nu)=\frac{1}{\Gamma(\alpha\nu+1)}\int_0^p e^{-x}x^{\alpha\nu}dx.$

This assumption would lead us to consider Le Roy's generalisation of Borel's method (Bromwich, *Infinite Series*, p. 299. Le Roy appears to confine himself to the case in which α is an integer).

Here

$$\lim_{n\to\infty} S(n, p) = \lim_{n\to\infty} \sum_0^{[n]} \frac{u_\nu}{\nu!} \int_0^p e^{-x} x^\nu dx = \int_0^p e^{-x} u(x)\, dx,$$

if Borel's associated function

$$u(x) = \sum_0^\infty \frac{u_\nu}{\nu!} x^\nu$$

is an integral function. Hence the repeated limit (2) takes the form

$$\int_0^\infty e^{-x} u(x)\, dx,$$

so that *the definition by the repeated limit* (2) *is equivalent to Borel's definition.** And once more we obtain an infinity of other definitions corresponding to simultaneous limits along particular paths.

Consider, *e.g.*, the series

$$1 - 1 + 1 - \dots .$$

If we use the formula

$$\frac{1}{\nu!} \int_0^p e^{-x} x^\nu dx = 1 - e^{-p} \left(1 + p + \frac{p^2}{2!} + \dots + \frac{p^\nu}{\nu!}\right),$$

we find that

$$S(n, p) = e^{-p} \left(p + \frac{p^3}{3!} + \dots + \frac{p^{[n]}}{[n]!}\right),$$

or $$= 1 - e^{-p} \left(1 + \frac{p^2}{2!} + \dots + \frac{p^{[n]-1}}{([n]-1)!}\right),$$

according as $[n]$ is odd or even.

Now it is not difficult to prove that

$$1 + \frac{p^2}{2!} + \dots + \frac{p^{2k}}{2k!} \backsim p + \frac{p^3}{3!} + \dots + \frac{p^{2k+1}}{(2k+1)!} \backsim \tfrac{1}{2} e^p,$$

if $$p \to \infty, \qquad k/p \to \infty$$

—more precisely if

$$k = p + q, \qquad q/\sqrt{p} \to \infty .$$

---

* Borel (3). Borel's definition is *less* powerful than Le Roy's, but on the whole more powerful than Euler's. But it will not sum *all* series for which Abel's limit exist; not even all series summable by Cesàro's simplest method (see G. H. Hardy, first paper quoted in § 1, and Bromwich, *Infinite Series*, p. 320).

Hence if $\quad n=p+q, \quad q/\sqrt{p} \to \infty,$

the series is summablo to sum $\frac{1}{2}$.

It may be shown without difficulty that

$$1+p+\frac{p^2}{2!}+\ldots+\frac{p^p}{p!} \backsim \tfrac{1}{2}e^p.*$$

Using this result we can prove that, if $[n]=p$, $S(n, p)$ oscillates between the limits $\frac{1}{4}$ and $\frac{3}{4}$. By taking $n=p+q$, where

$$q/\sqrt{p} \to \beta,$$

β being a suitably chosen constant, we can make $S(n, p)$ oscillate between γ and $1-\gamma$, where γ is any positive number between 0 and 1. The limits of oscillation are in fact

$$\tfrac{1}{2}\left\{1 \mp \frac{1}{\sqrt{(2\pi)}}\int_\beta^\infty e^{-\frac{1}{2}t^2}\,dt\right\}.$$

IV.

Some general considerations relating to double and repeated limits.

§ 17. The reader will have by now (cf. § 11) a sufficiently clear idea of the general relations subsisting between the various methods of summation included in a general scheme such as was defined in §§ 4, 5. In fig. 2 we have represented these relations roughly by a figure. The lines 1–6 correspond to methods of increasing efficacy; the line 6 to the most powerful definition (Abel's limit for the Cesàro-Riesz scheme of type n or for the scheme of § 13, Le Roy's or Borel's definition for the schemes of §§ 11, 12, or § 17). The line 3 corresponds to the ordinary definition of the sum of a convergent series; the lines to the left of it to methods which apply only to convergent series. In this section we propose to indicate the relation to these ideas of some expounded by F. London in an extremely interesting paper on double limits.†

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* Mr. Littlewood informs us that the property that *the terms up to the greatest term are asymptotically equivalent to half the whole series* is a characteristic of large classes of integral power-series.

† "Ueber Doppelfolgen und Doppelreihen," *Math. Ann*, Bd. LIII., S. 322. This paper should be compared with two by Pringsheim ("Zur Theorie der zweifach unendlichen Zahlenfolgen," *ibid.*, S. 289, and "Elementäre Theorie der unendlichen Doppelreihen," *Münchener Sitzungsberichte*, Bd. XXVII., S. 101).

§ 18. Let $S(n, p)$ be any function of two positive integral variables n and p.* Let there be a functional relation

$$F(n, p) = 0, \quad n = N(p), \quad p = P(n),$$

which possesses the following properties:—

(1) To every integral value of n (p) corresponds at least one (not necessarily only one) value of p (n).

(2) N and P are steadily increasing functions of p and n respectively.†

(3) N and P tend to ∞ with p and n respectively.

The functional relations can evidently be represented by an infinity of points

$$(n_i, p_i) \quad i = 0, 1, 2, \ldots,$$

where $$n_i \leqq n_{i+1} \leqq n_i + 1, \qquad p_i \leqq p_{i+1} \leqq p_i + 1;$$

and is graphically represented by a broken line L (fig. 3). We shall call this broken line *the path* F (cf. § 5). Should $S(n, p)$ tend to a limit when n and p tend to infinity along the line L, we shall say that S *converges for the path* F.

The region H_F between the axis of n and the path F we shall call the *horizontal region for* F: the region V_F between the axis of p and the path F we shall call the *vertical region for* F.

The double sequence

$$(S(n, p))$$

will be said to be *horizontally convergent* if it is possible so to determine F that S converges for all paths F_H which lie ultimately‡ inside or on the boundary of H_F. Similarly we define *vertical convergence*.

Then London has proved that, on the assumption

$$\lim_{n \to \infty} S(n, p)$$

exists for every particular value of p, *the necessary and sufficient condition for the existence of the repeated limit*

$$\lim_{p \to \infty} \lim_{n \to \infty} S(n, p)$$

lies in the horizontal convergence of the double sequence (S).

---

* There is no difficulty in applying London's results to the case in which $p \to \infty$ continuously, or through an enumerable sequence of values not necessarily integral, or to the case in which negative values of n are taken into account.

† If $p_2 > p_1$ *any* of the values of $N(p_2)$ is at least as great as *any* of the values of $N(p_1)$.

‡ For values of n (or p) greater than some definite value. There is no loss of energy in supposing these paths subject to the same restrictions as F itself.

§ 19. This conclusion remains valid if n and p are continuous variables, as the reader may easily satisfy himself by an examination of London's argument.

Similarly the necessary and sufficient condition for the existence of the other repeated limit lies in the vertical convergence of the double sequence. From these results, and the corresponding results for the quarter plane $n<0$, $p>0$, it at once follows that *if the repeated limit* (2) *exists, the series* Σu_ν *is summable for all paths* F *lying ultimately in a certain region above the positive axis* n; *and similarly for the repeated limit* (2′). *And if the repeated limit* (1) *exists, i.e., if the series is convergent, then the series is summable for all paths* F *lying ultimately in a certain region enclosing on both sides the axis of* p.

Finally, if we assume that $S(n,p)$ is a continuous function of n and p (as will usually be the case), we can apply a modification of London's argument to show that *if the series is summable along any path* F, *it is summable along all paths lying ultimately within a certain region enclosing* F *on both sides.*

This remark is of importance as showing that, to put it roughly, *the region of summability of the series is essentially an* OPEN *region.*

In the light of these general remarks, we proceed to consider more particularly the most interesting* of our general systems.

V.

General theorems concerning the methods of Cesàro-Riesz.

§ 20. We recall the expression for the n^{th} partial sum-function corresponding to the methods of Cesàro-Riesz, viz.,

$$S(n, p) = \sum_{\nu=0}^{[n]} u_\nu \left\{1 + \frac{\lambda(\nu)}{\lambda\,|n|}\right\}^{\lambda(n)/p};$$

$[n]$ and $\lambda(n)$ have the meanings explained in § 6.

An important property which every method of summation of oscillatory series should possess† is that of applicability to

* In spite of its comparatively narrow range, the especial interest of this system is due to the fact that it includes so many paths corresponding to familiar definitions.

† For example, if we were dealing with the sum of a convergent series and a summable series, it would obviously be convenient to treat the two series together by one method of summation.

every convergent series: any method of summation which can be considered as at all *natural* may be expected to have this property. Furthermore, the value so arrived at for the "sum" of the convergent series must be the same as the value of the ordinary sum (which we denote by s). Evident complications will arise if the latter conditions be not satisfied.

A method of summation which has these two properties combined is said to satisfy the "condition of consistency." We proceed to show that the methods of Cesàro-Riesz, in their most general form,* satisfy this condition.

We shall prove, in fact, that *the existence of the repeated limit*

$$(1) \qquad \lim_{n\to\infty}\ \lim_{p\to\infty} S(n, p) = s,$$

or, what is the same thing, the convergence of Σu_ν *to the sum* s, *involves the existence of the Pringsheim double limit*

$$(2) \qquad \lim_{n, p\to\infty} S(n, p)$$

in the positive quadrant; and therefore involves the existence of a limit for ALL *paths* F, *and the existence of the other repeated limit (i.e., of Abel's limit for the Dirichlet's series* $\Sigma u_\nu e^{-x\lambda(\nu)}$), *and the equality of all these limits to* s.

§ 21. This will be deduced from the following general theorem.

THEOREM. *Let*

$$S(n, p) = \sum_0^n a_\nu b_\nu(n, p),$$

where $b_\nu(n, p)$ *satisfies the following conditions:*

(i) $$0 \leqq b_\nu(n, p) \leqq 1;$$

(ii) *when* n *and* p *are fixed the sequence* $b_0(n, p), b_1(n, p), \ldots, b_n(n, p)$ *is monotonic, whether increasing or decreasing;*

(iii) $b_\nu(n, p) \to 1$ *as* $p \to \infty$, *and, moreover, when* N *has been fixed, we can choose* n_0 *so that* $b_\nu(n, p) \to 1$ *uniformly for* $\nu = 1, 2, \ldots, N-1$ $(n \geqq n_0)$.

---

* "The most general form" means that the limit of $S(n, p)$ may be taken along *any path whatever* in the positive quadrant of the $\lambda(n)-p$-plane. The negative upper quadrant is excluded from consideration here.

Then the existence of the repeated limit (1) *involves the existence of the double limit** (2).

Let
$$c_\nu(n, p) = 1 - b_\nu(n, p),$$
$$T(n, p) = \sum_0^n a_\nu c_\nu(n, p).$$

We have to prove that if Σa_ν is convergent then, given ϵ, we can find n_0, p_0 so that
$$|T(n, p)| < \epsilon \quad (n \geqq n_0;\ p \geqq p_0);$$
for
$$T(n, p) = \sum_0^n a_\nu - S(n, p)$$
$$= s + \epsilon_n - S(n, p),$$
where $\epsilon_n \to 0$ as $n \to \infty$, independently of p.

Now
$$T(n, p) = \sum_0^{N-1} a_\nu c_\nu + \sum_N^n a_\nu c_\nu,$$
and by Abel's well-known lemma†
$$\left| \sum_N^n a_\nu c_\nu \right| < H\bar{c} \leqq H,$$
where $\bar{c}$ is the upper limit of the terms c_ν and is therefore not greater than 1; and H is the upper limit of $\sum_N^m a_\nu$ for $m = N+1$ to n. Since Σa_ν is convergent, we can choose N_0 so that $H < \epsilon$, *i.e.*,

(α)
$$\left| \sum_N^n a_\nu c_\nu \right| < \epsilon$$
for $N \geqq N_0$, $n \geqq N$, and all values of p.

Also, when N is fixed, in virtue of condition (iii), we can choose n_0, p_0 so that
$$|c_\nu| < \epsilon'$$
for $\nu = 0, 1, \ldots, N-1$, $n \geqq n_0$, $p \geqq p_0$, and therefore

(β)
$$\left| \sum_0^{N-1} a_\nu c_\nu \right| < \epsilon.$$

---

* And therefore, of course, if $\lim_{n\to\infty} S(n, p)$ exists, the existence and equality to s of the other repeated limit. Evidently $\lim_{n\to\infty} S(n, p)$ does exist in the case of the Cesàro-Riesz methods when Σu_ν is convergent.

† Cf. Bromwich, *Infinite Series*, § 23.

From (α) and (β) it follows that

$$|T(n, p)| < 2\epsilon$$

for $n \geqq n_0$, $p \geqq p_0$. Thus the theorem is established.

§ 22. In the particular case in question we have

$$b_\nu(n, p) = \left\{1 - \frac{\lambda(\nu)}{\lambda|n|}\right\}^{\lambda(n)/p}.$$

Since n and p are in the positive quadrant, $\lambda(n)$ and p are positive. Heuce, conditions (i) and (ii) are satisfied. Also $b_\nu(n, p) \to 1$ as $p \to \infty$, and we have therefore only to verify that condition (iii) is fully satisfied.

Now
$$0 < c_\nu(n, p) = 1 - \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^{\lambda(n)/p}$$
$$< 1 - \left\{1 - \frac{\lambda(N)}{\lambda(n)}\right\}^{\lambda(n)/p},$$

if $0 \leqq \nu < N$. When n is fixed we can choose n_0 so that $\lambda(n) > 2\lambda(N)$ for $n \geqq n_0$. The last expression is

$$1 - \exp\left[-\frac{\lambda(N)}{p} - \frac{\{\lambda(N)\}^2}{2p\,\lambda(n)} - \ldots\right]$$
$$< 1 - \exp\left[-\frac{\lambda(N)}{p} - \frac{\{\lambda(N)\}^2}{p\lambda(n)} - \ldots\right]$$
$$= 1 - \exp\left[-\frac{\lambda(N)}{p} - \frac{\{\lambda(N)\}^2}{p\{\lambda(n) - \lambda(N)\}}\right]$$
$$< 1 - \exp\left[-\frac{2\lambda(N)}{p}\right],$$

which tends to zero as $p \to \infty$. Thus condition (iii) is fully satisfied.

We have therefore proved that, if Σa_ν is convergent, and has the sum s, then

$$S(n, p) = \sum_0^n u_\nu \left\{1 - \frac{\lambda(\nu)}{\lambda(n)}\right\}^{\lambda(n)/p}$$

tends to the limit s when n and p tend to infinity *in any way whatever*, simultaneously or successively, in the positive quadrant

If we make $n \to \infty$ first we obtain the well-known generalization of Abel's theorem, that the convergence of Σa_ν involves

$$\Sigma a_\nu e^{-x\lambda(\nu)} \to s$$

as $x \to 0$. If we make $\lambda(n) = \alpha p$ we obtain the condition of consistency for Riesz's most general definition with index α.

§ 23. The methods of Cesàro-Riesz thus satisfy (in the positive quadrant) what we have called the "condition of consistency." But the conception of consistency may be generalized, along the lines adumbrated in § 11, so as to require that a series summable (R, λ, ω) along any particular path F shall be summable (with the same sum) when the limit is taken along any other path F_H ultimately lying in the region between F and the positive axis of $\lambda(n)$.*

As we stated in § 11, we have not succeeded in showing that the methods of Cesàro-Riesz satisfy this generalized condition of consistency when F and F_H are quite unrestricted. If, however, we restrict F to be a straight line, so that

$$\lambda(n)/p = \alpha,$$

where $\alpha > -1$ (*i.e.*, the path F makes with the positive $\lambda(n)$-axis an angle which is less than $3\pi/2$), it is not difficult to establish the desired result: and this is so without any restriction on the form of the path F_H, provided that this lies in the region specified. Thus, if the equation to the path F_H is expressed in the form

$$\lambda(n)/p = h(n),$$

the only condition which F_H must satisfy is

$$h(n) > \alpha,$$

for $n \geqq n_0$.

We therefore state the following theorem :—

If Σu_n is summable along any path F corresponding to the equation $\lambda(n)/p = \alpha$ $(\alpha > -1)$, then it is also summable, with the same sum s, along any path F_H ultimately lying in the region between F and the positive $\lambda(n)$-axis, so that if F_H corresponds to the equation $\lambda(n)/p = h(n)$, we have $h(n) > \alpha$, for $n \geqq n_0$.

---

* We are now considering the negative as well as the positive quadrant. And we are now regarding $\lambda(n)$, and not n, as one of the independent variables in our geometrical representation (§ 9).

In the succeeding proof, for convenience in writing out our formulæ, $h(n)$ will be frequently abbreviated to h, where no confusion is likely to arise; h will therefore always mean $h(n)$.

§ 24. Our proof will be based on the formulæ

$$(n-l)^h = \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\,\Gamma(\alpha+1)} \int_l^n (n-k)^{h-\alpha-1}(k-l)^\alpha\, dk,$$

$$\{\lambda|n| - \lambda(l)\}^h = \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\,\Gamma(\alpha+1)}$$

$$\times \int_l^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1}\{\lambda(k)-\lambda(l)^\alpha\lambda'(k)\, dk,$$

which hold for $-1<\alpha<h$.

Summing from $l=0$ to $l=[n]$, and inverting the order of summation and integration, we obtain the equation

$$\sum_{l=0}^{[n]} u_l\{\lambda|n| - \lambda(l)\}^{h(n)} = \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\,\Gamma(\alpha+1)}$$

$$\times \int_0^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1}\Big[\sum_{l=0}^{[k]} u_l\{\lambda(k)-\lambda(l)\}^\alpha\Big]\lambda'(k)\, dk.$$

Now, by hypothesis, we have

$$\sum_{l=0}^{[k]} u_l\{\lambda(k)-\lambda(l)\}^\alpha = (s+\epsilon_k)\,\}\lambda(k)\}^\alpha,$$

where $\epsilon_k \to 0$ as $k\to\infty$. Hence

$$\sum_{l=0}^{[n]} u_l\{\lambda|n| - \lambda(l)\}^{h(n)} = \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\,\Gamma(\alpha+1)}$$

$$\times \int_0^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1}\{\lambda(k)\}^\alpha(s+\epsilon_k)\lambda'(k)\, dk$$

$$= s\,\{\lambda|n|\}^{h(n)} + I,$$

where I denotes the expression

$$\frac{\Gamma(h+1)}{\Gamma(h-\alpha)\,\Gamma(\alpha+1)} \int_0^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1}\{\lambda(k)\}^\alpha\,\epsilon_k\lambda'(k)\, dk.$$

Choose k_0 so that, for $k \geqq k_0$, $|\epsilon_k|<\epsilon$, the latter being an arbitrarily assigned small positive number. Divide the range of integration of the last integral into two parts $(0, k_0)$

and $(k_0, |n|)$, and denote the corresponding two parts of I by I_1 and I_2 respectively. Then

$$I_1 = \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)} \int_0^{k_0} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1} \{\lambda(k)\}_\alpha \epsilon_k \lambda'(k)\, dk,$$

and

$$I_2 < \epsilon \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)} \int_{k_0}^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1} \{\lambda(k)\}^\alpha \lambda'(k)\, dk.$$

In I_1 a finite upper limit K can be found to the expression $\epsilon_k \{\lambda(k)\}^\alpha$ over the range $(0, k_0)$. Hence

$$|I_1| < K \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)} \int_0^{k_0} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1} \lambda'(k)\, dk$$

$$\leqq K' \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)} \{\lambda|n|\}^{h-\alpha-1} \quad ,$$

where K' is independent of n. Consequently

$$|I_1 \{\lambda|n|\}^{-h(n)}| \leqq K' \frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)} \{\lambda|n|\}^{-\alpha-1}.$$

There are now two cases of importance to consider:—

(*a*) If $h(n)$ has a finite upper limit as $n \to \infty$, the same is true of

$$\frac{\Gamma(h+1)}{\Gamma(h-\alpha)\Gamma(\alpha+1)};$$

if $h-\alpha$ should tend to zero, $\Gamma(h-\alpha) \to \infty$, and the above expression tends to zero. Also, since $-\alpha-1<0$, $\{\lambda|n|\}^{-\alpha-1} \to 0$ as $n \to \infty$, for as $n \to \infty$ so does $\lambda|n|$. Hence, in case (*a*),

$$\lim_{n\to\infty} I_1 \{\lambda|n|\}^{-h(n)} = 0.$$

(*b*) If, however, $h(n) \to \infty$ as $n \to \infty$, we proceed as follows: A constant N can be found such that, for $n \geqq N$,

$$\frac{\Gamma(h+1)}{\Gamma(h-\alpha)} < 2h^{\alpha+1}.$$

Therefore, for $n \geqq N$,

$$|I_1 \{\lambda|n|\}^{-h(n)}| < \frac{2}{\Gamma(\alpha+1)} \left\{\frac{h(n)}{\lambda(n)}\right\}^{\alpha+1},$$

which tends to zero as $n \to \infty$, since $\alpha + 1 > 0$ and

$$h(n)/\lambda(n) = 1/p,$$

while $p \to \infty$ as $n \to \infty$.

These two cases are the only ones of any particular interest. But it is of course possible (even when p is a monotonic function of n) that

$$h(n) = \lambda(n)/p$$

should oscillate as $n \to \infty$. In this case we can combine the arguments used under (a) and (b) above; we can, in fact, determine K and N so that

$$\frac{\Gamma(h+1)}{\Gamma(h-\alpha)} < K + 2h^{\alpha+1} \quad (n \geqq N),$$

and the argument proceeds substantially as before. Thus, in all cases,

$$\lim I_1 \{\lambda |n|\}^{-h(n)} = 0.$$

Again

$$\int_{k_0}^{|n|} \{\lambda|n| - \lambda(k)\}^{h-\alpha-1} \{\lambda(k)\}^{\alpha} \lambda'(k)\, dk$$
$$< \frac{\Gamma(h-\alpha)\Gamma(\alpha+1)}{\Gamma(h+1)} \{\lambda|n|\}^{h(n)}.$$

Hence $$|I_2| < \epsilon \{\lambda |n|\}^{h(n)}.$$

But (as we have just seen) we can determine n_0, when k_0 is fixed, so that

$$|I_1| < \epsilon \{\lambda |n|\}^{h(n)} \quad (n \geqq n_0).$$

Hence $$|I| < 2\epsilon \{\lambda |n|\}^{h(n)} \quad (n \geqq n_0),$$

and so, finally

$$\lim_{n \to \infty} \sum_{l=0}^{[n]} u_l \left\{1 - \frac{\lambda(l)}{\lambda|n|}\right\}^{h(n)} = s.$$

Thus the theorem is proved.

VI.

Some corollaries and additional results.

§ 25. The general theorem of § 24 includes as a particular case:—

I. *If Σu_ν is summable (R, λ, α) with sum s, it is summable (R, λ, β), with the same sum, for $\beta > \alpha > -1$.*

II. *If Σu_ν is summable (R, λ, α), and the series $\Sigma u_\nu e^{-x\lambda(\nu)}$ is convergent for $x > 0$, then*

$$\Sigma u_\nu e^{-x\lambda(\nu)} \rightarrow s$$

*as $x \rightarrow 0$.**

For then $\lim\limits_{n\rightarrow\infty} S(n, p)$ exists, and so, in virtue of London's results (§§ 18, 19), the repeated limit

$$\lim_{p\rightarrow\infty} \lim_{n\rightarrow\infty} S(n, p)$$

exists and is equal to s.

III. *If* $\lim\limits_{x\rightarrow 0} \Sigma u_\nu e^{-x\lambda(\nu)} = s$, *the series Σu_ν is summable (R, λ, ω) for* SOME $\omega < \lambda$.

This is an immediate corollary from London's results of §§ 18, 19.

IV. *If $\lambda'(n)/\lambda(n)$ tends steadily to zero, and*

$$\frac{\lambda(n)\, u_n}{\lambda'(n)} \rightarrow 0,$$

then the existence of the repeated limit

$$\lim_{p\rightarrow\infty} \lim_{n\rightarrow\infty} S(n, p)$$

(i.e., of $\lim\limits_{x\rightarrow 0} \Sigma u_\nu e^{-x\lambda(\nu)}$*) involves that of the Pringsheim double limit and of the other repeated limit of Σu_ν (i.e., the convergence of Σu_ν).*

This follows at once from Landau's extension of Tauber's theorem, that $na_n \rightarrow 0$, together with $\Sigma a_n x^n \rightarrow s$ as $x \rightarrow 1$, involves the convergence of Σa_n.† It is not difficult to give a direct proof of the existence of the double limit; but this of course follows from the convergence of Σa_n and the theorem of §§ 20–22.

For further extensions of Tauber's theorem we may refer to a paper by Mr. Littlewood.‡

§ 26. The following theorem is of interest, serving as it does to substantiate the remark made in § 2 that there are methods of summation which will sum *any* convergent series, but *no* oscillatory series.§

* This is proved in the special case of $\alpha = 1$ by G. H. Hardy (20). The above two theorems are no doubt familiar to Dr. Riesz.

† E. Landau (24), G. H. Hardy (20).

‡ J. E. Littlewood (25).

§ *Strictly* we should say no *finitely* oscillatory series, for that is all we prove in the succeeding theorem; but it is improbable that any infinitely oscillatory series (of a natural character) will be summable in the region defined, if no finitely oscillatory series is so summable.

No finitely oscillatory series is summable (R, λ, ω) in the region immediately adjacent to, and to the right of, the p-axis, such that

$$\lambda(n)/p \equiv \omega(n) < 1/\log\lambda(n).$$

For, suppose K to be the upper limit of the absolute values of the partial sum of any finitely oscillatory series which is summable (R, λ, ω).

We have

$$\lim_{n\to\infty} \sum_{\nu=0}^{[n]} u_\nu\{\lambda(n)-\lambda(\nu)\}^{\omega(n)} = s,$$

if s is the "sum." Also we may write

$$\sum_{\nu=0}^{[n]} u_\nu[\{\lambda(n)-\lambda(\nu)\}^{\omega(n)}-1] = \sum_{\nu=0}^{[n]} u_\nu b_\nu.$$

As ν increases b_ν decreases and therefore, by Abel's lemma,

$$\left|\sum_{\nu=0}^{[n]} u_\nu b_\nu\right| < Kb_0 = K[\{\lambda(n)-\lambda(0)\}^{\omega(n)}-1],^*$$

which tends to zero as $n\to\infty$ if

$$\omega(n) < 1/\{\log\lambda(n)\}.$$

But

$$\lim_{n\to\infty}\sum_{\nu=0}^{n} u_\nu b_\nu = 0$$

is equivalent to $\Sigma\, u_\nu \to s$ as $n\to\infty$ by reason of the definition of b_ν. Hence, within the region

$$\omega(n) < 1/\{\log\lambda(n)\},$$

no finitely oscillatory series is summable. It is easy to assign a region within which any assigned infinitely oscillating series cannot possibly be summable (R, λ, ω). To do this, we now regard K as a function of n (*i.e.*, K_n is the upper limit of all the partial sums of Σu_ν up to the n^{th}), then the condition, that the expression on the right of the inequality

$$\left|\sum_{\nu=0}^{[n]} u_\nu b_\nu\right| < K_n[\{\lambda(n)-\lambda(0)\}^{\omega(n)}-1]$$

may tend to zero is evidently

$$\omega(n) < 1/\{K_n \log\lambda(n)\}.$$

---

* Bromwich, *Infinite Series*, p. 54.

Therefore the assigned series cannot possibly be summable by any method of Cesàro-Riesz, for which the path along which $\lambda(n)$ and p tend to infinity lies in the region thus defined.

§ 27. It has already been shown* that the series

$$\Sigma n^s e^{ain},$$

where s may be *any real or complex number whatever* and $0 < \alpha < 2\pi$, is summable (R, n, r) if the real part of r is greater than that of s. In this result *complex* orders of summation are considered, but no very interesting results seem to follow. Such orders evidently cannot be represented on our $\{\lambda(n), p\}$ plane, and we shall therefore not consider them here.

The analysis used in the paper cited may easily be made to yield more general results than are there enunciated. As these generalizations afford a good illustration of the ideas set forth in this paper, we shall discuss them now.

We shall directly quote the formula for $S(n, p)$ arrived at in the place already mentioned, referring the reader thither for the details of the proof. We have, when $n/p = r$,

$$S(n, p)$$

$$= -\frac{\Gamma(r)\Gamma(s)}{4\pi^2 n^r}\left\{\int_{C'}\left[\int_C \frac{(-x)^{-s-1}e^{-(x-\alpha i)}}{1-e^y e^{-(x-\alpha i)}}dx\right](-y)^{-r-1}e^{-ny}dy\right.$$

$$\left. - e^{n\alpha i}\int_{C'}\left[\int_C \frac{(-y)^{-r-1}e^{-y}}{1-e^{x-\alpha i}e^{-y}}dy\right](-x)^{-s-1}e^{-nx}dx\right\}.$$

Here C is a contour in the plane of the complex variable x, commencing at $+\infty$, circulating round the origin in the counter-clockwise direction, and returning again to $+\infty$; and $(-x)^{-s}$ is to mean $e^{-s\log(-x)}$, where the real value of $\log(-x)$ is to be taken when x is negative, and the logarithm is to be rendered single-valued by the stipulation that the variable is not to cross the real axis at any point on the positive side of the origin. C' is a similar contour in the plane of the complex variable y, and a corresponding convention applies to $(-y)^{-r}$. All the integrals are absolutely convergent, on account of the presence of e^{-x} or e^{-y} in the integrand.

The first term in the above expression for $S(n, p)$ may be transformed, by writing $ny = y'$, into

$$-\frac{\Gamma(r)\Gamma(s)}{4\pi^3}\int_{C'}\left[\int_C \frac{(\quad x)^{-s-1}e^{-(x-\alpha i)}}{1-e^{y/n}e^{-(x-\alpha i)}}dx\right](-y)^{-r-1}e^{-y}dx,$$

* S. Chapman (8).

where, for convenience, the accents have been dropped. It is then easy to prove* that, as $n \to \infty$, this tends to

$$\frac{ie^{ai}}{2\pi}\Gamma(s)\int_C \frac{(-x)^{-s-1}e^{-x}}{1-e^{-(x-ai)}}\,dx,$$

provided that r (which we consider to be not a constant, but a function of n) has some finite upper limit.†

The second term can similarly be written

$$\frac{+e^{nai}}{n^{r-s}}\frac{\Gamma(r)\Gamma(s)}{4\pi^2}\int_{C'}\left[\int_C \frac{(-y)^{-r-1}e^{-y}}{1-e^{-y}e^{-ai}e^{x/n}}\,dy\right](-x)^{-s-1}e^{-x}\,dx.$$

Owing to the presence of the factor e^{ain} this term is an oscillatory function of n, and can therefore only tend to a limit by its modulus tending to zero. This, therefore, is the condition that the series $\Sigma n^s e^{ain}$ may be summable (R, n, r), and the sum, if it exists, will be

$$\frac{ie^{ia}}{2\pi}\Gamma(s)\int_C \frac{(-x)^{-s-1}e^{-x}}{1-e^{-(x-ai)}}\,dx.$$

The integral in our last expression for the second term of $S(n, p)$ has, when multiplied by $\Gamma(r)\Gamma(s)$, a finite upper limit as $n \to \infty$. Hence the second term tends to zero, provided that $n^{s-r} \to 0$, *i.e.*, that $e^{(s-n)\log n} \to 0$. This is true if

$$r(n) - s > 1/\log n.$$

This condition is both necessary and sufficient. Also, it is easy to see that the convergence of the second term to zero is uniform if $0 < \epsilon \leq \alpha \leq 2\pi - \epsilon < 2\pi$; thus the series

$$\Sigma n^s \cos \alpha n, \quad \Sigma n^s \sin \alpha n$$

are uniformly summable over the range $\epsilon \leq \alpha \leq 2\pi - \epsilon$.

These results should be compared with those obtained in §§ 14–15, where a different method of summation was applied to the series $1^s - 2^s + 3^s - \ldots$, which then proved not to be summable along any *rectilinear* path.

§ 28. More general results may be obtained by a method already used in § 15, which we shall illustrate, as there, by an application to the series

$$e^{\sqrt{1}} - e^{\sqrt{2}} + e^{\sqrt{3}} - \ldots.$$

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* *Loc. cit.*, § 28.

† It would be easy to state a less restrictive condition.

We have

$$S(n, p) = 1 + \sum_1^{[n]} (-1)^\nu e^{\sqrt{\nu}} \left(1 - \frac{\nu}{n}\right)^{n/p}$$

$$= 1 + \frac{n^{-n/p}}{2\pi i} \left\{ \left(\int_{n-i\infty}^{n+i\infty} - \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \right) \frac{\pi}{\sin z\pi} e^{\sqrt{z}} (n-z)^{n/p} dz \right\},$$

where $\sqrt{z}$ is positive when z is positive, and $(n-z)^{n/p}$ is positive for $0 < z < n$.*

Arguing as in § 15, we see that everything depends on proving that

$$n^{-n/p} \int_{n-i\infty}^{n+i\infty} \frac{\pi}{\sin z\pi} e^{\sqrt{z}} (n-z)^{n/p} dz \to 0.$$

We use the inequalities (§ 15)

$$\left| \frac{\pi}{\sin z\pi} \right| < K e^{-|\eta|\pi}, \quad |e^{\sqrt{z}}| < e^{\sqrt{n} + \sqrt{|\eta|}}.$$

Since, when $z = n + i\eta$,

$$|n - z|^{n/p} = |\eta|^{n/p},$$

we have to formulate conditions under which

$$n^{-n/p} e^{\sqrt{n}} \int_0^\infty e^{-\pi\eta + \sqrt{\eta}} \eta^{n/p} d\eta \to 0.$$

It is easy to see that

$$(n/p) - \sqrt{n} \to \infty$$

is a sufficient condition; and, more generally,

$$(n/p) - \log \phi(n) \to \infty$$

is, in a large class of cases, a sufficient condition for the summability of

$$\phi(1) - \phi(2) + \phi(3) - \dots .†$$

§ 29. Suppose we have any series Σu_ν, convergent or not; let it be so diluted‡ that the ν^{th} term takes the $\mu(\nu)^{\text{th}}$ place

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* Strictly speaking, the singular point $z = n$ should be excluded by a small semicircle, whose radius is then made to tend to zero. The amplitude of $(n-z)^{n/p}$ suffers a discontinuity at $z = n$.

† Cf. § 15. These conditions are unnecessarily stringent, and may easily be widened.

‡ This term was introduced in the paper by S. Chapman (8). See § 27 of that paper.

where, of course, $\mu(\nu)$ is an integer greater than ν. Let Σv_n be the new series; then $v_n = u_\nu$ when $n = \mu(\nu)$, and when n has not this form, $v_n = 0$.

Let λ be the function inverse to μ, so that if

$$n = \mu(\nu),$$

then
$$\nu = \lambda(n).$$

Evidently $\lambda(n)$ is less than n.

Apply the method of summation (R, λ, ω) to the series Σv_n. Let $S(n, \omega)$ be the corresponding n^{th} partial sum-function. Then

$$S(n, \omega) = \sum_{m=0}^{[n]} v_m \left(1 - \frac{\lambda(m)}{\lambda(n)}\right)^{\omega(n)}$$

$$= \sum_{\nu=0}^{[\nu_0]} u_\nu \left(1 - \frac{\nu}{\nu_0}\right)^{\overline{\omega}(\nu_0)},$$

where $\nu_0 = \lambda(n)$ and $\overline{\omega}(\nu_0) = \omega[\mu(\nu_0)]$. But

$$\sum_{\nu=0}^{[\nu_0]} u_\nu \left(1 - \frac{\nu}{\nu_0}\right)^{\overline{\omega}(\nu_0)}$$

is the ν_0^{th} partial sum function for the series Σu_ν with the method of summation $(R, n, \overline{\omega})$. Also it is to be noted that the curve on the plane $\{\lambda(n), p\}$, represented by the equation

$$\omega(n) = \lambda(n)/p,$$

is exactly the same as the curve on the plane (ν, p), repre- by the equation

$$\overline{\omega}(\nu) = \nu/p,$$

since $\overline{\omega}(\nu) = \omega(n)$ and $\nu = \lambda(n)$. We thus have proved the theorem that *the necessary and sufficient condition that the diluted series Σv_m may be summable (R, λ, ω) along any path $\omega(n) = \lambda(n)/p$ in the $\{\lambda(n), p)$ plane is that the original, undiluted series Σu_ν shall be summable $(R, n, \overline{\omega})$ along the geometrically identical path $\overline{\omega}(\nu) = \nu/p$, where $\overline{\omega}(\nu) = \omega(n)$ and $\nu = \lambda(n)$, in the (n, p) plane, provided that the ν^{th} term of the original series takes the $\mu(\nu)^{\text{th}}$ place in the diluted series, and that λ is the function inverse to ν. Also the two sums are the same.*

We shall consider, as an example of this theorem, the case when $\mu(\nu)=2^\nu$. Then $\lambda(n)=\log n$ (the base of the logarithm being 2, of course). Hence, if Σu_ν is summable (R, n, ω), then Σv_m is summable $(R, \log n, \bar{\omega})$. If we suppose that Σu_ν is the series $1-1+1-\ldots$, which, as we have shown, is summable (R, n, ω), provided that

$$\omega = n/p \succ 1/\log n,^*$$

it follows that the diluted series Σv_m, for which

$$\Sigma v_m x^m = \Sigma (-1)^n x^{2^n},$$

is summable $(R, \log n, \bar{\omega})$, where

$$\bar{\omega} = \log n/p \succ 1/\log\log n.$$

This series well illustrates the greater power of Riesz's logarithmic method of summation $(R, \log n, p)$, as compared with the method (R, n, p). The series Σv_m is not summable by the latter method,† though it is summable $(R, \log n, p)$.

The Dirichlet's series

$$\Sigma v_m e^{-x\lambda(n)} = \Sigma v_m e^{-x \log n} = \Sigma u_\nu e^{-\nu x}$$

is evidently convergent for $x>0$ and tends to $\frac{1}{2}$ as $x \to 0$, as might also have been inferred from the theorem proved above.

§ 30. The substance of the preceding sections can of course be applied, with modifications of detail, to the theory of summable integrals. But the general theory runs on lines so nearly the same that it is hardly worth while to go over the ground again, even in the form of a statement of results.

Integrals are almost always more tractable than series, and we have obtained some results concerning them, the analogues of which for series we are at present wholly unable to prove. But, in view of the length to which this paper has already extended, we think it better to reserve these theorems for discussion on some future occasion.

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* In the notation of the Infinitärcalcül (for an exposition of which see the Cambridge Tract on the subject by G. H. Hardy) $z \succ y$ means that $z/y \to \infty$ as $x \to \infty$, if z and y are positive monotonic functions of x, tending to infinity with x.

† $\Sigma u_m x^m = \Sigma(-1)^n x^{2^n}$ has no definite limit as $n \to \infty$ [G. H. Hardy (16)]. Hence (§25) it is not summable (R, n, α) for any value of α. M. Riesz (32) has stated the theorem that summability (R, n, α) *always* involves summability $(R, \log n, \alpha)$.

Fig. 1.

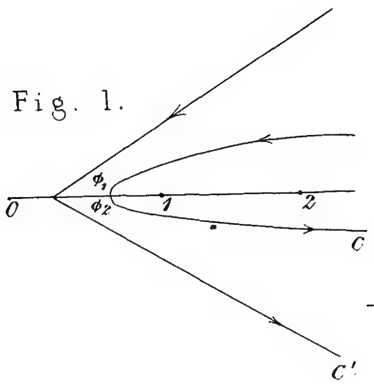


Fig. 2.

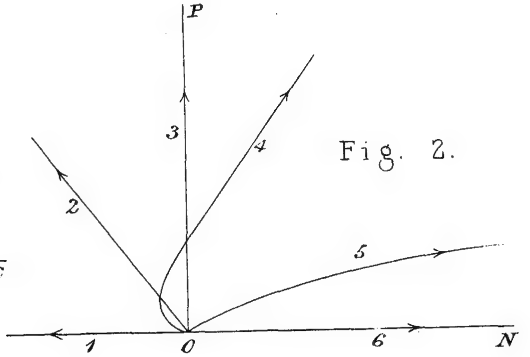
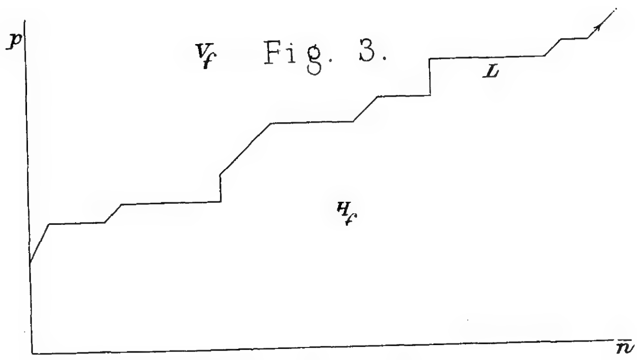


Fig. 3.



CORRECTIONS

p. 185, *last line.* For (i) read (1).

p. 186, *line* 1. For (i)′ read (1)′.

p. 187, *line* 15. For 'are less than unity' read 'are in general less than unity'.

p. 189, *line* 7. For $|F(n)|$ read $|F(n)-s|$.

p. 190, *line* 18. Here, and throughout the paper, for repeated limit (1), (1)′, (2) or (2)′ read repeated limit (1), (1)′, (2) or (2)′ of § 4 (the 2nd (i), (i)′ of § 4 being taken to be (1), (1)′).

p. 193, *line* 5. Insert (1) on the left.

—— *line* 13. For (1) read 1.

—— *line* 16. For $\lim\limits_{\delta\to\infty}$ read $\lim\limits_{\delta\to 0}$.

p. 200, *3rd footnote.* For 'energy' read 'generality'.

p. 201, *line* 4 *up.* For $+$ read $-$.

p. 204, *line* 12. For n read N.

p. 207, *line* 3. For $\{\ \}_\alpha$ read $\{\ \}^\alpha$.

p. 209, *line* 10. For $\omega < \lambda$ read $\omega \succ 1$; see p. 215, footnote.

p. 210. In the inequalities $\omega(n) < 1/\log\lambda(n)$ (3 times) and $\omega(n) < 1/\{K_n \log\lambda(n)$, for $<$ read $\prec$.

p. 212, *line* 17. For $e^{(s-n)\log n}$ read $e^{(s-r)\log n}$.

—— *line* 18. For $>$ read $\succ$.

FIG. 3. Read H_F and V_F.

COMMENTS

The methods of summability defined in § 4 are based on 'Hardy's principle', that of using an inverted repeated limit, where the initial repeated limit evaluates convergent series only. The paper was written shortly before Toeplitz's theorem† was published. After Hahn's proof‡ that the conditions of Dedekind, discussed in 1907, § 2, are necessary and sufficient for a series-to-function transformation to preserve convergence, it is easy to replace the conditions here by ones that are necessary and sufficient. For example, necessary and sufficient conditions for the repeated limit

$$\lim_{p\to\infty}\lim_{n\to\infty}\sum_{\nu\leqslant n} f(n,p,\nu)u_\nu$$

to be a regular summability method are:

† *Prace Mat.–Fiz.* 22 (1911), 113–19.

‡ *Monatsh. für Math. u. Phys.* 32 (1922), 3–88.

$$\text{(a)} \quad \lim_{n\to\infty} f(n,p,\nu) = \phi(p,\nu),$$

$$\text{(b)} \quad \sum_{\nu\leqslant n-1} |f(n,p,\nu)-f(n,p,\nu+1)|+|f(n,p,[n])| \leqslant K(p),$$

for the inner limit to exist and equal $\sum \phi(p,\nu)u_\nu$, whenever $\sum u_\nu$ converges, and

$$\text{(c)} \ \lim_{p\to\infty} \phi(p,\nu) = 1,$$

$$\text{(d)} \ \sum_{\nu\geqslant 0} |\phi(p,\nu)-\phi(p,\nu+1)| \leqslant K,$$

for the repeated limit to exist and equal $\sum u_\nu$, whenever this converges. Since only sufficient conditions are considered, there is no loss of generality in replacing (b) and (d) by

$$\text{(b)}' \quad K(p) \geqslant f(n,p,\nu) \geqslant f(n,p,\nu+1) \geqslant 0,$$

and
$$\text{(d)}' \quad K \geqslant \phi(p,\nu) \geqslant \phi(p,\nu+1) \geqslant 0,$$

as in the Abel–Dirichlet conditions; cf. the theorems of §§ 7 and 21, where monotony conditions are introduced.

The extended Riesz means defined in § 6 by the factor (2), with $\lambda(n)$ satisfying (1), are unsatisfactory when $n \to -\infty$, since the index is negative and the factor becomes unbounded, whenever the *continuous* variable n passes through a negative integral value $-N$, even if the value $n = -N$ itself is omitted. Thus the inner limit

$$\lim_{n\to-\infty} S(n,p)$$

cannot exist, except in the trivial case where $u_\nu = 0$ for all $\nu \geqslant A$. Similarly, the means (R,λ,α), $\alpha < 0$, can only sum terminating series; see the Comments on 1911, 1. On the other hand, the *discontinuous* means (R^*,λ,α), obtained by restricting the variable n to integral values and taking the sums over $0 \leqslant \nu \leqslant n-1$, do have a meaning for $\alpha < 0$.

The main investigation consists of examples, in which the parameters n, p either tend to infinity separately or together 'along a path'. In the extended Riesz case, the 'type' $\lambda(n)$ is given, while the 'order' $\omega = \lambda(n)/p$ is variable. Some tentative conclusions, based on London's theorem,|| § 18, are obtained, concerning the relative strength of methods with different 'variable orders' (curvilinear orders); see the propositions in §§ 11, 19, 20, and 23. In 1916, 5, Hardy obtains roughly analogous results concerning the relative strength of methods with different 'types'; cf. remarks at the ends of §§ 12 and 19. He says, in 1916, 5, that he 'had conjectured the truth' of the 'second theorem of consistency' 'when engaged, in collaboration with Mr Chapman', on the present paper.

The method defined by (2) of § 12 is Lindelöf's method;†† see D.S., pp. 77–9 and 190–1. An analogous method for integrals is given in 1904, 4, § 18.

§ See also the Comments on 1907, 2.

|| *Math. Annalen* 53 (1900), 322–70.

†† *J. de math. pures et appl.* (5), 9 (1903), 213–70.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XXX.

A theorem concerning summable integrals.

§ 1. The following theorem, concerning series summable by Cesàro's method of mean values, was proved independently by H. Bohr and myself.*

If (i) Σa_n *is summable* (Cr), (ii) $f_n \to 0$, (iii) *the series*

$$\Sigma n^r |\Delta^{r+1} f_n|$$

is convergent, then the series $\Sigma a_n f_n$ *is summable* (Cr), *and its sum is equal to that of the absolutely convergent series*

$$\Sigma S_n^r \Delta^{r+1} f_n.$$

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* See *Proc. Lond. Math. Soc.*, vol. vi. (1908), p. 257, and vol. viii. (1910), p. 277; and Bohr, *Comptes Rendus*, January 11, 1909, and *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen 1910.

Here S_n^r is defined by the equations

$$S_n = a_0 + a_1 + \ldots + a_n, \quad S_n^1 = s_0 + s_1 + \ldots + s_n,$$
$$S_n^k = S_0^{k-1} + S_1^{k-1} + \ldots + S_n^{k-1},$$

so that $$r!\, S_n^r / n^r \to s,$$

where s is Cesàro's sum of the series Σa_n.

§ 2. This theorem suggests the truth of an analogous theorem for integrals.

Suppose $f(x)$ integrable and absolutely integrable throughout any finite interval, and let

$$f_1(x) = \int_a^x f(t)\, dt, \quad f_2(x) = \int_a^x f_1(t)\, dt, \; \ldots,$$

so that we may write symbolically

$$f_k(x) = \left(\int_a^x dt\right)^k f(t),$$

then we shall say that the integral

$$(1) \qquad \int_a^\infty f(x)\, dx$$

is *summable* (r, x)* to sum s if

$$(2) \qquad r!\, f_{r+1}(x) / x^r \to s$$

as $x \to \infty$. In virtue of a well-known formula† this definition may also be presented in the form

$$(2') \qquad \int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\, dt \to s.$$

The theorem then runs as follows:—*Suppose that* (i) *the integral* (1) *is summable* (r, x), (ii) $\phi(x)$ *has a derivative* $\phi^{(r+1)}(x)$ *integrable and absolutely integrable in any finite interval*, (iii) *the integral*

$$\int_a^\infty x^r \,|\phi^{(r+1)}(x)|\, dx$$

is convergent, (iv) $\phi(x) \to 0$ *as* $x \to \infty$; *then will the integral*

$$(3) \qquad \int_a^\infty f(x)\, \phi(x)\, dx$$

be summable (r, x), *and its sum will be equal to the value of the absolutely convergent integral*

$$(4) \qquad (-1)^{r+1} \int_a^\infty f_{r+1}(x)\, \phi^{(r+1)}(x)\, dx.$$

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* I hope in a later note to develop more systematically some properties of integrals summable by this and by more general methods the reason for this notation will then appear.

† Jordan, *Cours d'Analyse*, t 3, p. 59

It is convenient to suppose $a > 0$, and it is easy to see that no generality is lost by this assumption.

In order to prove this theorem we require certain lemmas.

§ 3. LEMMA 1. *If the integral* (1) *is summable* (r, x) *to sum* s, *it is summable* $(r+1, x)$ *to the same sum. If it is convergent and has the value* s, *it is summable* (r, x) *to sum* s *for any value of* r.

In fact, from (2), it follows, by the use of familiar theorems, that

$$(r+1)!\, f_{r+2}(x)/x^{r+1} \to s.$$

LEMMA 2. *The formula*

$$\left(\int_a^x dt\right)^r (f\phi) = f_r\phi - \binom{r}{1}\left(\int_a^x dt\right)(f_r\phi')$$
$$+ \binom{r}{2}\left(\int_a^x dt\right)^2 (f_r\phi'') - \ldots + (-1)^r \left(\int_a^x dt\right)^r (f_r\phi^{(r)})$$

is true for all positive integral values of r.

To prove this we observe that

$$\left(\int_a^x dt\right)^{k+1} (f_r\phi^{(k)}) = \left(\int_a^x dt\right)^k \left(\int_a^x f_r\phi^{(k)}\, dt\right)$$
$$= \left(\int_a^x dt\right)^k \left(f_{r+1}\phi^{(k)} - \int_a^x f_{r+1}\phi^{(k+1)}\, dt\right)$$
$$= \left(\int_a^x dt\right)^k (f_{r+1}\phi^{(k)}) - \left(\int_a^x dt\right)^{k+1} (f_{r+1}\phi^{(k+1)}).$$

If now we assume the truth of the lemma for $r = s$, integrate once more, and apply the formula just proved to each term, we find that the lemma is true for $r = s+1$. As it is plainly true for $r = 1$, it is therefore true in general.

LEMMA 3. *The conditions* (iii) *and* (iv) *of the theorem imply*

$$\phi' \to 0,\ \phi'' \to 0,\ \ldots,\ \phi^{(r)} \to 0,$$

and the convergence of all the integrals

$$\int_a^\infty x^{r-\rho}\, |\phi^{(r-\rho+1)}(x)|\, dx$$

$(\rho = 1, 2, \ldots, r)$. This was shown in Note XXIX.

LEMMA 4. *If* $\chi(x)$ *is limited and integrable in any finite interval, and tends to zero as* $x \to \infty$, *and* α *and* β *are real and greater than* -1, *then*

$$x^{-\alpha-\beta-1}\int_a^x t^\alpha (x-t)^\beta \chi(t)\, dt \to 0.$$

If we put $t=xu$, we obtain

$$\int_{a/x}^{1} u^{a}(1-u)^{\beta}\chi(xu)\,du,$$

which is numerically less than

$$\int_{0}^{1} u^{a}(1-u)^{\beta}|\chi(xu)|\,du.$$

Choose τ so that $|\chi(t)|<\epsilon$ for $t\geqq\tau$. When τ is fixed we have

$$|\chi(xu)|<K \quad (0\leqq u\leqq \tau/x).$$

Hence

$$\int_{0}^{1} u^{a}(1-u)^{\beta}|\chi(xu)|\,du < K\int_{0}^{\tau/x} u^{a}du + \epsilon\int_{0}^{1} u^{a}(1-u)^{\beta}du,$$

which is less than a constant multiple of ϵ when x is large enough.

§4. We can now proceed to the proof of the theorem. We have

$$\frac{r!}{x^r}\left(\int_a^x dt\right)^{r+1}(f\phi)=\frac{r!}{x^r}\sum_{\nu=0}^{r+1}(-1)^{\nu}\binom{r+1}{\nu}\left(\int_a^x dt\right)^{\nu}(f_{r+1}\phi^{(\nu)})$$

$$=\frac{r!}{x^r}\sum_{\nu=0}^{r+1}(-1)^{\nu}\binom{r+1}{\nu}\eta_{\nu},$$

say, by Lemma 2. Here the operation $\left(\int_a^x dt\right)^0$, operating on a function of t, is to be interpreted as a substitution of x for t.

Now the integral (4) is absolutely convergent, since $|f_{r+1}(x)|<Kx^r$. It follows from Lemma 1 that

$$\frac{r!}{x^r}\eta_{r+1}\rightarrow\int_a^{\infty} f_{r+1}(x)\,\phi^{(r+1)}(x)\,dx.$$

In order to complete the proof of the theorem, we have therefore only to show that

$$\eta_{\nu}/x^r\rightarrow 0, \quad (\nu=0,1,\ldots,r),$$

In the first place

$$|\eta_0|/x^r=|f_{r+1}(x)\,\phi(x)/x^r|<K|\phi(x)|\rightarrow 0.$$

When $\nu>0$,

$$\eta_{\nu}=\left(\int_a^x dt\right)^{\nu}(f_{r+1}\phi^{(\nu)})=\frac{1}{(\nu-1)!}\int_a^x (x-t)^{\nu-1}f_{r+1}\phi^{(\nu)}\,dt,$$

which is numerically less than a constant multiple of

$$\zeta_{\nu}=\int^{x}(x-t)^{\nu-1}t^{r}|\phi^{(\nu)}|\,dt.$$

Let $$\Phi_\nu(t) = \int_t^\infty u^{\nu-1} |\phi^{(\nu)}(u)| \, du;$$

the convergence of this integral follows from Lemma 3. Then if $\nu = 1$, we have

$$\zeta_1 = -\int_a^x t^r \Phi_1'(t) \, dt = a^r \Phi_1(a) - x^r \Phi_1(x) + r \int_a^x t^{r-1} \Phi_1(t) \, dt,$$

and so $\zeta_1/x^r \to 0$. If $\nu > 1$, we have

$$\zeta_\nu = -\int_a^x (x-t)^{\nu-1} t^{r-\nu+1} \Phi_\nu'(t) \, dt$$

$$= (x-a)^{\nu-1} a^{r-\nu+1} \Phi_\nu(a) - (\nu - 1) \int_a^x (x-t)^{\nu-2} t^{r-\nu+1} \Phi_\nu(t) \, dt$$

$$+ (r - \nu + 1) \int_a^x (x-t)^{\nu-1} t^{r-\nu} \Phi_\nu(t) \, dt = \zeta_{\nu,1} + \zeta_{\nu,2} + \zeta_{\nu,3},$$

say. It is evident that $\zeta_{\nu,1}/x^r \to 0$, and that $\zeta_{\nu,2}/x^r$ and $\zeta_{\nu,3}/x^r$ also tend to zero follows from Lemma 4, by putting first

$$\alpha = \nu - 2, \quad \beta = r - \nu + 1, \quad \alpha + \beta + 1 = r,$$

and then

$$\alpha = \nu - 1, \quad \beta = r - \nu, \quad \alpha + \beta + 1 = r.$$

The theorem is thus established.

§ 5. Suppose, in particular,

$$\phi(x) = x^{-s},$$

where s is complex and has its real part positive. Then

$$\phi^{(r+1)}(x) = (-1)^{r+1} s(s+1)\dots(s+r) \, x^{-s-r-1},$$

and the conditions of the theorem are plainly satisfied. It follows that *if the integral*

$$(5) \qquad \int_a^\infty x^{-s} g(x) \, dx \quad (a > 0)$$

is summable (r, x) *for any particular value of* s, *then it is summable* (r, x) *for all values of* s *whose real part is greater. From this we deduce the existence of lines of summability*

$$R(s) = s_0, \quad R(s) = s_1, \quad R(s) = s_2, \dots$$

such that $$s_r - 1 \leqq s_{r+1} \leqq s_r;$$

the integral (5) is summable (r, x) to the right of the line $R(s) = s_r$. All this corresponds precisely to what Bohr has proved concerning Dirichlet's series.

COMMENTS

The theorem in § 2 is an analogue for infinite integrals of the Bohr–Hardy theorem; see 1908, 1 and 1910, 1, and the Comments on 1908, 1. As in the Bohr–Hardy theorem, hypothesis (i) may be replaced by: (i)′ *the integral* (1) *is bounded* (r, x), or alternatively, hypothesis (iv) may be replaced by: (iv)′ $\phi(x) \to \lim$ *as* $x \to \infty$.

Cossar† extended the result to non-integral orders of summability, and Borwein‡ showed that the conditions are (in a sense) necessary and sufficient.

The proof of Lemma 3 is given by Hardy in 1911, 6 (in Vol. V).

† *J. London Math. Soc.* 16 (1941), 56–68.

‡ *J. London Math. Soc.* 25 (1950), 302–15.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*.

XXXI.

The uniform convergence of Borel's integral.

1. Borel defines the sum of the power series

(1) $$\Sigma a_n x^n$$

as

(2) $$s(x) = \int_0^\infty e^{-t} u(tx)\, dt,$$

where

(3) $$u(x) = \Sigma \frac{a_n x^n}{n!}.$$

In applying this definition we have to distinguish two cases.

(*a*) *The radius of convergence of* (1) *is positive.* In this case (3) is convergent for all values of x.

(*b*) *The radius of convergence of* (1) *is zero.* In this case (3) may converge for all values of x, for some only, or for none. In the first case the original definition applies. In the second case (3) defines a branch of an analytic function, regular about $x=0$, and it may happen that this branch is regular along the line described by tx, when x has some particular value and t varies along the real axis from 0 to ∞. We can then still apply the formula (2), regarding $u(tx)$ as defined by the power series (3) and its analytic continuation along this line. In the third case the sum of (1) can only be defined by some extension of Borel's definition.

2. It has been proved by various writers* that, *if the integral* (2) *is convergent for* $x=1$, *it is uniformly convergent for* $0<\delta \leqq x \leqq 1$, however small be δ. It has been stated

* See Phragmen, *Comptes Rendus*, t. 132, p. 1396; Hardy, *Quarterly Journal*, vol. xxxv., p. 44; Bromwich, *Infinite Series*, p. 291 (where the result is made more complete).

that the integral is uniformly convergent for $0 \leqq x \leqq 1$, and application has been made of theorems in this form.* But the argument used for the interval $\delta \leqq x \leqq 1$ does not apply to an interval which includes the origin. My object in writing this note is to fill up this gap in the theory.

I observe first that in Case (a) the desired extension of the proof presents no difficulty whatever. For, suppose that the radius of convergence of (1) is ρ. Then, if σ is any number greater than $1/\rho$,

$$|a_n| = \epsilon_n \sigma^n,$$

where $\epsilon_n \to 0$ as $n \to \infty$; and from this it follows immediately that

$$e^{-\sigma tx} |u(tx)| \to 0$$

as $t \to \infty$. Thus

$$|u(tx)| < Ke^{\sigma tx},$$

where K is a constant. We can choose δ so that

$$\sigma\delta < \tfrac{1}{2};$$

and it follows, by comparison with

$$\int_0^\infty Ke^{-\frac{1}{2}t}\,dt,$$

that (2) is uniformly convergent for $0 \leqq x \leqq \delta$. As it is known to be uniformly convergent for $\delta \leqq x \leqq 1$, it is uniformly convergent for $0 \leqq x \leqq 1$.

An extension of this argument is sufficient to prove that *Borel's integral is uniformly convergent throughout any closed region lying entirely inside the "polygon of summability."*†

3. In Case (b), however, when there is no circle of convergence for (1), nor polygon of summability for (3), this argument breaks down. And as the origin is in general a singular point for the function defined by the integral (2), this is not surprising. I shall therefore prove the following theorem.

* *E.g.*, by the present writer, *Proc. L.M.S.*, vol. viii., p. 282.

† Bromwich, *l.c.*, p. 296.

THEOREM. *Let $(u)\,x$ be a real function of x, continuous for all positive values of x. Then if the integral*

$$\int_0^\infty e^{-t} u(tx)\,dt$$

is convergent for $x=1$, it is uniformly convergent for $0 \leqq x \leqq 1$, that is to say, given ϵ we can find H_0 so that

$$\left| \int_{H_1}^{H_2} e^{-t} u(tx)\,dt \right| < \epsilon,$$

for $H_2 \geqq H_1 \geqq H_0$, $0 \leqq x \leqq 1$.

As the integral is known to be uniformly convergent for $\delta \leqq x \leqq 1$ for any positive δ, we may replace 1, in the inequalities contained in the enunciation, by δ.

We can choose ξ and K so that

$$(4) \qquad |u(x)| < K,$$

for $0 \leqq x \leqq \xi$. We can then choose positive numbers δ, β so that

$$(5) \qquad \delta + (1/\beta) < \tfrac{1}{2},$$

$$(6) \qquad \beta x \log(1/x) < \xi,$$

for $0 \leqq x \leqq \delta$. Thus we might take $\beta = 3$, and then choose η so that

$$\delta < \tfrac{1}{6}, \quad 3x \log(1/x) < \xi \quad (0 \leqq x \leqq \delta).$$

Then

$$(7) \qquad |u(tx)| < K,$$

for $0 \leqq t \leqq \beta \log(1/x)$, $0 \leqq x \leqq \delta$.

Now

$$\int_{H_1}^{H_2} e^{-t} u(tx)\,dt = \left(\int_{H_1}^{\beta\log(1/x)} + \int_{\beta\log(1/x)}^{H_2} \right) e^{-t} u(tx)\,dt$$

if $H_1 < \beta \log(1/x) < H_2$. If $H_1 > \beta \log(1/x)$ or $H_2 < \beta \log(1/x)$ no such splitting up of the integral is necessary, and the argument is simplified.

In the first place

$$(8) \qquad \left| \int_{H_1}^{\beta\log(1/x)} e^{-t} u(tx)\,dt \right| < K \int_{H_1}^\infty e^{-t}\,dt = K e^{-H_1}.$$

In the second place

$$\int_{\beta\log(1/x)}^{H_2} e^{-t}\,u(tx)\,dt = \frac{1}{x}\int_{\beta x\log(1/x)}^{H_2 x} e^{-w/x}\,u(w)\,dw$$

Writing $$e^{w-(w/x)}\,e^{-w}$$

for $e^{-w/x}$ and applying the second Mean Value Theorem, we obtain

$$x^{\beta-1}\,e^{\beta x\log(1/x)}\int_{\beta x\log(1/x)}^{W} e^{-w}\,u(w)\,dw,$$

where $W > \beta x\log(1/x)$. Since

$$x^{\beta-1} = e^{-(\beta-1)\log(1/x)} < e^{-(\beta-1)H_1/\beta},$$

this is less than

(9) $$Me^{\xi-(\beta-1)H_1/\beta},$$

where M is an upper limit for

$$\int_{\tau_1}^{\tau_2} e^{-w}\,u(w)\,dw.$$

Thus, from (8) and (9), we obtain

(10) $$\left|\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt\right| < Ke^{-H_1} + Me^{\xi-(\beta-1)H_1/\beta}.$$

So far we have supposed $H_1 < \beta\log(1/x) < H_2$. If $H_1 \geqq \beta\log(1/x)$, we have

$$\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt = \frac{1}{x}\int_{H_1 x}^{H_2 x} e^{-w/x}\,u(w)\,dw,$$

which is numerically less than

(11) $$Me^{-H_1+H_1 x}/x < Me^{-H_1+H_1\delta+(H_1/\beta)} < Me^{-\frac{1}{2}H_1},$$

since $$1/x = e^{\log(1/x)} < e^{H_1/\beta}, \quad \delta + (1/\beta) < \tfrac{1}{2}.$$

Finally, if $H_2 \leqq \beta\log(1/x)$, we have

(12) $$\left|\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt\right| < Ke^{-H_1}.$$

From (10), (11), and (12) it follows that

$$\left| \int_{H_1}^{H_2} e^{-t} u(tx)\, dt \right|$$

is less than one or other of the numbers

$$Ke^{-H_1}, \quad Me^{-\frac{1}{2}H_1}, \quad Ke^{-H_1} + Me\xi^{-(\beta-1)H_1/\beta}.$$

It is evident that, when K, ξ, β, and δ are fixed, we can choose H_0 so that each of these numbers is less than ϵ for $H_2 \geqq H_1 \geqq H_0$, $0 \leqq x \leqq \delta$. Thus the theorem is proved.

4. In order to establish the uniform convergence, for $0 \leqq x \leqq 1$, of Borel's integral (2), we have only to split up $u(tx)$ into its real and imaginary components and apply the theorem to the two integrals thus obtained.

CORRECTIONS

p. 162, *line* 5 *up*. Omit 'for (3)'.

p. 163, *line* 1. For $(u)x$ read $u(x)$.

—— *line* 17. For η read δ.

COMMENTS

The theorem in § 3 supplements the results of Phragmén† and Hardy (1904, 4, § 7) concerning the analogue for Borel summability of Abel's limit theorem, as well as the account in Bromwich (1st edn.); see the Comments on 1904, 4. The theorem fills a gap in 1910, 1, where it is assumed; see the Comments on 1910, 1. A simple proof of the theorem, due to Landau,‡ is given in D.S., Theorem 130.

The extension of Borel's method, used in case $(b)_{\text{ii}}$, is called the (B^*) method in D.S., p. 192. It was given by Borel§ as an extension of *absolute summability* (in the sense of 1904, 4), i.e. with the integral (2), and the integrals with u', u'',... in place of u, all assumed to be absolutely convergent.

In case (b), if the integral (2) converges for $x = x_0$ ($\neq 0$), then it converges on the segment $(0, x_0)$ and defines a branch of an analytic function regular inside the circle on $(0, x_0)$ as diameter; Phragmén (loc. cit.), D.S., Theorem 132. The point $x = 0$ is a singular point for the branch. For otherwise the branch would be expansible in a power series (1′): $\sum b_n x^n$ near the origin. But it would follow, by Lerch's uniqueness theorem,‖ that $\sum b_n x_0^n t^n/n! = u(x_0 t)$ for small $t > 0$. In case $(b)_{\text{iii}}$, this is contradictory, since $u(x_0 t)$ has no power series expansion. In case $(b)_{\text{i}}$ or $(b)_{\text{ii}}$ we would have

$$\sum b_n x_0^n t^n/n! = \sum a_n x_0^n t^n/n!$$

for small $t > 0$, and hence $a_n = b_n$ ($n \geqslant 0$). This is contradictory, since (1) has zero radius of convergence. Borel (loc. cit) obtained the same result under his more restrictive conditions.

The result stated at the end of § 2 is proved in D.S., Theorem 133. The proof in Bromwich (1st edn.) is incomplete; see the Comments on 1910, 1.

Hardy gives some illustrative examples in 1914, 9.

† *Comptes rendus* 132 (1901), 1396–9.

‡ *Acta Math.* 42 (1920), 95–8.

§ *Ann. de l'École norm. sup.* (3), 16 (1899), 9–131 (89–91); Borel (2nd edn.), p. 143.

‖ *Acta. Math.* 27 (1903), 339–51; Widder, *The Laplace Transform*, Ch. II, § 6. Princeton University Press, 1941.

ON THE MULTIPLICATION OF DIRICHLET'S SERIES

By G. H. Hardy.

[Received May 18th, 1911.—Read June 8th, 1911.]

1. In this paper I propose to generalise some results communicated to the Society in 1908.*

As in my former communication, I denote by A and B the series

$$a_1+a_2+a_3+\ldots, \qquad b_1+b_2+b_3+\ldots,$$

and by C the series
$$c_1+c_2+c_3+\ldots,$$

where c_n is a function of the a's and b's, to be defined more precisely in a moment. I shall also use the letters A, B, C to denote the sums of the series, when they are convergent.

I shall denote by
$$A_x$$
the finite sum
$$\sum_{n\leqslant x} a_n;$$
here x is not restricted to be integral. Similarly I define B_x, C_x.

2. When the series C is the product-series of A and B, formed in accordance with Cauchy's rule, we have

$$c_p = a_1 b_p + a_2 b_{p-1} + \ldots + a_p b_1 = \sum_{(m+n=p+1)} a_m b_n.$$

This was the only case that I considered in my former paper.

Cauchy's rule for multiplication is, however, only one among an infinity. We are led to it by arranging the formal product of the power series
$$\Sigma a_m x^m, \qquad \Sigma b_n x^n$$
in ascending powers of x, and then putting $x=1$. It is the same thing to say that we arrange the formal product of the Dirichlet's series

$$\Sigma a_m e^{-ms}, \qquad \Sigma b_n e^{-ns}$$

* "The Multiplication of Conditionally Convergent Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410.

according to the ascending order of the sums

$$m+n,$$

associating together all the terms for which $m+n$ has the same value, and then put $s=0$. It is clear that we arrive at a generalisation of our conception of multiplication by considering the more general Dirichlet's series

$$\Sigma a_m e^{-\lambda_m s}, \quad \Sigma b_n e^{-\lambda_n s},*$$

and arranging their formal product according to the ascending order of the sums

$$\lambda_m+\lambda_n.\dagger$$

Let

$$\nu_1,\ \nu_2,\ \ldots,\ \nu_p,\ \ldots$$

be the ascending sequence defined by the possible values of $\lambda_m+\lambda_n$. Then the Dirichlet's product of the series A, B, according to the rule defined by the sequence λ_n, is

$$c_1+c_2+c_3+\ldots,$$

where

$$c_p = \sum_{(\lambda_m+\lambda_n=\nu_p)} a_m b_n.$$

Thus, if $\lambda_m = \log m$, so that the Dirichlet's series are ordinary Dirichlet's series,

$$\nu_p = \log p,$$

and

$$c_p = \sum_{mn=p} a_m b_n = \sum_{(d)} a_d b_{p/d},$$

the summation being extended to all the divisors of p.

3. The three classical theorems relating to ordinary multiplication have their analogues for the general form of Dirichlet's multiplication.

(1) **Analogue of Abel's Theorem.**—*If all three series are convergent, then* $C = AB$.

(2) **Analogue of Cauchy's Theorem.**—*If A and B are absolutely convergent, then C is absolutely convergent.*

(3) **Analogue of Mertens' Theorem.**—*If A is absolutely and B conditionally convergent, then C is convergent.*

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* Here, of course, (λ_m) is any increasing sequence whose limit is infinity.

† For a general account of the theory, see Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd. 2, S. 750. Landau remarks that there is no loss of generality in adopting the same sequence (λ_m) in both series: for if we had two sequences (λ_m), (μ_n), we could combine them into one, regarding some of the a's and b's as zero.

Of these results (2) is almost obvious, while (3) was deduced by Landau as a corollary of a more general theorem of Stieltjes.* The analogue of Abel's theorem was also proved by Landau, by considerations drawn from the theory of analytic functions. An elementary proof was afterwards found independently by Phragmen, Bohr, and Riesz,† who deduce it from the following theorem :—

(4) **Analogue of Cesàro's Theorem.**—*If A and B are convergent, then C is summable $(R, 1, \nu)$ to sum AB: that is to say,*

$$\frac{(\nu_2-\nu_1)C_1+(\nu_3-\nu_2)C_2+\ldots+(\nu_p-\nu_{p-1})C_{p-1}}{\nu_p} \to AB,$$

as $p \to \infty$.

4. In my former paper I proved the following theorems (for multiplication by Cauchy's rule).

(5) *If A and B are convergent, and*

$$na_n \to 0, \qquad nb_n \to 0,$$

then C is convergent.

(6) *The same result holds under the more general conditions*

$$|na_n| < K, \qquad |nb_n| < K.$$

I propose now to establish the analogues of these theorems for the general form of Dirichlet's multiplication. It might be thought that, as (5) is a special case of (6), it would not be worth while to prove it independently, as in my previous paper. This view would, I think, be mistaken. Theorem (5) above, and its generalisation, can be proved by a very much simpler argument than seems to be called for by (6) and its generalisation; and the simpler proof of the less general theorem affords a good deal more information about the behaviour of the product series than can be obtained in the more general case. It therefore seems worth while to keep the two distinct.‡

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* Stieltjes, *Nouvelles Annales*, Sér. 3, t. VI, p. 210; Landau, *Rendiconti di Palermo*, t. XXIV, p. 81; *Handbuch*, S. 752.

† Landau, *Handbuch*, S. 762 and 904; Riesz, *Comptes Rendus*, July 9, 1909; Bohr, *Nachrichten der König. Gesellschaft der Wiss. zu Göttingen*, 1909, S. 247.

‡ These theorems may be compared with Tauber's theorem (the converse of Abel's theorem on the continuity of power series) and its extension given recently in these *Proceedings* by Mr. Littlewood—a similar distinction presents itself in the case of the two latter theorems.

5. THEOREM 1.—*If A and B are convergent, and*

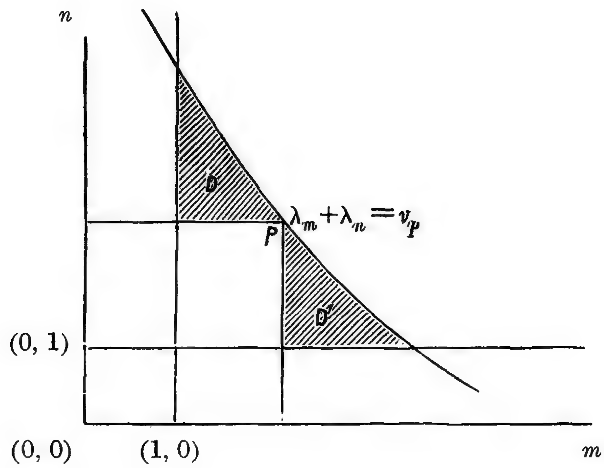
$$\frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} \to 0, \qquad \frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}} \to 0,$$

then the product series C, formed by the rule of Dirichlet's multiplication corresponding to the sequence (λ_n), is convergent.

We have
$$c_p = \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n,$$
and
$$C_p = \Sigma\, a_m b_n,$$
the summation being bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

I shall suppose that $\lambda_1 = 0$; this hypothesis in no way affects the generality of the result, and simplifies the expression of the proof a little.



Let us draw the curve
$$\lambda_m + \lambda_n = \nu_p$$
in the (m, n) plane (see the figure), and on it take the point P whose coordinates are
$$m_p = n_p = \bar{\lambda}(\tfrac{1}{2}\nu_p),$$
where $\bar{\lambda}$ is the function inverse to λ. Then

$$C_p - A_{m_p} B_{n_p} = \Sigma\, a_m b_n,$$

where the summation extends to all points (m, n) inside the regions D, D' shaded in the figure, including those on the curved (but not on the

straight) boundaries of those regions. Now

$$\sum_{(D)} a_m b_n = \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \lambda(\nu_p)} b_n A_{\bar{\lambda}(\nu_p-\lambda_n)}.$$

There is a constant K such that

$$|A_x| < K$$

for all values of x. Moreover we can choose p so that

$$|b_n| < \epsilon(\lambda_n - \lambda_{n-1})/\lambda_n$$

for $n \geqslant \lambda(\frac{1}{2}\nu_p)$. Then

$$\begin{aligned} \Big|\sum_{(D)} a_m b_n\Big| &< \epsilon K \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \bar{\lambda}(\nu_p)} \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \\ &< \epsilon K \left\{1 + \sum_{\bar{\lambda}(\frac{1}{2}\nu_p)+1 < n \leqslant \bar{\lambda}(\nu_p)} \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right)\right\}^* \\ &< \epsilon K \left\{1 + \log\frac{\lambda\bar{\lambda}(\nu_p)}{\lambda\bar{\lambda}(\frac{1}{2}\nu_p)}\right\} \\ &= \epsilon K(1+\log 2). \end{aligned}$$

It follows that $$\sum_{(D)} a_m b_n \to 0$$

as $p \to \infty$. Similarly we can shew that the sum of the terms inside (D') tends to zero. Hence

$$C_p - A_{m_p} B_{n_p} \to 0, \tag{1}$$

and the theorem follows.

It should be observed that the same argument proves that

$$C_p - A_{\bar{\lambda}(\alpha\nu_p)} B_{\bar{\lambda}(\beta\nu_p)} \to 0, \tag{2}$$

if α and β are any positive numbers such that $\alpha + \beta = 1$. Moreover the truth of (1) and (2) depends *only* on the existence of an upper limit for $|A_x|$ and $|B_x|$, and not on the actual convergence of A and B. Thus we have in reality proved more than is actually contained in the enunciation of the theorem.

6. I shall now proceed to the proof of the generalised form of Theorem 6 of § 4. Here we find it necessary to pursue an entirely

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* For, if $u = (\lambda_n - \lambda_{n-1})/\lambda_n$, we have $0 < u < 1$, and

$$u < \log\left(\frac{1}{1-u}\right) = \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right).$$

different and much less direct form of argument, and the touch of extra precision, pointed out at the end of the last paragraph, cannot be obtained. We shall also find it necessary to subject the absolute generality of the sequence (λ_n) to a restriction, viz., that

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0.*$$

7. THEOREM II.—*If* $$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0,$$

then the result of Theorem I is still valid when we assume only that

$$\left| \frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} \right| < K, \qquad \left| \frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}} \right| < K.$$

We know, by Theorem 4 of § 4, that

$$\frac{(\nu_2 - \nu_1) C_1 + (\nu_3 - \nu_2) C_2 + \ldots + (\nu_p - \nu_{p-1}) C_{p-1}}{\nu_p}$$

tends to the limit AB. This expression may also be written in the form

$$C_p - \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p}.$$

Hence the necessary and sufficient condition that the product series should be convergent is that

$$\frac{\bar{C}_p}{\nu_p} = \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p} \to 0.$$

Now $$\bar{C}_p = \Sigma\, \nu_p \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n = \Sigma (\lambda_m + \lambda_n)\, a_m b_n,$$

where the summation is bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

That is to say,

$$\bar{C}_p = \sum_{\lambda_n \leqslant \nu_p} \lambda_n b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} a_m + \sum_{\lambda_n \leqslant \nu_p} b_n \,.\! \sum_{\lambda_m + \lambda_n \leqslant \nu_p} \lambda_m a_m$$
$$= {}_1\Gamma_p + {}_2\Gamma_p,$$

* I return to the subject of this restriction later on. It is interesting to observe that Mr. Littlewood has found it necessary to make a similar restriction in one of his theorems concerning the converse of Abel's theorem.

say.* I shall first prove that

(1) $$ {}_1\Gamma_p/\nu_p \to 0. $$

Let m_n be the largest integral value of m for which

$$ \lambda_m+\lambda_n \leqslant \nu_p, $$

and q the largest value of n for which $\lambda_n \leqslant \nu_p$, so that $m_1 = q$. Then

$$ {}_1\Gamma_p = \sum_1^q \lambda_n b_n A_{m_n}. $$

Since A is convergent, we may write

$$ A_x = A+\epsilon_x \quad (\epsilon_x \to 0), $$

$$ \frac{{}_1\Gamma_p}{\nu_p} = \Delta_p+\Delta'_p, $$

where
$$ \Delta_p = \frac{A}{\nu_p}\sum_1^q \lambda_n b_n \to 0,\dagger $$

and

(2) $$ \Delta'_p < \frac{K}{\nu_p}\sum_1^q (\lambda_n-\lambda_{n-1})\,|\,\epsilon_{m_n}\,|.\ddagger $$

Choose M so that
$$ |\,\epsilon_x\,| < \epsilon \quad (x \geqslant M). $$

Then we shall have $m_n \geqslant M$, if

$$ \lambda_M+\lambda_n \leqslant \nu_p, $$

or if
$$ n \leqslant \bar{\lambda}(\nu_p-\lambda_M). $$

Suppose that this condition is satisfied for $n = 1, 2, \ldots, N$, but that

$$ \lambda_{N+1} > \nu_p-\lambda_M. $$

It is plain that, when M has been fixed, $N \to \infty$ with p.

Then the right-hand side of (2) is less than

$$ \frac{\epsilon K}{\nu_p}\sum_1^N (\lambda_n-\lambda_{n-1})+K\left(\frac{\lambda_{N+1}-\lambda_N}{\nu_p}\right)+\frac{K'}{\nu_p}\sum_{N+2}^q (\lambda_n-\lambda_{n-1}) $$

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* In the special case when

$$ \lambda_m = m, \quad \lambda_n = n, \quad \nu_p = p+1, $$

${}_1\Gamma_p$ and ${}_2\Gamma_p$ reduce to the expressions X and Y of my former paper (*l.c.*, pp. 415, 416).

† Since B is convergent, so that $(\lambda_1 b_1+\lambda_2 b_2+\ldots+\lambda_q b_q)/\lambda_q \to 0$, and $\lambda_q \leqslant \nu_p$.

‡ It is convenient to agree that $\lambda_0 = \lambda_1 = 0$.

(where K' is any number not less than the product of K by the greatest value of $|\epsilon_x|$)

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'}{\nu_p}(\lambda_q - \lambda_{N+1})$$

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'\lambda_M}{\nu_p},$$

since $$\lambda_q \leqslant \nu_p, \quad \lambda_{N+1} > \nu_p - \lambda_M.$$

But when M has been fixed, each of the last two terms tends to zero as $p \to \infty$; and so
$$\Delta'_p \to 0,$$

which establishes the truth of the assertion (1).

It remains to prove that

(3) $${}_2\Gamma_p/\nu_p \to 0.$$

Now $${}_2\Gamma_p = \sum_{\lambda_n \leqslant \nu_p} b_n \sum_{\lambda_m+\lambda_n \leqslant \nu_p} \lambda_m a_m$$

$$= b_1 \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m + b_2 \sum_{\lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots + b_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m$$

$$= B_1 \sum_{\nu_p - \lambda_2 < \lambda_m \leqslant \nu_p} \lambda_m a_m + B_2 \sum_{\nu_p - \lambda_3 < \lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots$$

$$+ B_{q-1} \sum_{\nu_p - \lambda_q < \lambda_m \leqslant \nu_p - \lambda_{q-1}} \lambda_m a_m + B_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m.$$

If in this equation we write
$$B_x = B + \epsilon_x,$$

so that $\epsilon_x \to 0$, we obtain $$\frac{{}_2\Gamma_p}{\nu_p} = \Delta_p + \Delta'_p,$$

where $$\Delta_p = \frac{B}{\nu_p} \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m \to 0,$$

and $$\nu_p |\Delta'_p| < K|\epsilon_1| \sum_{\nu_p - \lambda_2 < \lambda_m \leqslant \nu_p} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K|\epsilon_n| \sum_{\nu_p - \lambda_{n+1} < \lambda_m \leqslant \nu_p - \lambda_n} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K|\epsilon_q| \sum_{\lambda_m \leqslant \nu_p - \lambda_q} (\lambda_m - \lambda_{m-1}).$$

Choose N so that $$|\epsilon_x| < \epsilon \quad (x \geqslant N).$$

Then

$$|\Delta'_p| < \frac{K'}{\nu_p} \sum_{\nu_p-\lambda_N<\lambda_m\leqslant\nu_p} (\lambda_m-\lambda_{m-1}) + \frac{\epsilon K}{\nu_p} \sum_{\lambda_m\leqslant\nu_p-\lambda_N} (\lambda_m-\lambda_{m-1}),$$

where K' is defined as before.

Let M be the largest value of m for which

$$\lambda_m+\lambda_N \leqslant \nu_p,$$

so that

$$\lambda_{M+1} > \nu_p-\lambda_N.$$

Then

$$|\Delta'_p| < \epsilon K + \frac{K'}{\nu_p}(\lambda_{M+1}-\lambda_M) + \frac{K'}{\nu_p}\sum_{M+2}^{q}(\lambda_m-\lambda_{m-1})$$

$$< \epsilon K + K'\left(\frac{\lambda_{M+1}-\lambda_M}{\lambda_{M+1}}\right) + \frac{K'\lambda_N}{\nu_p};$$

and it follows, as in our previous discussion of ${}_1\Gamma_p$, that $\Delta'_p \to 0$ and that the assertion (3) is true. Thus the proof of Theorem II is completed.

8. It will be observed that the condition that $(\lambda_n-\lambda_{n-1})/\lambda_n \to 0$ is only used twice in the above proof, and then in a way that rather suggests the possibility of avoiding it. But I have not been able to free Theorem II of this condition. Nor does the point seem to be of importance.

When the sequence (λ_n) does not satisfy this condition, the conditions

$$\left|\frac{\lambda_n a_n}{\lambda_n-\lambda_{n-1}}\right| < K, \ \ldots$$

tell us nothing (in any interesting case) except that $|a_n|$ and $|b_n|$ have finite upper limits. But this is, of course, already involved in the fact of the convergence of A and B: we know, in fact, that a_n and b_n tend to zero, so that the conditions of Theorem I will be satisfied. Thus there appears to be no particular purpose to be served by attempting a proof of the more general theorem.*

If $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to a limit other than zero, Theorem I suffices to tell us that *any* two convergent series may be multiplied by the corresponding rule of Dirichlet's multiplication. It is interesting to verify this conclusion in a particular case. Let

$$\lambda_n = 2^n.$$

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* In all cases of interest $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to zero or to some limit (obviously not greater than unity). This limit is zero if

$$\lambda_n \prec e^{\delta n},$$

positive but less than unity if λ_n is (roughly) of increase e^{An}, unity if

$$\lambda_n \succ e^{\Delta n}.$$

Here δ and Δ denote respectively arbitrarily small and arbitrarily large positive numbers.

Then $$\lambda_m+\lambda_n > \lambda_\mu+\lambda_\nu,$$

if $$m \geqslant n, \quad \mu \geqslant \nu, \quad \text{and} \quad m > \mu.$$

The Dirichlet's product of A and B is

$$a_1 b_1+(a_1 b_2+a_2 b_1)+a_2 b_2+(a_1 b_3+a_3 b_1)+(a_2 b_3+a_3 b_2)+a_3 b_3 \\ +(a_1 b_4+a_4 b_1)+\dots,$$

the principle of the method being that a suffix m does not appear at all until all possible combinations of two lesser suffixes are exhausted.* It will easily be verified that the mere convergence of A and B is enough to ensure the convergence of the product series.†

* The rule is exactly the same for

$$\lambda = 3^n, 4^n, \dots, 2^{2^n}, \dots .$$

† It is perhaps worth pointing out that the reasons which make Theorem II (as an addition to Theorem I) trivial in the case of very "high" indices λ_n, do not apply to the problem of completing Mr. Littlewood's results concerning "Tauber's theorem" in the case of such high indices.

CORRECTIONS

p. 397, *line* 10 *up*. Read p/d.

p. 399, *line* 1. For Theorem 1 read Theorem I.

—— *last line*, and *p.* 400, 1*st line*. For 'including those on the curved (but not on the straight) boundaries of those regions' read 'including those on the boundaries of those regions, except for the horizontal boundary of D and the vertical boundary of D'''; cf. 1914, 11, p. 139.

p. 401, *line* 9. For § 4 read § 3.

—— *line* 6 *up*. For $\sum \nu_p \sum_{\lambda_m+\lambda_n=\nu_p}$ read $\sum_1^p \nu_r \sum_{\lambda_m+\lambda_n=\nu_r}$.

p. 402, *last line*, and *p.* 403, *lines* 3–4. For K, in the middle term, read K'.

COMMENTS

The restriction $(\lambda_n-\lambda_{n-1})/\lambda_n \to 0$ in Theorem II, § 7, was removed by Rosenblatt,† and the theorem was further generalized by Neder,‡ with the conditions

$$\sum_{\frac{1}{2}x\leqslant\lambda_n\leqslant x} |a_n| = O(1), \qquad \sum_{\frac{1}{2}x\leqslant\lambda_n\leqslant x} |b_n| = O(1).$$

In 1914, 11, Hardy and Littlewood use conditions similar to Neder's, with o in place of O, in the proof of their Theorem B, which also include the conditions of Theorem I here.

If the condition on b_n, in Theorems I or II, is given with μ_n in place of λ_n, it does not follow that, by inserting zero terms, the sequences λ_n and μ_n may be replaced, without loss of generality, by a single sequence. For the ratios $(\lambda_n-\lambda_{n-1})/\lambda_n$ and $(\mu_n-\mu_{n-1})/\mu_n$ will, in general, be decreased by this process. Landau's remark (§ 3, footnote) does not apply in this context. However, Hardy shows in 1927, 10 that the theorems do remain true with the modified hypotheses, multiplication being defined with $\lambda_m+\mu_n$ in place of $\lambda_m+\lambda_n$. He remarks that the modified theorems can be obtained from Neder's theorem. In fact, if Neder's conditions and the multiplication rule are restated, with sequences λ_n and μ_n, zero terms may be inserted, without altering the conditions or the product, and so the λ_n and μ_n may be replaced by a single sequence.

In the proof of Theorem II, the expression ${}_2\Gamma_2$ is the same as ${}_1\Gamma_2$, with a and m interchanged with b and n, and both may be treated similarly.

The remark in § 8 shows effectively that if $\underline{\lim}\, \lambda_{n+1}/\lambda_n > 1$, then the conditions of Theorem I are satisfied, whenever $\sum a_n$ and $\sum b_n$ are convergent; cf. H.R., Theorem 36, and 1926, 5, Theorem 1.

† *Jahresber. d. Deutschen Math.-Verein.* 23 (1914), 80–4; see also Landau, ibid., 29 (1920), 238.

‡ *Proc. London Math. Soc.* (2), 23 (1925), 172–84.

GENERALISATIONS OF A LIMIT THEOREM OF MR. MERCER.

By G. H. HARDY.

1. IT was proved by Cauchy that if

(1) $$x_{n+1} - x_n \to s,$$

as $n \to \infty$, then also

(2) $$x_n / n \to s;$$

or, what amounts to the same thing, that if

(1′) $$s_n \to s,$$

then

(2′) $$\frac{s_1 + s_2 + \ldots + s_n}{n} \to s.$$

The converses of these theorems are, of course, in general untrue.*

From (1) and (2) it follows that

(3) $$x_{n+1} - x_n - a(x_n/n) \to (1-a)s.$$

2. It has been shown by Mr. Mercer† that, if a is real and less than 1, (1) and (2) are both consequences of (3); but if $a > 1$ this is not the case. This theorem of Mr. Mercer's seems likely to have interesting applications to the theory of summable series, with which it is obviously connected. But I have found myself that when the relation (3) occurs in this theory it is with $a > 1$, exactly the case excluded in Mr. Mercer's theorem. In the attempt to extend the theorem to this case (naturally with some additional condition imposed upon x_n) I have been led to a series of theorems which seem to me of some intrinsic interest.

* In the *Proc. Lond. Math. Soc.*, vol. viii., pp. 302 *et seq*, I proved that (1′) follows from (2′) if $s_n = a_1 + a_2 + \ldots + a_n$ and $|na_n| < K$. Landau has extended this theorem to the case in which the last inequality is replaced by the less stringent inequality $na_n > -K$; see *Proc Matematyczno-Fizycznych*, pp. 97 *et seq.*

† *Proc. Lond. Math. Soc.*, vol v., pp. 206 *et seq.*

3. I shall write, after Landau,

$$f(n)=O(\phi) \text{ or } f(x)=O(\phi)$$

if ϕ is a positive function of n or x, and

$$|f|<K\phi$$

as n or x tends to infinity; and

$$f(n)=o(\phi) \text{ or } f(x)=o(\phi)$$

if $f/\phi \to 0$. Thus $f=o(1)$ means the same as $f \to 0$.

THEOREM 1.* *If a is a real or complex constant whose real part is not equal to unity, and*

$$f'(x)-\frac{a}{x}f(x)=o(1),$$

then

$$f(x)=Cx^a+o(x),$$

where C is a constant. If the real part of a is less than unity, C is zero.

Let $$f=\phi x^a.$$

Then $$f'-(af/x)=\phi' x^a=o(1),$$

and so, if $a=\alpha+i\beta$, $$\phi'=o(x^{-\alpha}).$$

If $\alpha<1$, we have

$$\phi=\phi(0)+\int_0^x o(t^{-\alpha})\,dt$$
$$=o(x^{1-\alpha}),$$
$$f=\phi x^a=o(x).$$

If $\alpha>1$, $\int^\infty \phi' dx$ is convergent, and so ϕ tends to a limit C as $x\to\infty$. Also

$$\phi=C-\int_x^\infty o(t^{-\alpha})\,dt$$
$$=C+o(x^{1-\alpha}),$$
$$f=\phi x^a=Cx^a+o(x).$$

If $\alpha=1$, we can only prove that

$$f=o(x\log x).$$

That the result of the theorem is not true in this case may be seen by considering the example

$$f=x^{1+i\beta}(\log x)^p,$$

where $0<p<1$.

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* We suppose f and f' continuous in any finite interval; the same applies to f, f', f'', ... in some later theorems. If a is complex, x^a has its principal value.

The theorem may be transformed into a variety of forms. Thus, if we put $f(x) = xF(x)$, we find that

$$xF'(x) - AF(x) = o(1)$$

involves $$F(x) = Cx^A + o(1).$$

Corollary. If

$$f'(x) - \frac{a}{x} f(x) = l + o(1),$$

then $$f(x) = Cx^a + \frac{l}{1-a} x + o(x).$$

4. THEOREM 2. *If*

$$f(n+1) - f(n) - \frac{a}{n} f(n) = o(1),$$

and $\alpha \neq 1$, *then* $$f(n) = C\frac{\Gamma(n+a)}{\Gamma(n)} + o(n),$$

where C is a constant. If $\alpha < 1$, C is zero.

Let $$f(n) = \frac{\Gamma(n+a)}{\Gamma(n)} \phi(n).$$

Then

$$f(n+1) - f(n) - \frac{a}{n} f(n) = \{\phi(n+1) - \phi(n)\} \frac{\Gamma(n+a+1)}{\Gamma(n+1)}.$$

Hence, as $\dfrac{\Gamma(n+a+1)}{\Gamma(n+1)} \sim n^a$, we obtain

$$\phi(n+1) - \phi(n) = o(n^{-\alpha}).$$

If $\alpha < 1$, we have

$$\phi(n) = \phi(0) + \sum_0^{n-1} o(\nu^{-\alpha})$$
$$= o(\nu^{1-\alpha}),$$

and $f(n) = o(n)$. If $\alpha > 1$, the series

$$\sum^\infty \{\phi(n+1) - \phi(n)\}$$

is convergent, and $\phi(n)$ tends to a limit as $n \to \infty$. Also

$$\phi(n) = \phi(\infty) - \sum_n^\infty o(\nu^{-\alpha})$$
$$= C + o(n^{1-\alpha}),$$

and $$f(n) = C\frac{\Gamma(n+a)}{\Gamma(n)} + o(n).$$

Corollary. If

$$f(n+1)-f(n)-\frac{a}{n}f(n)=l+o(1),$$

then
$$f(n)=C\frac{\Gamma(n+a)}{\Gamma(n)}+\frac{l}{1-a}+o(n).$$

If a is real and less than 1, this is Mr. Mercer's theorem.

Theorems 1 and 2 may be generalised by replacing $o(1)$, in the formula from which we start, by $o(\psi)$, where ψ is an increasing function of x or n; but the theorems thus obtained seem to be less interesting. There are, however, two other directions in which interesting generalisations may be found.

5. THEOREM 3. *If the roots r, s of the equation*

$$x(x-1)-ax-b=0$$

are distinct, and neither has its real part equal to 2, *and*

$$f''(x)-\frac{a}{x}f'(x)-\frac{b}{x^2}f(x)=o(1),$$

then
$$f(x)=Cx^r+Dx^s+o(x^2),$$

where C and D are constants.

Let
$$f=\phi x^r.$$

Then
$$f''-\frac{a}{x}f'-\frac{b}{x^2}f=\phi''x^r+(2r-a)\phi' x^{r-1},$$

and so
$$\phi''-\frac{s-r-1}{x}\phi'=o(x^{-\rho}),$$

where ρ is the real part of r.

Now let
$$\phi'=\psi x^{s-r-1}.$$

Then we find, as in § 3, that

$$\psi' x^{s-r-1}=o(x^{-\rho}),$$
$$\psi'=o(x^{1-\sigma}),$$

where σ is the real part of s. Hence, provided $\sigma\neq 2$, we get

$$\psi=D+o(x^{2-\sigma})$$
$$\phi'=Dx^{s-r-1}+o(x^{1-\rho}),$$

and, provided $s \neq r$, $\rho \neq 2$,

$$\phi = C + Dx^{s-r} + o(x^{2-\rho}),^*$$
$$f = Cx^r + Dx^s + o(x^2).$$

Corollary. *If* $r = s$,

$$f(x) = x^r (C + D \log x) + o(x^2).$$

It is easy to see that the result of the theorem ceases to be true when ρ or σ is equal to 2. Thus, if $a = 3$, $b = -4$, we have $r = s = 2$. It will be found that

$$f = x^2 (\log x)^m \quad (0 < m < 2)$$

satisfies the original relation.

6. THEOREM 4. *If*

$$\Delta^2 f_n - \frac{a}{n+1} \Delta f_n - \frac{b}{(n+1)\,n} f_n = o(1),$$

then
$$f_n = C \frac{\Gamma(n+r)}{\Gamma(n)} + D \frac{\Gamma(n+s)}{\Gamma(n)} + o(n^2).$$

Here
$$\Delta f_n = f_{n+1} - f_n,$$

and r, s are defined as before, and are subject to the same restrictions.

Let
$$f_n = \frac{\Gamma(n+r)}{\Gamma(n)} \phi_n.$$

We find, by easy calculations,

$$\Delta f_n = \frac{\Gamma(n+r+1)}{\Gamma(n+1)} \Delta\phi_n + \frac{r}{n} \frac{\Gamma(n+r)}{\Gamma(n)} \phi_n,$$

$$\Delta^2 f_n = \frac{\Gamma(n+r+2)}{\Gamma(n+2)} \Delta^2\phi_n + 2 \frac{r}{n+1} \frac{\Gamma(n+r+1)}{\Gamma(n+1)} \Delta\phi_n + \frac{r(r-1)}{(n+1)\,n} \frac{\Gamma(n+r)}{\Gamma(n)} \phi_n.$$

Hence

$$\frac{\Gamma(n+r+2)}{\Gamma(n+2)} \Delta^2\phi_n + \frac{2r-a}{n+1} \frac{\Gamma(n+r+1)}{\Gamma(n+1)} \Delta\phi_n = o(1),$$

or
$$\Delta^2\phi_n - \frac{s-r-1}{n+r+1} \Delta\phi_n = o(n^{-\rho}).$$

* The value of the arbitrary constant D is changed, of course, on integration.

Now let
$$\Delta\phi_n = \frac{\Gamma(n+s)}{\Gamma(n+r+1)}\psi_n.$$

Then, substituting in the last equation, we find
$$\frac{\Gamma(n+s+1)}{\Gamma(n+r+2)}\Delta\psi_n = o(n^{-\rho}),$$
or
$$\Delta\psi_n = o(n^{1-\sigma}).$$

Hence, provided neither ρ nor σ is equal to 2, we obtain
$$\psi_n = D + o(n^{2-\sigma}),$$
$$\Delta\phi_n = D\frac{\Gamma(n+s)}{\Gamma(n+r+1)} + o(n^{1-\rho}),$$
$$\phi_n = C + D\Sigma\frac{\Gamma(n+s)}{\Gamma(n+r+1)} + o(n^{2-\rho})$$
$$= C + D\frac{\Gamma(n+s)}{\Gamma(n+r)} + o(n^{2-\rho}),*$$
$$f_n = C\frac{\Gamma(n+r)}{\Gamma(n)} + D\frac{\Gamma(n+s)}{\Gamma(n)} + o(n^2).$$

If $r=s$, the result is slightly different, viz.,
$$f_n = \frac{\Gamma(n+r)}{\Gamma(n)}\{C + D\psi(n+r)\} + o(n^2),$$
where
$$\psi(u) = \frac{d}{du}\log\Gamma(u).$$

7. It is clear that we may proceed indefinitely in this way, and establish corresponding propositions relating to functions defined by approximate differential or difference equations of any order.

There is also a corresponding series of propositions related to the differential equation
$$A\frac{d^ky}{dx^k} + B\frac{d^{k-1}y}{dx^{k-1}} + \ldots = 0,$$
as those already proved are to the equation
$$Ax^k\frac{d^ky}{dx^k} + Bx^{k-1}\frac{d^{k-1}y}{dx^{k-1}} + \ldots = 0.$$

Thus, to take only the two simplest cases, we have:

* The value of the arbitrary constant D is, of course, changed on summation.

THEOREM 5. *If*

$$f'(x) - af(x) = o(1),$$

and $R(a) \neq 0$, *then* $f(x) = Ce^{ax} + o(1)$.

THEOREM 6. *If* $\Delta f_n - af_n = o(1)$

and $|a+1| \neq 1$, *then*

$$f_n = C(a+1)^n + o(1).$$

If, in Theorem 5, we put $a = -1$, we obtain the result that $f + f' \to 0$ involves $f \to 0$, a result employed in Bromwich's *Infinite Series*, where an alternative proof is given.* From Theorem 6 it follows that

$$f_{n+1} - Af_n \to 0$$

involves $f_n \to 0$, if $|A| < 1$. If $f_{n+1} - Af_n \to l$, then of course $f_n \to l/(1-A)$.

8. There is another line of generalisation which leads to more interesting results.

I suppose that $\lambda(x)$ is a function of x which has a continuous derivative and tends steadily to ∞ with x.

THEOREM 7. *If*

$$\frac{f'(x)}{\lambda'(x)} - a\frac{f(x)}{\lambda(x)} = o(1),$$

and $\alpha \neq 1$,† *then* $f = C\lambda^a + o(\lambda)$.

This is merely a repetition of Theorem 1, as appears at once if we take λ as a new independent variable. It is inserted to lead up to the corresponding theorem concerning functions of an integral variable.

I shall now subject λ to a restriction, viz., that

$$\Sigma \left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_n}\right)^2$$

where $\lambda_n = \lambda(n)$, is convergent.

Let
$$\rho_n = (\lambda_{n+1} - \lambda_n)/\lambda_n,$$

$$\Phi_n(a) = (1 + a\rho_0)(1 + a\rho_1)\ldots(1 + a\rho_{n-1}),$$

so that $\Phi_n(1) = (1+\rho_0)(1+\rho_1)\ldots(1+\rho_{n-1}) = \lambda_n/\lambda_1$.

Then

$$\log \Phi_n(a) - a \log \Phi_n(1) = \sum_{1}^{n-1} \{\log(1 + a\rho_\nu) - a\log(1+\rho_\nu)\},$$

---

* p. 272: see also Hardy, *Quarterly Journal*, vol. xxxv., pp. 31 *et seq*, for an equivalent result.

† α is, as before, the real part of a.

and the series on the right-hand side is convergent when extended to infinity. Hence

$$\log \Phi_n(a) - a \log (\lambda_n/\lambda_1)$$

tends to a limit as $n \to \infty$, and so

$$\Phi_n(a) \sim A\lambda_n^a,$$

where A is a constant.

Now consider the equation

$$\frac{f_{n+1}-f_n}{\lambda_{n+1}-\lambda_n} - a\frac{f_n}{\lambda_n} = o(1).$$

Putting $$f_n = \phi_n \Phi_n(a),$$

and observing that

$$\frac{\Phi_{n+1}(a)-\Phi_n(a)}{\lambda_{n+1}-\lambda_n} - a\frac{\Phi_n(a)}{\lambda_n} = 0,$$

we obtain $$\frac{\phi_{n+1}-\phi_n}{\lambda_{n+1}-\lambda_n}\Phi_n(a) = o(1),$$

$$\phi_{n+1}-\phi_n = o\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_n^a}\right).$$

There is no further difficulty in completing the proof of the following theorem.

THEOREM 8. *If λ_n is an increasing sequence such that*

$$\Sigma\rho_n^2 = \Sigma\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right)^2$$

is convergent, and

$$\frac{f_{n+1}-f_n}{\lambda_{n+1}-\lambda_n} - a\frac{f_n}{\lambda_n} = o(1),$$

then $$f_n = C\Phi_n(a) + o(\lambda_n),$$

where $$\Phi_n(a) = (1+a\rho_1)(1+a\rho_2)+\ldots(1+a\rho_{n-1}).$$

We can at once write down the generalised forms of Theorems 7 and 8, in which $o(1)$ on the right-hand side is replaced by $l + o(1)$. In particular, we have:

THEOREM 9. *If*

$$\frac{f_{n+1}-f_n}{\lambda_{n+1}-\lambda_n} - a\frac{f_n}{\lambda_n} \to l,$$

and $R(a) < 1$, then $$\frac{f_n}{\lambda_n} \to \frac{l}{1-a}.$$

CORRECTIONS

p. 145, *line* 8 *up*. If $a \neq -1, -2,...,$ for $\phi(0)+\sum_{0}^{n-1}$ read $\phi(1)+\sum_{1}^{n-1}$;

$\phi(0)$ is not defined. If $a = -M$ $(M = 1, 2,...)$, read $\phi(M+1)+\sum_{M+1}^{n-1}$;

$\phi(n)$ is not defined for $n \leqslant M$.

—— *line* 7 *up*. For $o(\nu^{1-a})$ read $o(n^{1-a})$.

COMMENTS

A Mercerian theorem is, broadly speaking, a statement about the inverse (or inverses) of a given transformation. A typical proof is one that obtains the result without actually finding the inverse. If $a \neq -1, -2,...,$ Mercer's theorem may be associated with the transformation (i): $t_n = x_{n+1}-x_n-ax_n/n$ $(n \geqslant 1)$. The inverse is

$$x_n = \frac{\Gamma(n+a)}{\Gamma(n)}\left(C+\sum_{1}^{n-1}\frac{\Gamma(r+1)}{\Gamma(r+1+a)}t_r\right) \quad (n \geqslant 1),$$

where $x_1 = C\Gamma(1+a)$ is arbitrary. If $a = -M$ $(M = 1, 2,...)$, and the transformation (ii): $t_n = x_{n+1}-x_n+Mx_n/n$ is defined for $n \geqslant M+1$, the inverse is

$$x_n = \frac{(n-1-M)!}{(n-1)!}\left(C+\sum_{M+1}^{n-1}\frac{r!}{(r-M)!}t_r\right) \quad (n \geqslant M+1),$$

where $x_{M+1} = C/M!$ is arbitrary. On the other hand, if (ii) is defined for $n \geqslant 1$, then $x_{M+1} = t_M$, which is not arbitrary, and the inverse is

$$x_n = \frac{(n-1-M)!}{(n-1)!}\sum_{M}^{n-1}\frac{r!}{(r-M)!}t_r \quad \text{for } n \geqslant M+1,$$

$$x_n = (-1)^{n-1}\binom{M-1}{n-1}\left(C+\sum_{1}^{n-1}(-1)^r t_r\Big/\binom{M-1}{r}\right) \quad \text{for } 1 \leqslant n \leqslant M,$$

where $x_1 = C$ is arbitrary.

In 1913 Schur† made an important modification in Mercer's theorem by introducing the matrix transformation

$$T_\alpha = \alpha I+(1-\alpha)H, \quad \alpha > 0,$$

where $y = T_\alpha(s)$ denotes

$$y_n = \alpha s_n+(1-\alpha)\frac{s_0+...+s_n}{n+1} \quad (n \geqslant 0).$$

† *Math. Annalen* 74 (1913), 447–58.

This corresponds to (iii): $t_n = x_n - x_{n-1} - ax_n/(n+1)$ $(n \geqslant 0)$, where $a < 1$ and $x_{-1} = 0$. Schur's transformation is regular and has a unique inverse, which is also regular. He showed that, for k a positive integer, the Cesàro means C_n^k are transformed to the Hölder means H_n^k, by the succession of reversible transformations $T_{1/k}$, $T_{1/(k-1)}, \ldots$, T_1. This gave a new proof of the Knopp–Schnee equivalence theorem; see the Comments on 1907, 6. Schur's form of Mercer's theorem is adopted by Hardy in D.S., pp. 104–7.

A theorem of Mercerian type is given in 1904, 4, § 4; see the Comments on 1904, 4. The 'high indices' theorem (1926, 5) may also be regarded as a Mercerian theorem; see the Comments on 1926, 5. See also 1912, 7, § 3.

In the proof of Theorem 3, it would be natural to put $x = e^t$, $f(x) = g(t)$, $x\,d/dx = d/dt$, so that the differential equation becomes

$$\left(\frac{d}{dt}\left(\frac{d}{dt}-1\right)-a\,\frac{d}{dt}-b\right)g(t) = o(e^{2t}).$$

But the proof given is a model for that of Theorem 4.

NOTE ON A THEOREM OF CESÀRO.

By *G. H. Hardy.*

1. On p. 53 of his *Introduction to the Theory of Infinite Series* Dr. Bromwich states and proves the following theorem:

Let a_n be a positive and steadily decreasing function of n, whose limit, as $n \to \infty$, is zero; and let p_n be the number of positive terms and q_n the number of negative terms in the first n terms of the series

$$(1) \qquad \pm a_1 \pm a_2 \pm a_3 \pm \ldots,$$

so that $p_n + q_n = n$. Then if the series (1) *is convergent, but not absolutely convergent, the ratio*

$$p_n / q_n$$

cannot tend to any limit other than unity—or, what is the same thing, the ratio

$$(p_n - q_n) / n$$

cannot tend to any limit other than zero.

The theorem is due to Cesàro (*Rendiconti della Accademia dei Lincei*, ser. 4, t. 4, p. 133). The proof given by Dr. Bromwich is based upon one given by Bagnera (*Bulletin des Sciences Mathématiques*, sér. 2, t. 12, p. 227).

I find that Cesàro proved a good deal more than this; and, as the theorem in question is an exceedingly curious and interesting one, I think it may be worth while to make a few remarks about it.

2. The theorem itself assigns no criterion for the existence of the limit p_n/q_n; it merely states that *if* it exists it must be unity. Cesàro, however, went on to consider the question of the existence of the limit, and in the paper already quoted states the further result: "*If the series Σa_n is not less divergent than the harmonic series, i.e., if na_n is ultimately greater than some positive constant, then p_n/q_n certainly tends to a limit (that is to say, unity).*"

The proof that he gives appears, however, to be faulty.*

* See the argument (*l.c.*, p. 135) beginning "S' il est impossible de trouver..."; and that beginning "En effet, si $(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)/n$ n'admet pas une limite..."

Cesàro, however, returned to the subject in a later paper (*Nouvelles Annales*, sér. 3, t. vii., p. 405), and gave a valid and much simpler proof as follows. Let

$$u_n = \pm a_n = \epsilon_n a_n, \quad s_n = u_1 + u_2 + \ldots + u_n.$$

Then, as $n \to \infty$, $s_n \to s$, say; and by a well-known theorem of Cauchy and Stolz, if A_n is a function of n which tends steadily to infinity with n, we have

$$\frac{A_1 s_1 + (A_2 - A_1) s_2 + \ldots + (A_n - A_{n-1}) s_n}{A_n} \to s,$$

or

$$s_n - \frac{A_1 u_2 + A_2 u_3 + \ldots + A_{n-1} u_n}{A_n} \to s,$$

or

$$\frac{A_1 u_2 + A_2 u_3 + \ldots + A_{n-1} u_n}{A_n} \to 0.$$

But also, as $u_n \to 0$, we have

$$\frac{A_1 u_1 + (A_2 - A_1) u_2 + \ldots + (A_n - A_{n-1}) u_n}{A_n} \to 0,$$

and combining these relations we obtain

(2) $$\frac{A_1 u_1 + A_2 u_2 + \ldots + A_n u_n}{A_n} \to 0.$$

Now let $A_n = 1/a_n$, and we obtain

(3) $$(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n) a_n \to 0,$$

or

(3′) $$(p_n - q_n) a_n \to 0.$$

If $na_n > K$, for $n > n_0$, it follows that

(4) $$(p_n - q_n)/n \to 0,$$

and the theorem is proved. It should be observed, however, that Cesàro has proved *more* than the theorem, the relation (3′) giving in general more information than (4).

3. The following proof of the theorem of § 2, though perhaps less elegant than Cesàro's, is in some ways more direct and even simpler.

We have*

(5) $$s_n = \sum_1^{n-1} (p_\nu - q_\nu)\,\Delta a_\nu + (p_n - q_n)\,a_n$$

and

(6) $$s_n - s_m = -(p_m - q_m)\,a_{m+1} + \sum_{m+1}^{n-1} (p_\nu - q_\nu)\,\Delta a_\nu + (p_n - q_n)\,a_n.$$

Suppose that, if possible, $(p_n - q_n)\,a_n$ has not the limit zero. Then we can find a positive number δ such that either

$$(p_n - q_n)\,a_n > \delta$$

for an infinity of values of n, or

$$(p_n - q_n)\,a_n < -\delta$$

for an infinity of values of n; let us say the former. Let $n_1, n_2, \ldots$ be such an infinity of values of n. Then, *unless* $p_n - q_n$ *is ultimately of constant sign*, we can associate with each n_i the greatest m_i for which $p_m - q_m \leqq 0$, and $m_i \to \infty$ with n_i. But then, from (6),

$$s_{n_i} - s_{m_i} > \delta,$$

which contradicts the hypothesis that s_n has a limit.

If, however, $p_n - q_n$ is ultimately of constant sign, let U, L be the maximum and minimum limits of $(p_n - q_n)\,a_n$, so that $0 \leqq L \leqq U$. Then we can find an infinity of values n_i such that

$$(p_n - q_n)\,a_n > U - \epsilon \quad (n = n_i),$$

and another of values m_i such that

$$(p_m - q_m)\,a_m < L + \epsilon \quad (m = m_i),$$

ϵ being an arbitrary positive number; and, from (6),

$$s_{n_i} - s_{m_i} > U - L - 2\epsilon,$$

for all pairs n_i, m_i such that $n_i > m_i$. And this again contradicts the hypothesis that s_n has a limit, unless $U = L$. There remains only the possibility that

$$(p_n - q_n)\,a_n \to l > 0.$$

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* Bromwich, *Infinite Series*, p. 53.

But then it follows, from (5), that

$$\Sigma \frac{a_\nu - a_{\nu+1}}{a_\nu}$$

is convergent, and therefore that

$$\Pi\left(1-\frac{a_\nu - a_{\nu+1}}{a_\nu}\right) = \Pi\left(\frac{a_{\nu+1}}{a_\nu}\right)$$

is convergent; and as $a_n \rightarrow 0$ this is untrue. Thus Cesàro's theorem is established. The proof just given has the advantage of proceeding directly from first principles.

4. The question also arises as to whether Cesàro's result fairly represents the *maximum* of information of this kind about $p_n - q_n$ that can be obtained from a knowledge of the convergence of $\Sigma(\pm a_n)$. The following examples indicate that this is, substantially, the case.

(i) Let $a_n = n^{-s}$, where $0 < s < 1$, and let us start from the convergent series

$$1^{-s} - 2^{-s} + 3^{-s} - \dots. \tag{7}$$

Take a number t, greater than unity: it is convenient to suppose s and t *irrational.**

Of the two integers nearest to n^t, one less than and one greater than n^t, one is odd and one even. Let us denote the even one by $\phi(n)$. And now let us alter the series (7) by changing the sign of

$$u_{\phi(n)} \quad (n = 1, 2, \dots)$$

from *minus* to *plus*.† We thus obtain a series which is convergent or divergent according as

$$\Sigma\{\phi(n)\}^{-s}$$

is convergent or divergent: and it is easy to see that this series converges or diverges with Σn^{-st}; *i.e.*, converges if and only if

$$t > 1/s. \tag{8}$$

* The purpose of this hypothesis is merely to avoid certain slight and entirely irrelevant complications, which do not make the least difference to the result.

† It may happen, for some of the first values of n, that successive values of $\phi(n)$ coincide. Thus, if $t = \frac{3}{2}$, $\phi(1) = \phi(2) = 2$. In what follows we ignore this possibility, which is plainly without effect on the result.

Now, if n is even, the value of $p_n - q_n$ is plainly $2m$, where m is determined by

$$\phi(m) \leq n < \phi(m+1).$$

That is to say, $p_n - q_n$ is of order $n^{1/t}$, and the condition

$$(p_n - q_n)\, a_n \to 0$$

reduces to
$$n^{-s+(1/t)} \to 0,$$

or $t > 1/s$. Comparing this result with (8) we see that the series converges or diverges according as Cesàro's condition is or is not satisfied.

(ii) Consider the series

$$(9) \qquad \Sigma(\pm n^{-s}) = \sum_{1^t}^{2^t-1} n^{-s} - \sum_{2^t}^{3^t-1} n^{-s} + \sum_{3^t}^{4^t-1} n^{-s} - \dots,$$

where $0 < s < 1$, and t is a positive integer greater than unity. The k^{th} group of terms is

$$(-1)^{k-1}\left\{\left(\sum_{1}^{(k+1)^t-1} - \sum_{1}^{k^t-1}\right) n^{-s}\right\},$$

and a little elementary calculation shows that the contents of the large bracket may be expressed in the form

$$tk^{(1-s)t-1} + R_k,$$

where the order of R_k, as a function of k, is that of $k^{(1-s)t-2}$. It follows that the series (9) is convergent if, and only if, $(1-s)\,t - 1 < 0$, *i.e.*, if $t < 1/(1-s)$.

Now, if $n = (2m)^t - 1$, it is clear that

$$p_n - q_n = (2^t - 1^t) - (3^t - 2^t) + \dots + \{(2m)^t - (2m-1)\}^t = \tfrac{1}{2}t(2m)^{t-1} + \dots,$$

as appears from a little easy calculation, the neglected terms being of order $t-2$. Similarly, if $n = (2m+1)^t - 1$, we find

$$p_n - q_n = -\tfrac{1}{2}t\,(2m)^{t-1} + \dots.$$

In other words, the oscillations of $p_n - q_n$ about zero are of order m^{t-1} or $n^{1-(1/t)}$; and Cesàro's condition is satisfied if, and only if,

$$n^{-s+1-(1/t)} \to 0,$$

or $t < 1/(1-s)$. In other words, the series converges or oscillates according as Cesàro's condition is or is not satisfied.

These two examples seem to show sufficiently clearly what is the answer to the question raised at the beginning of this section: it would not be difficult to formulate general theorems.

5. In Cesàro's equation (2) of § 2 we may, instead of putting $A_n = 1/a_n$, put $A_n = b_n/a_n$, where b_n is a function of n which tends to zero less rapidly than a_n. Then

$$\frac{\epsilon_1 b_1 + \epsilon_2 b_2 + \ldots + \epsilon_n b_n}{(b_n/a_n)} \to 0.$$

Thus the convergence of

$$\pm\frac{1}{1} \pm \frac{1}{2} \pm \ldots \ldots$$

involves the relation

$$\frac{1}{\sqrt{n}}\left(\pm\frac{1}{\sqrt{1}} \pm \frac{1}{\sqrt{2}} \pm \ldots \pm \frac{1}{\sqrt{n}}\right) \to 0.$$

This also enables us to obtain some information in the case in which $na_n \to 0$, when Cesàro's relation (3′) gives no information, $p_n - q_n$ being certainly less than n. For example, if we take $b_n = 1/n$, we obtain

$$na_n\{\epsilon_1 + \tfrac{1}{2}\epsilon_2 + \ldots + (1/n)\epsilon_n\} \to 0;$$

and if
$$a_n > K/(n \log n),$$

we may replace a_n in this relation by $1/(n \log n)$, so obtaining

$$\{\epsilon_1 + \tfrac{1}{2}\epsilon_2 + \ldots + (1/n)\epsilon_n\}/\log n \to 0.$$

In other words, if $\Sigma(\pm a_n)$ is convergent and Σa_n diverges at least as rapidly as

$$\Sigma\frac{1}{n \log n},$$

the numbers $\epsilon_n = \pm 1$ have the mean value zero in the sense of M. Riesz;* just as, when the series Σa_n diverges at least as rapidly as $\Sigma(1/n)$, they must have the mean value zero in the ordinary sense.

* *Comptes Rendus*, July 5, 1909; see also *Proc. L.M.S.*, vol. viii., p. 301, *et seq.*

In the first of the two papers of Cesàro quoted by Hardy,† and also in a later paper,‡ Cesàro deduced the theorem stated in § 1 from a case of the Cesàro–Hardy theorem:§ *if* ϵ_n *is any sequence such that*

$$(\epsilon_1+\epsilon_2+...+\epsilon_n)/n \to s,$$

a_n *is positive and decreasing*|| *and* $\sum a_r = \infty$, *then*

$$\frac{a_1\epsilon_1+...+a_n\epsilon_n}{a_1+...+a_n} \to s.$$

In particular, if $\sum a_r \epsilon_r$ converges, then s must be zero. The theorem of § 1 is the case where $a_n \to 0$, $\epsilon_n = \pm 1$.

In § 2 the result that (2) *holds when* $\sum u_n$ *converges and* A_n *increases to infinity* was given by Kronecker.††

In Cesàro's paper (1), there is a passage following his 'faulty proof' of the theorem discussed in § 2, that seems to establish the result correctly. Starting from the formula

$$s_n = (\epsilon_1+...+\epsilon_n)a_{n+1}+\sum_{i=1}^{n}(\epsilon_1+...+\epsilon_i)(a_i-a_{i+1}),$$

where $s_n = a_1\epsilon_1+...+a_n\epsilon_n$, he writes: 'Par un calcul inverse on trouve

$$\epsilon_1+...+\epsilon_n = \frac{s_n}{a_{n+1}}-\sum_{i=1}^{n}\left(\frac{1}{a_{i+1}}-\frac{1}{a_i}\right)s_i,$$

et l'on en déduit sans peine

$$\lim(\omega_n-\tfrac{1}{2})na_n = 0.\text{'}$$

Here $\omega_n-\frac{1}{2} = (\epsilon_1+...+\epsilon_n)/n$ and a_n decreases to zero.

Suppose that Cesàro's formulae are rearranged so that the nth term of the sum is combined with the term outside. Then, if $\epsilon_1+...+\epsilon_n = p_n-q_n$, $p_n+q_n = n$, his first formula becomes Hardy's formula (5), § 3, while his second formula becomes the inverse of (5):

$$(p_n-q_n)a_n = s_n-a_n\sum_{i=1}^{n-1}\left(\frac{1}{a_{i+1}}-\frac{1}{a_i}\right)s_i.$$

This provides an alternative to Hardy's proof, and illustrates the Mercerian character of the theorem; cf. 1912, 5.

† Cesàro (1), *Atti d. Reale Accad. d. Lincei Rend.* (4), 4 (1888), 133–8; (2), *Nouvelles Ann. de Math.* (3), 7 (1888), 401–7.

‡ Cesàro (3), *Bull. des. sci. math.* (2), 13 (1889), 51–4.

§ See 1907, 5, § 4. In case (a) of the theorem, take $b_n = 1$, $c_n = a_n$, $s_n = \epsilon_n$.

|| In Cesàro's version $a_n \to 0$.

†† *Comptes rendus* 103 (1886), 980–7; see D.S., Theorem 26.

THE RELATIONS BETWEEN BOREL'S AND CESÀRO'S METHODS OF SUMMATION

By G. H. Hardy *and* J. E. Littlewood.

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I.

1. The results obtained in this paper are developments of an idea that has been prominent in much recent work on the theory of divergent series.

Consider the series

$$a_0+a_1+a_2+\dots. \tag{1}$$

This series may converge, or may possess a "sum" according to one or other of a large variety of definitions of what is meant by the "sum" of a series: Hölder's, Cesàro's, or Riesz's definitions by mean values, Euler's definition as the limit of a power series, Borel's exponential definition, and so forth. To each of these definitions correspond certain limits of applicability. Thus Hölder's and Cesàro's definitions can never be successful unless

$$n^{-k}a_n \to 0$$

for some value of k; Euler's definition requires that $\Sigma a_n x^n$ should converge for $|x|<1$; Borel's that

$$\Sigma \frac{a_n x^n}{n!}$$

should be an integral function of x. Roughly, we may say that any method of summation will fail if the series to which it is applied is *too divergent.** Or, in other words, to any method corresponds a certain upper limit of its power, the specification of which is a problem generally not difficult and often uninteresting.

It is only recently that it has been observed that the range of applicability of all methods of summation is limited from below, so to say, as well as from above. Methods fail, not only when the series to which it is attempted to apply them is *too divergent*, but also if it is *too nearly convergent*: not only is their *power* limited, but also their *delicacy*. The theorems which express this latter fact are more subtle than those which express the "limitation from above," and take a rather different form: they assert that, if a series is too nearly convergent, it cannot be summable unless it is *actually* convergent.

The theorems of this character which correspond to Cesàro's (or Hölder's) and to Euler's methods were discussed by us in two recent papers in these *Proceedings.*† It follows from a well known theorem of Tauber‡ that a series for which

$$na_n \to 0 \tag{2}$$

cannot be summable by any of these methods unless it is convergent. In these papers we showed that this condition may be replaced by the more general condition

$$|na_n| < K. \tag{2'}$$

Similar results hold for Riesz's more general methods: these results will also be found in the papers referred to.

The primary object of this paper was to establish the analogue of Tauber's theorem for Borel's exponential method of summation. But this problem has led us on to a number of others, some of which are discussed here, while to others we hope to return later. There is one important point with regard to which our results are not complete. We

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* In these general explanations (as in the phrase "Theory of Divergent Series") it is convenient to use *divergent* as meaning simply non-convergent. In detailed work it is essential to distinguish between divergent and oscillatory series.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301; Littlewood, *ibid.*, Vol. 9, p. 434. The results of the first of these papers may be deduced as corollaries from those of the second: they have been extended, in a somewhat different direction, by Landau (*Prac Matematyczno-Fizycznych*, t. 21, p. 97), who shows that, when a_n is real, it is enough to suppose $na_n < K$ or $na_n > -K$, and makes an interesting application of the result to the Theory of Prime Numbers.

‡ If $\Sigma a_n x^n \to s$, as $x \to 1$, and $na_n \to 0$ as $n \to \infty$, then Σa_n is convergent (to sum s). For a proof see Littlewood, *l.c. supra*, and Bromwich, *Infinite Series*, p. 251.

prove (Theorem 1 below) that a series for which

(3) $$\sqrt{n} \,.\, a_n \to 0$$

cannot be summable by Borel's method, unless it is convergent. It can hardly be doubted that this result (the analogue of Tauber's theorem) is susceptible of the same generalisation: that is to say, that (3) may be replaced by

(3′) $$|\sqrt{n} \,.\, a_n| < K.$$

But this we have not at present succeeded in proving; and the difficulties attendant on the generalisation of Tauber's theorem suggest forcibly that the proof may not be at all easy to find.*

We shall use the symbols

$$K, \ \epsilon, \ \delta, \ O, \ o, \ \ldots,$$

in special senses for an explanation of which we must refer elsewhere.†

II.

2. Borel gave two definitions of the sum of the series

(1) $$a_0+a_1+a_2+\ldots.$$

According to his first definition the sum is

(2) $$s = \lim_{x\to\infty} e^{-x} \Sigma \frac{s_n x^n}{n!},$$

where $$s_n = a_0+a_1+\ldots+a_n.$$

According to the second it is

(2a) $$\int_0^\infty e^{-x} a(x)\,dx,$$

where

(2b) $$a(x) = \Sigma \frac{a_n x^n}{n!}.$$

These definitions are not exactly equivalent, the second being slightly

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* In the abstract of this paper which appeared in the "Notes and Corrections" to Vol. 9 of the *Proceedings*, we implied that we were in possession of a proof. We have since discovered that our belief that we had found one was mistaken.

† See Hardy, "Orders of Infinity," *Camb. Math. Tracts*, No. 12. The only symbol whose use is not explained there is the small o, introduced by Landau (*Handbuch der Lehre von der Verteilung der Primzahlen*, p. 61). We write

$$f = o(\varphi),$$

if ϕ is positive and $f/\phi \to 0$.

more general.* They are certainly equivalent whenever $a_n \to 0$. For the present we shall adopt the first definition as fundamental; if (2) holds, we shall say that the series (1) is *summable* (B) to sum s.

3. Our first object is to prove the following theorem:—

THEOREM 1.—*If* Σa_n *is summable* (B), *and*

$$\sqrt{n}\,.\,a_n \to 0,$$

then Σa_n *is convergent.*

This theorem, however, is only one of a hierarchy of theorems connecting Borel's with Cesàro's methods of summation. To establish these we shall require a number of lemmas.

We shall write

$$\begin{aligned} s_n &= a_0+a_1+\dots+a_n, \\ s_n^1 &= s_0+s_1+\dots+s_n, \\ s_n^2 &= s_0^1+s_1^1+\dots+s_n^1, \\ \dots &\quad \dots \quad \dots, \end{aligned}$$

so that Σa_n is summable (Ck) if

$$k!\, s_n^k / n^k \to s.\dagger$$

LEMMA 1.—*If* $c_n \sim An^a$ *as* $n \to \infty$, *then*

$$e^{-x} \Sigma c_n \frac{x^n}{n!} \sim Ax^a,$$

as $x \to \infty$.

This is, of course, well known.

LEMMA 2.—*We have identically*

$$e^{-x} \sum_0^\infty s_n^k \frac{x^{n+k}}{(n+k)!} = \left(\int_0^x dx\right)^k \left(e^{-x} \sum_0^\infty s_n \frac{x^n}{n!}\right).$$

We use the identity

$$s_n^k = s_n + \begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1} + \begin{bmatrix} k \\ 2 \end{bmatrix} s_{n-2} + \dots + \begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

where

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{p(p+1)\dots(p+q-1)}{q!}.$$

* Bromwich, *Infinite Series*, p. 297.

† We need hardly point out that s_n^k does not stand for a power of s_n.

The coefficient of s_ν on the left-hand side is

$$e^{-x}\left\{\frac{x^{\nu+k}}{(\nu+k)!}+\begin{bmatrix}k\\1\end{bmatrix}\frac{x^{\nu+k+1}}{(\nu+k+1)!}+\begin{bmatrix}k\\2\end{bmatrix}\frac{x^{\nu+k+2}}{(\nu+k+2)!}+\dots\right\},$$

and on the right-hand side is

$$\left(\int_0^x dx\right)^k\left(e^{-x}\frac{x^\nu}{\nu!}\right).$$

That these two expressions are identical may be verified immediately by induction.

LEMMA 3.—*If* Σa_n *is summable* (B) *to sum* s, *then*

$$e^{-x}\sum^{\infty} s_n^k \frac{x^n}{(n+k)!}\to\frac{s}{k!}.$$

This is an immediate corollary from Lemma 2.

4. Now suppose Σa_n summable (B), so that

$$e^{-x}\Sigma s_n^k\frac{x^n}{(n+k)!}\to\frac{s}{k!} \tag{3}$$

for any value of k. And let us suppose that there is a number a such that

$$s_n^{k-1}=o(n^a).\text{*} \tag{4}$$

Then, if for shortness we write $s_n^k=t_n$, we have

$$\Delta t_n\equiv t_n-t_{n+1}=-s_{n+1}^{k-1}=o(n^a). \tag{5}$$

If, in (3), we put $x=m$, we have

$$S_1+S_2\to s/k!, \tag{6}$$

where $\quad S_1=e^{-m}t_m\Sigma\dfrac{m^n}{(n+k)!},\qquad S_2=e^{-m}\Sigma(t_n-t_m)\dfrac{m^n}{(n+k)!}.$

By Lemma 1,

$$S_1=m^{-k}t_m(1+\epsilon_m). \tag{7}$$

* If $k=0$, $s_n^{k-1}=a_n$.

We proceed to discuss S_2, which we write in the form

(8) $$e^{-m}\left(\sum_{0}^{(1-H)m}+\sum_{(1-H)m}^{(1+H)m}+\sum_{(1+H)m}^{\infty}\right)=S_3+S_4+S_5,$$

say. Here H is a positive constant less than unity, and $\sum_{\mu}^{\nu}$ denotes a summation extended to all integral values n such that $\alpha<n<\beta$: it is convenient to suppose H *irrational*, so that $(1-H)m$ and $(1+H)m$ cannot be integral, but the limits of summation may be left indefinite to the extent of a term or two without any effect on the argument.

To obtain upper limits for S_3, S_4, and S_5, we use the inequalities

(9) $$\begin{cases} |t_n-t_m|<Km^K & (in\ S_3), \\ |t_n-t_m|\leqslant \epsilon_m m^a|n-m| & (in\ S_4), \\ |t_n-t_m|<Kn^K & (in\ S_5), \end{cases}$$

which are immediate consequences of inequalities already established.

Let $u_n=m^n/n!$. Then we find at once, by an elementary application of Stirling's theorem, that

$$e^{-m}u_\mu<e^{-Km},\qquad e^{-m}u_\nu<e^{-Km},$$

where u_μ and u_ν are the last u which occurs in S_3 and the first which occurs in S_5.

Hence $$|S_3|<Km^Ke^{-m}\sum_{0}^{(1-H)m}n^{-k}u_n$$
$$<m^Ke^{-m}(u_\mu+u_{\mu-1}+\dots+u_0)$$
$$<m^Ke^{-Km}\{1+(1-H)+(1-H)^2+\dots\}$$

(10) $$<e^{-Km}.$$

Similarly, $$|S_5|<Ke^{-m}\sum_{(1+H)m}^{\infty}n^{K-k}u_n$$
$$<e^{-m}\sum_{(1+H)m}^{\infty}n^Ku_n$$
$$<m^Ke^{-Km}\left\{1+\frac{2^K}{1+H}+\frac{3^K}{(1+H)^2}+\dots\right\}$$

(11) $$<e^{-Km}.$$

Finally we have to consider S_4, which we write in the form

$$S_4=\sum_{n>m}+\sum_{m>n}=S_6+S_7.$$

We have
$$|S_6| = \left| e^{-m} \sum_{m}^{(1+H)m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$
$$< Km^{-k} e^{-m} \sum_{m}^{(1+H)m} |t_n - t_m| \frac{m^n}{n!}$$
$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m+r}}{(m+r)!} \,*$$
$$< m^{a-k} e^{-m} \frac{m^m}{m!} \epsilon_m \sum \frac{rm^r}{(m+1)(m+2)\dots(m+r)}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \sum \frac{r}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)}$$

Now, for $1 \leqslant r < Hm$, we have
$$1\Big/\left(1+\frac{r}{m}\right) = \exp\left(-\frac{r}{m}+\frac{r^2}{2m^2}-\dots\right) < e^{-r/2m},$$
$$r\Big/\left\{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\dots\left(1+\frac{r}{m}\right)\right\} < r\exp\left(-\frac{1}{2m}-\frac{2}{2m}-\dots-\frac{r}{2m}\right)$$
$$< re^{-r^2/4m}.$$

Hence
$$|S_6| < m^{a-k-\frac{1}{2}} \epsilon_m \sum_{0}^{\infty} re^{-r^2/4m}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \int_0^\infty xe^{-x^2/4m}\,dx \dagger$$
$$< m^{a-k+\frac{1}{2}} \epsilon_m$$
(12)
$$= o(m^{a-k+\frac{1}{2}}).$$

Again,
$$|S_7| = \left| e^{-m} \sum_{(1-H)m}^{m} (t_n - t_m) \frac{m^n}{(n+k)!} \right|$$
$$< Km^{-k} e^{-m} \sum_{(1-H)m}^{m} |t_n - t_m| \frac{m^n}{n!}$$
$$< m^{a-k} e^{-m} \epsilon_m \sum_{1}^{Hm} r \frac{m^{m-r}}{(m-r)!}$$
$$< m^{a-k-\frac{1}{2}} \epsilon_m \sum r \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{r-1}{m}\right).$$

In this case we use the inequality
$$1-\frac{r}{m} < e^{-r/m},$$

* Using (5).

† When m is large, the integral is of order m. The difference between the integral and the series is less than a constant multiple of the maximum of $xe^{-x^2/4m}$, which is of order $\sqrt{m}$.

and the argument proceeds as before ; so that

(13) $$|S_7| = o(m^{a-k+\frac{1}{2}}).$$

From (4), (7), (10), (11), (12), and (13), we deduce

(14) $$s_m^k(1+\epsilon_m)/m^k+o(m^{a-k+\frac{1}{2}}) \rightarrow s/k!.$$

From this relation we can deduce Theorem 1 and the other theorems referred to in § 3.

5. Suppose first $a = k-\frac{1}{2}$. Then

(15) $$s_m^k/m^k \rightarrow s/k!.$$

Hence we obtain

THEOREM 2.—*If* Σa_n *is summable* (*B*), *and*

$$s_n^{k-1} = o(n^{k-\frac{1}{2}}),$$

then Σa_n *is summable* (*Ck*).

This reduces to Theorem 1 when $k = 0$ (when s_n^{-1} must be interpreted as meaning a_n).*

Next, suppose $a > k-\frac{1}{2}$. Then we obtain

THEOREM 3.—*If* Σa_n *is summable* (*B*) *and*

$$s_n^{k-1} = o(n^a)$$

where $a > k-\frac{1}{2}$, *then* $s_n^k = o(n^{a+\frac{1}{2}})$.†

THEOREM 4.—*If* Σa_n *is summable* (*B*), *and*

$$s_n^{k-1} = o(n^k),$$

then Σa_n *is summable* $(C, k+1)$.

For, by Theorem 3, we have $s_n = o(n^{k+\frac{1}{2}})$, and Theorem 4 accordingly follows from Theorem 2. The special case in which $k = 0$ is particularly interesting and deserves a separate statement.

THEOREM 4*a*.—*If* Σa_n *is summable* (*B*), *and* $a_n \rightarrow 0$, *then* Σa_n *is summable* (*C*1).

By a repeated application of the argument which led to Theorem 4, we deduce

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* The theorem was stated for the particular cases in which $k = 0, 1$ in the Abstract of this paper referred to above.

† That $s_n^k = o(n^{a+1})$ is trivial. The point of the theorem lies in the reduction of $a+1$ to $a+\frac{1}{2}$ as a result of the hypothesis of Borel summability.

THEOREM 5.—*If* Σa_n *is summable* (B), *and*

$$s_n^{k-1} = o(n^{k+\frac{1}{2}(r-1)}),$$

then Σa_n *is summable* $(C,\ k+r)$.

6. If $\Sigma a_n x^n$ is convergent for $|x| < 1$, and the function $f(x)$ represented by the sum of the series is regular for $x = 1$, the series Σa_n is summable (B). It therefore follows from Theorems 1 and 4*a* that

(i) *If* $f(x)$ *is regular for* $x=1$, *and* $\sqrt{n}.a_n \to 0$, *then* Σa_n *is convergent.*

(ii) *If* $f(x)$ *is regular for* $x=1$, *and* $a_n \to 0$, *then* Σa_n *is summable* $(C1)$.

Each of these corollaries of our theorems is included in Fatou's theorem* that, *if* $f(x)$ *is regular for* $x = 1$, *and* $a_n \to 0$, *then* Σa_n *is convergent.* But we have, of course, assumed much less than regularity for $x = 1$.

If Σa_n is summable by Cesàro's means, or, more generally, if Abel's limit exists, we can only infer convergence if

$$a_n = O(1/n).$$

To assume that Σa_n is summable (B) is to assume *more* than that it is summable (C)† or by Abel's limit, but less than that $f(x)$ is regular. To this corresponds the fact that $\sqrt{n}.a_n \to 0$ asserts less than $a_n = O(1/n)$, but more than $a_n \to 0$.

7. The results of § 5 may be represented conveniently by means of a diagram.

If $a_n =$	$s_n =$	$s_n^1 =$	$s_n^2 =$	...	*then the series is*
$o(n^{-\frac{1}{2}})$					*convergent*
$o(1)$	$o(n^{\frac{1}{2}})$				*summable* $(C1)$
$o(n^{\frac{1}{2}})$	$o(n)$	$o(n^{\frac{3}{2}})$			,, $(C2)$
$o(n)$	$o(n^{\frac{3}{2}})$	$o(n^2)$	$o(n^{\frac{5}{2}})$		,, $(C3)$
...	...	...	...	...	...

* Fatou, *Thèse* (Paris, 1906) and *Acta Mathematica*, t. 30, p. 389. A simpler proof, and a series of important generalisations, have been given by Riesz, *Crelle's Journal*, Bd. 140, S. 89, and *Comptes Rendus*, Nov. 22, 1909; see also Landau, *Prac Matematyczno-Fizycznych*, t. 21, p. 151.

† See Theorem 6 below for a precise statement.

This diagram at once suggests that there should be further theorems corresponding to the spaces which we have not filled in, such as :—

(a) if $s_n = o(1)$, and Σa_n is summable (B), then Σa_n is convergent;

(b) if $s_n^1 = o(\sqrt{n})$, and Σa_n is summable (B), then Σa_n is convergent;

and so on. These theorems are all trivial or false. Thus (a) is obviously trivial: the same is true of all the theorems which correspond to the vacant spaces which have two sides in common with those filled in in the diagram.

On the other hand, (b) is false. For, if $s_n^1 = o(\sqrt{n})$, we have

$$(s_0+s_1+\ldots+s_n)/(n+1) = o(1/\sqrt{n});$$

and so we can *deduce* from the condition $s_n^1 = o(\sqrt{n})$ that Σa_n is summable (B) to sum 0.* Hence the theorem suggested would show that $s_n^1 = o(\sqrt{n})$ by itself implies convergence to zero, and this is untrue, as s_n may well be of the form

$$\epsilon_n \sqrt{n} - \epsilon_{n-1}\sqrt{(n-1)}$$

without tending to zero.

A very interesting general conclusion may be drawn from the theorems comprised in our diagram, viz.,

THEOREM 6.—*If Σa_n is summable (B), and*

$$a_n = o(n^k)$$

for some value of k, then Σa_n is summable by Cesàro's means of sufficiently high order.

In the language of § 1, we may express this by saying that *Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so.*

A particular case of Theorem 6 deserves special notice. It is well

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* If
$$\frac{s_0+s_1+\ldots+s_n}{n+1} = s + o\left(\frac{1}{\sqrt{n}}\right),$$
then Σa_n is summable (B) to sum s. See Hardy, *Quarterly Journal*, Vol. 35, p. 40; Bromwich, *Infinite Series*, pp. 319–322. It may be shown more generally (cf. Bromwich, *l.c.*) that
$$k!\, s_n^k/n^k = s + o(1/\sqrt{n})$$
implies the same conclusion: we have thus a general condition which enables us to infer Borel from Cesàro summability. For some examples of series summable by Cesàro's, but not by Borel's, method, see Hardy, *l. c. supra* and *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 290, and §§ 10, 11 below.

known that a power series is summable (B) at any regular point on its circle of convergence. It therefore follows that, *if $f(x)$ is regular for $x=1$, and $a_n = o(n^k)$, then Σa_n is summable (C).* This result has been found by Riesz.*

8. It is natural to enquire whether the preceding results may be extended to non-integral orders of Cesàro summation.† The necessary analysis is not difficult, but, as its conclusions are obvious generalisations of those already established, we shall be content to sketch the argument very shortly.

In the first place, Lemma 1 of § 3 is quite independent of any assumption as to the arithmetic nature of a.

Secondly, Lemma 2 may be replaced by the equation

$$e^{-x}\sum_0^\infty s_n^k \frac{x^{n+k}}{\Gamma(n+k+1)} = \frac{1}{\Gamma(k)}\int_0^x (x-t)^{k-1} e^{-t}\left(\sum_0^\infty s_n \frac{t^n}{n!}\right) dt,$$

where
$$s_n^k = s_n + \begin{bmatrix} k \\ 1 \end{bmatrix} s_{n-1} + \ldots + \begin{bmatrix} k \\ n \end{bmatrix} s_0,$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q+1)}.$$

This equation may be shown to hold for any positive value of k. From it may be deduced the analogue of Lemma 3, viz., that

$$e^{-x}\sum_0^\infty s_n^k \frac{x^n}{(n+k)!} \to \frac{s}{\Gamma(k+1)},$$

if Σa_n is summable (B). We then deduce equation (14) of § 4,‡ precisely as in that section. We thus obtain Theorems 2, 3, 4, 5, freed from the restriction that k is an integer. The effect of this is to replace each set of theorems, corresponding to a set of spaces lying on a line parallel to the principal diagonal of the diagram of § 7, by a *continuous* series of theorems.

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* *L.c. supra* (p. 9). Riesz assumes more than we do, and so obtains a more precise result: in fact, he establishes summability (Ck), whereas all that can be deduced from our hypothesis is summability $(C, 2k+1)$.

† For the theory of such methods of summation, see Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369; Hardy and Chapman, *Quarterly Journal*, Vol. 42, p. 181; and various writings of Bohr, Knopp, and Riesz quoted in the latter paper. A later note by Riesz appeared in the *Comptes Rendus*, June 12th, 1911.

‡ Of course with $\Gamma(k+1)$ for $k!$.

9. So far we have confined ourselves to Borel's definition (2). The question remains whether our results remain valid when this definition is replaced by the integral definition expressed by (2a) and (2b).

If $a_n \to 0$, the two definitions are certainly equivalent. For the necessary and sufficient condition for equivalence is that

$$e^{-x} a(x) \to 0.$$

It is, however, not difficult to see that all our results still hold when Borel's integral definition is adopted.

For, if $$a_0 + a_1 + a_2 + \dots$$

is summable by the integral definition, then

$$0 + a_0 + a_1 + a_2 + \dots$$

is summable by the definition (2).* Moreover, if the first series satisfies one of our conditions

$$a_n = o(n^{-\frac{1}{2}}), \quad a_n = o(1), \quad s_n = o(n^{\frac{1}{2}}), \quad \dots,$$

the second satisfies a corresponding condition, and is accordingly summable by the appropriate one of Cesàro's means. But the two series are completely equivalent in regard to the application of Cesàro's method. Thus all our results apply as well to one of Borel's definitions as to the other.

III.

10. A good deal of light may be thrown on the foregoing theorems by the study of a particular series, viz., the series

(15) $$\Sigma n^{-b} e^{in^a},$$

where a and b are real and $0 < a < 1$.†

This series is convergent if $a+b > 1$, summable $(C1)$ if $2a+b > 1$, summable $(C2)$ if $3a+b > 1$, and so on.‡ If $b = 1-a$, it is finitely oscillating. In this case, if $a < \frac{1}{2}$, we have

$$n^{-b} e^{in^a} = o(n^{-\frac{1}{2}});$$

* Hardy, *Quarterly Journal*, Vol. 35, p. 34.

† We might equally well consider the more general series

$$\Sigma n^{-b} e^{iAn^a}.$$

‡ Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 142.

and so, by Theorem 1, the series cannot be summable (B). We are thus led to expect, in regard to the summability (B) of the series, different results according as $a > \frac{1}{2}$ or $a < \frac{1}{2}$. We shall, in fact, prove that *the series* (15) *is summable* (B) *for all values of* b *if* $a > \frac{1}{2}$, *but is never summable* (B) *when* $a < \frac{1}{2}$, *except when it is convergent.*

The proof of this result is tedious rather than difficult, and we shall content ourselves with sketching its main features.

In the first place, we can easily verify that, if

$$a_n = n^{-b} e^{in^a},$$

then
$$s_n = -(i/a)\, n^{1-a-b} e^{in^a} + \sum_{(\nu)} K n^\nu e^{in^a} + C + o(1),$$

where the summation extends over a finite number of values of ν, all less than $1-a-b$, and C is a constant arising from the application of the Euler-Maclaurin sum-formula.

We can now prove that, *if* $a > \frac{1}{2}$,

$$(16) \qquad e^{-x} \sum_0^\infty n^\nu e^{in^a} \frac{x^n}{n!} \to 0$$

as $x \to \infty$, *for all values of* ν. It will then follow that the series (15) is summable to sum C. Let

$$m = [x].$$

Then it is easy to see that we may replace the left-hand side of (16) by

$$(17) \qquad e^{-x} \sum_{-\mu}^{\mu} (m+r)^\nu e^{i(m+r)^a} \frac{x^{m+r}}{(m+r)!},$$

where
$$\mu \sim m^{\frac{1}{2}+\delta}.$$

We then show, by a straightforward but tedious process of approximation, that, if we write $x = m+f$, and keep f constant as $m \to \infty$, we can write (17) in the form

$$(18) \qquad K m^{\nu-\frac{1}{2}} e^{im^a} \sum_{-\mu}^{\mu} \left[e^{iam^{a-1} r - (r^2/2m)} \{1 + \Sigma K r^A m^B + \Sigma O(r^\alpha m^\beta)\} \right],$$

where the number of terms in each sum is finite, A is integral, and

$$\nu + \tfrac{1}{2}\alpha + \beta < 0$$

for each pair of indices α, β.

Next, it is easy to see that

$$Km^{\nu-\frac{1}{2}}e^{im^a}\sum_{-\mu}^{\mu}\{e^{iam^{a-1}r-(r^2/2m)}O(r^a m^\beta)\} = O(m^{\nu+\beta-\frac{1}{2}})\int_{-\infty}^{\infty}e^{-r^2/2m}|r|^a\,dr$$

$$= O(m^{\nu+\frac{1}{2}a+\beta}) = o(1).$$

Thus these terms may be neglected, and everything is reduced to showing that a number of terms of the type

$$(19)\qquad m^p\sum_{-\mu}^{\mu}r^q e^{iam^{a-1}r-(r^2/2m)},$$

where q is an integer, tend to zero. It is easily proved that the limits of summation may be replaced by ∞ and $-\infty$.

First suppose $q=0$. Then

$$\sum_{-\infty}^{\infty}e^{iam^{a-1}r-r^2/2m} = 1+2\sum_{1}^{\infty}e^{-r^2/2m}\cos r\theta = \vartheta_3(v,\tau),^*$$

where $\qquad \theta = am^{a-1}, \quad v=\theta/2\pi, \quad \tau = i/2m\pi.$

Now† $\qquad \vartheta_3(v,\tau) = \sqrt{\left(\frac{i}{\tau}\right)}\,e^{-\pi i v^2/\tau}\vartheta_3\left(\frac{v}{\tau},\ -\frac{1}{\tau}\right),$

$$\vartheta_3\left(\frac{v}{\tau},\ -\frac{1}{\tau}\right) = 1+2\sum_{1}^{\infty}e^{-2mr^2\pi^2}\cosh 2mr\theta\pi = 1+o(1).$$

Hence $\qquad \vartheta_3(v,\tau)\sim\sqrt{(2m\pi)}\,e^{-\frac{1}{2}m\theta^2} = \sqrt{(2m\pi)}\,e^{-\frac{1}{2}a^2m^{2a-1}},$

which tends to zero more rapidly than any power of m.

When q is not zero the argument is a little more complicated, but in essence the same: in this case we use the q-th derivative of the theta-function with respect to v.

Thus the left-hand side of (16) tends to zero if $x=m+f$, where $0\leqslant f<1$, and $m\to\infty$. Moreover it does so uniformly with respect to f, and so our proof is completed.

11. When $0<a<\frac{1}{2}$, the discussion is somewhat similar, but rather simpler. The essential difference lies in the fact that we can choose δ so that $iam^{a-1}r$ is small throughout a range of values of r of magnitude $m^{\frac{1}{2}+\delta}$, so that, in approximating to $e^{i(m+r)^a}$, we can use e^{im^a} instead of the

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* Tannery and Molk, *Fonctions Elliptiques*, t. II, p. 252.
† *Ibid.*, p. 263.

more accurate approximation

$$e^{im^a+iam^{a-1}r}.$$

The result is that the dominant term of our final result assumes the form

$$-\frac{i}{a\sqrt{(2\pi)}}\,m^{\frac{1}{2}-a-b}\,e^{im^a}\sum_{-\mu}^{\mu} e^{-r^2/2m} \sim -(i/a)\,m^{1-a-b}\,e^{im^a},$$

and the series (15) is summable if and only if $a+b>1$, *i.e.*, if and only if it is convergent.

12. If $a=\frac{1}{2}+\delta$, we require $b>\frac{1}{2}-\delta$ for convergence. Hence we can find a series of the type (15), summable (B), but not convergent, and such that $a_n = O(n^{-\frac{1}{2}+\delta})$. This affords a formal proof that the index $\frac{1}{2}$ of Theorem 1 cannot be replaced by any lower index. We can show similarly, by means of the series considered in §§ 10, 11, that the indices of the powers of n, which occur in Theorems 2–5, are as small as they can possibly be.

A much more difficult question remains. *Is Theorem 1 true if the condition $\sqrt{n}\,.\,a_n \to 0$ is replaced by $|\sqrt{n}\,.\,a_n| < K$, and can similar changes be made in the other theorems?* It has been proved recently* that a similar extension may be given to Tauber's converse of Abel's theorem, and it is natural to suppose that the extension is possible here also.

It is interesting to consider for a moment what light is thrown on this question by the series of §§ 10, 11. The crucial case is that in which the series oscillates finitely; *i.e.*, when

$$a_n = n^{a-1}e^{in^a}.$$

Then $\qquad a_n = o(n^{-\frac{1}{2}}) \quad (a<\frac{1}{2}), \qquad a_n = O(n^{-\frac{1}{2}}) \quad (a=\frac{1}{2}).$

In the first case the series is certainly not summable, by Theorem 1 or by the results of §§ 10, 11. And the question of interest is *whether*

$$\Sigma\,\frac{e^{i\sqrt{n}}}{\sqrt{n}}$$

is summable (B).†

The answer to this question is (as analogy would lead us to expect) in

* Littlewood, *l.c.*, p. 434.

† If not, Theorem 1 shows that $\Sigma n^{-b} e^{i\sqrt{n}}$ is never summable unless convergent.

the negative. In fact,

$$s_n = C - 2ie^{i\sqrt{n}} + o(1),$$

and it may be shown that

$$e^{-x} \sum e^{i\sqrt{n}} \frac{x^n}{n!} \sim e^{-(1/8)+i\sqrt{x}}.*$$

Thus the evidence obtained from this series points to the truth of the suggested generalisations. But, as we stated in § 1, we have not been able to find a satisfactory proof of them.

13. Theorem 1 has an interesting application to the problem of the multiplication of series. It is easy to prove that *if* Σa_n *and* Σb_n *are summable* (B), *and*

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

then the product series Σc_n, *formed in accordance with Cauchy's rule, is also summable* (B). But it is evident that

$$c_n = O(\log n/n) = o(1/\sqrt{n}),$$

and therefore, by Theorem 1, Σc_n is *convergent*. We thus obtain a simple proof of a known theorem.†

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* If $a < \frac{1}{2}$,

$$e^{-x} \Sigma e^{in^a} \frac{x^n}{n!} \sim e^{ix^a},$$

while, if $a > \frac{1}{2}$, the left-hand side tends exponentially to zero (see §§ 10, 11). In the critical case we obtain a result resembling the first, but differing owing to the presence of the factor $e^{-1/8}$. The correspondence between the oscillations of the original series and of Borel's integral is not so precise in this case as it is shown to be, when $a < \frac{1}{2}$, by the formula at the end of § 11. For an illustration of the corresponding phenomenon in connection with Tauber's theorem, see Littlewood, *l.c. supra*, p. 436: in the formulæ there given a constant term $\zeta(1+ai)$ is omitted, but the insertion of this term does not affect the argument.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410. We have obtained a number of further results on this subject, to which we hope to return shortly. In particular we have proved that any convergent series for which $a_n = O(1/n)$—and therefore any such series summable by any of Cesàro's means—is summable $(C, -1+\delta)$ for any positive δ. We are thus able, by the use of Mr. Chapman's negative indices of summability, to deduce the multiplication theorem referred to above from the theorem that a series of this type cannot be summable (C) without being convergent.

CORRECTIONS

p. 6, *line* 4. For $\alpha < n < \beta$ read $\mu < n < \nu$.

p. 8, *line* 7 *up*. Read s_n^k.

p. 9, *line* 12. For 'assumed' read 'used'.

p. 14, *line* 2 (in the integral) and *line* 6. Read $r^2/2m$.

p. 15, 2*nd footnote*. Read $e^{i\sqrt{n}}$.

COMMENTS

The contrasting principles of *power* and *delicacy*, § 1, which govern *limitation* and *Tauberian* properties, were first discussed by Hardy in 1910, 3; see also 1934, 5.

Theorem 1, §§ 3–4, is the o-Tauberian theorem for Borel summability. The corresponding O-Tauberian theorem, conjectured in § 12, is proved in 1916, 8 and again in 1943, 4; see D.S., Theorem 156. There is also a one-sided version, with $a_n > -Kn^{-\frac{1}{2}}$, which is included in a theorem of Schmidt,† with a simplified proof by Vijayaraghavan;‡ see D.S., Theorem 241.

In the first footnote to § 7, a generalization of a theorem of Hardy, proved in 1904, 4, § 6, is formulated. The conclusion is, however, shown to be false in 1916, 8, § 3.2, where it is proved that the Borel mean tends to s $(C, k-1)$; see the Comments on 1916, 8.

The extensions of Theorems 2–5 to non-integral orders, sketched in § 8, are obtained from the extension of formula (14) contained in the proof of Theorem 147 of D.S.

In 1916, 8, Hardy and Littlewood state that there are extensions of Theorems 2–5, with O in place of o in the hypotheses, and give two examples (Theorems 3.11 and 3.12). By using Vijayaraghavan's method, Lord§ has shown that, in each of Theorems 2–5 and their non-integral extensions, the condition on s_n^k may be replaced by a one-sided condition. For example, *if* $k > -1$, $p \geqslant \frac{1}{2}$, $s_n^k > -Kn^{k+p}$ *and* $\sum a_n$ *is summable* (B), *then* $\sum a_n$ *is summable* $(C, k+2p)$.

The statement (§ 10) that the series $\sum n^{-b}e^{in^a}$ is 'summable $(C, 2)$ if $3a+b > 1$, and so on', is not established by the argument in 1911, 1, § 17, but there is a proof in D.S., Theorem 84; see the Comments on 1911, 1.

Another way of obtaining the result in § 11 that, *when* $0 < a < \frac{1}{2}$, *the series* $\sum n^{-b}e^{in^a}$ *is summable* (B) *if and only if it is convergent*, is given in 1913, 2, Section IV, at the end of § 41.

The discussion in § 12 shows that the result obtained in § 11 also holds when $a = \frac{1}{2}$. The fact that $\sum n^{-b}e^{i\sqrt{n}}$ is not summable (B) when $b < \frac{1}{2}$, follows from the *line of Borel summability* property for ordinary Dirichlet series, proved in 1910, 1.

Hardy and Littlewood do not indicate how they prove the theorem stated in § 13. It can be obtained by applying Theorem (7) of 1908, 2, § 13, to the Borel integrals. But Theorem (7) is an analogue of the result deduced here (Theorem C of 1908, 2). The results mentioned in the footnote to § 13 are proved in 1913, 2, Section V, Theorems 37–8.

† *Schriften d. Königsberger gelehrten Ges.* 1 (1925), 205–56.

‡ *Proc. London Math. Soc.* (2), 27 (1928), 316–26.

§ *Proc. London Math. Soc.* (2), 38 (1935), 241–56.

CONTRIBUTIONS TO THE ARITHMETIC THEORY OF SERIES

By G. H. Hardy *and* J. E. Littlewood.*

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CONTENTS.

I.

Introduction and Summary of Results.

1. This paper represents a continuation of researches published in three recent papers in these *Proceedings*.† It resembles them in that the general character of the theorems which it contains is "Tauberian": they are theorems of the type whose first example was the beautiful converse of Abel's theorem originally proved by Tauber.‡

Section II contains a systematic development of an idea already to be

* Some of the results of this paper were communicated to the Society at the meeting of January 11th, 1912, and an Abstract of them published under the title "A New Condition for the Truth of the Converse of Abel's Theorem relating to Power Series." These results are now included in Section IV of the paper.

† Hardy, "Theorems relating to the Summability and Convergence of Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320.

Littlewood, "The Converse of Abel's Theorem," *ibid.*, Ser. 2, Vol. 9, pp. 434–448.

Hardy and Littlewood, "The Relations between Borel's and Cesàro's Methods of Summation," *ibid.*, Ser. 2, Vol. 11, pp. 1–16.

‡ A general explanation of the nature of a "Tauberian" theorem is given in § 1 of the last of the three papers just quoted.

found in the second of the three papers referred to.* It was there proved that, if $f(x)$ is a function of x, with first and second derivatives continuous for $x > x_0$, and if

$$f(x) = s+o(1), \qquad f''(x) = O(1),$$

as $x \to \infty$, then $f'(x) = o(1)$; and thus that, if $f(x)$ has derivatives of all orders, which remain limited as $x \to \infty$, then $f(x)$ cannot tend to a limit unless all its derivatives tend to zero. Here we discuss a number of theorems of this character more thoroughly. In Theorem 1 we show that if

$$f(x) = O(\phi), \qquad f''(x) = O(\psi),$$

where ϕ and ψ are increasing functions, then

$$f'(x) = O\{\sqrt{(\phi\psi)}\};$$

with corresponding results when one or other of the O's is replaced by o. From this theorem we deduce others involving derivatives of higher orders. All these theorems seem to us of considerable interest in themselves, apart from any applications to the theory of series.

So far we have considered increasing functions ϕ, ψ. It is important, for subsequent applications, to obtain theorems of a similar character in which ϕ and ψ are replaced by decreasing functions. These theorems, of which Theorem 5 is the first and most fundamental, are a little more delicate and require some additional restrictions. We have found it most convenient to continue to suppose that $\phi, \psi, \ldots$ are increasing functions, but to consider, instead of relations of the type $f^{(r)} = O(\psi)$, relations of the type $x^r f^{(r)} = O(\psi)$.

2. In Section III we proceed to apply the results of Section II to the theory of ordinary Dirichlet's series—series, that is to say, of the type $\Sigma a_n n^{-s}$. There are two theorems in this section more important than the rest. The first is Theorem 18, which asserts that, if Σa_n is summable (Cr), and $a_n = O(n^a)$, where $a > -1$, then $\Sigma a_n n^{-s}$ is summable (Ck), where $0 \leqslant k \leqslant r$, for

$$s = (a+1)(r-k)/(r+1).$$

When $a = -1$, this theorem reduces to the principal theorem of the first of the three papers quoted at the beginning of § 1. This last theorem we shall describe as the *general Cesàro-Tauber theorem*—the theorem of Tauberian type which is concerned with Cesàro's method of summation

* Littlewood, *l.c.*, p. 437.

and whose hypotheses involve an O. It will be convenient to extend this system of nomenclature, calling theorems *general* if they involve an O and *special* if they involve an o. Thus Tauber's theorem is the special Abel-Tauber theorem, its extension the general Abel-Tauber theorem, Theorem 1 of our paper on Borel's method of summation is the special Borel-Tauber theorem, and Fatou's well known theorem may be described as the Euler-Tauber theorem.*

The second principal theorem of Section II is Theorem 19. In this, we suppose, as in Theorem 18, that Σa_n is summable (Cr), but instead of assuming a hypothesis of the type $a_n = O(n^a)$, we suppose that $\Sigma a_n n^{-s}$ is known to be summable $(C, r-k)$, where $0 \leqslant k \leqslant r$, for $s = \beta > 0$. We then prove that the series is summable $(C, r-l)$, for $0 \leqslant l \leqslant k$, when $s = l\beta/k$. An interesting corollary of the special case of Theorem 19 in which $k = 2$, $l = 1$ is Bohr's theorem that the breadth of the strip between the r-th and $(r+1)$-th lines of summability is a decreasing function of r.

3. Theorem 26, the chief of the early theorems of Section IV, is an extension of the general Abel-Tauber theorem, and related to it much in the same way as some of the preceding theorems are related to the general Cesàro-Tauber theorem. But, in this section, we are, for the most part, concerned with theorems of a different character. These theorems now take a subordinate position in the paper, but it is from them that our investigations started. Their distinguishing characteristic is that their conditions involve the differences of successive terms of the series. Thus Theorem 27, the principal result of this section, states that a sufficient condition for the truth of the converse of Abel's theorem is that

$$\Delta a_n = a_n - a_{n+1} = O(a_n/n).$$

The proof, which is intricate and difficult, shows first that this condition, together with the existence of Abel's limit, involves

$$a_n = o(1/n);$$

the main result then follows from Tauber's theorem. In view of the difficulty of the proof, we have thought it worth while to discuss separately the corresponding theorem for Cesàro summability of the first order, the

* It is reasonably near the truth to represent Euler as defining the sum of Σa_n as the value of $f(x) = \Sigma a_n x^n$ for $x = 1$, when the function is regular there; cf. Bromwich, *Infinite Series*, pp. 265 *et seq*. Fatou's theorem is essentially a theorem for o: it becomes false if o is replaced by O. The general Borel-Tauber theorem, on the other hand, is no doubt true, but as yet unproved.

proof of which is fairly simple. We have also indicated an alternative form of condition, suggested by a theorem of Landau. We also show that, when $\Delta a_n = o(a_n/n)$, the series *cannot* converge, whence it follows that, for such a series, Abel's limit cannot exist. Further, $|na_n| \to \infty$, and $s_n \sim na_n$.

It is to be observed that these theorems become trivial when a_n is real, as then a_n cannot change its sign. They have interesting applications to real series, the real parts of complex series which satisfy their conditions, such as

$$\Sigma n^{-s} \cos(a \log n);$$

but these real series do not themselves satisfy the conditions.

We also establish the analogous theorem (Theorem 31) relating to Borel's method.

These theorems seem at first sight curious, but artificial. We are, however, inclined to think that they are in reality of some interest. This appears more clearly if we try to frame a theorem of this character but connected with Fatou's theorem. The theorem would evidently run: *if* $\Delta a_n = o(a_n)$, *and* $f(x) = \Sigma a_n x^n$ *is regular for* $x = 1$, *then* Σa_n *is convergent*. This is true; but for the reason that, if $\Delta a_n = o(a_n)$, $f(x)$ *cannot* be regular for $x = 1$.* For $\Delta a_n = o(a_n)$ is equivalent to $a_{n+1}/a_n \to 1$, and a well known theorem first rigorously established by Fabry,† tells us that $x = 1$ is certainly a singular point of $f(x)$. That the theorem thus suggested, though it proves in fact to be trivial, should appear as a link between two such important theorems, leads us to think that the artificiality of some of the theorems of this section is more apparent than real, and that conditions of the types here considered do correspond to important classifications of "slowly oscillating" series.

We conclude this section by pointing out an easy generalisation of Fatou's theorem, in which the condition $a_n = o(1)$ is replaced by $\Delta a_n = o(1)$ or $\Delta^r a_n = o(1)$, and by showing that the general Abel-Tauber theorem is a simple corollary of the theorem that $\Delta a_n = O(1/n^2)$, together with the existence of Abel's limit, implies the convergence of Σa_n. This theorem is an obvious consequence of the Abel-Tauber theorem, but it is rather surprising that the latter should be deducible from it in an elementary manner.

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* The theorem is true because a false proposition implies *all* propositions, and therefore any theorem with contradictory hypotheses is true (Whitehead and Russell, *Principia Mathematica*, Prop. 2·21).

† Fabry, *Annales de l'École Normale*, Ser. 3, Vol. 13; Hadamard, *La Série de Taylor*, pp. 23 *et seq.*

4. The contents of Section V are of a more miscellaneous character, and do not depend, as does nearly all the rest of the paper, on the general theorems of Section II. The chief theorem of §§ 44–50 is Theorem 37, which states that a convergent series for which $a_n = O(1/n)$ is summable $(C, -1+\delta)$ for all positive values of δ. This result is interesting in several ways. In the first place it shows that, for such series, *all** methods of Cesàro summation are equivalent, and so supplies an interesting completion of the Cesàro-Tauber and Abel-Tauber theorems. It also enables us to generalise and to connect with these theorems the "multiplicative" property of series for which $a_n = O(1/n)$.

In §§ 51–54 we discuss the analogue of Tauber's theorem for double series. We have had in our possession for several years a special form of this theorem (Theorem 42), in which the fundamental condition is taken to be

$$(m+n)^2 a_{m,n} \to 0.$$

The more general conditions employed in Theorem 40 were suggested† to us by Prof. W. H. Young, who had used them to establish the corresponding "double Cesàro-Tauber" theorem. Prof. Young's very opportune suggestion has led us to a further generalisation and enabled us for the first time to get a clear view of the analogy between the one- and two-dimensional cases. The paper concludes (§§ 55–58) by a discussion of certain further extensions of the ordinary form of Tauber's theorem. The main theorem here is Theorem 49, the purpose of which is to show the striking contrast which arises between Abel's theorem and its converse when paths which touch the circle of convergence are considered.

In spite of the length of the paper and the variety of the topics with which it deals, it leaves open many obvious questions for future discussion. The chief of these are connected with the general theorems of Section II. In the first place, there are clearly theorems of a similar character to be proved involving Riemann-Liouville derivatives of non-integral order, which should enable the results of Section III to be extended to non-integral orders of summability. There must also be theorems involving an arbitrary increasing function $\lambda(x)$, which should enable us to deal with the Rieszian methods of summation appropriate to general Dirichlet's

* Methods of non-integral order less than -1 can be defined, but, as Chapman (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 376) has shown, do not give results satisfactory from the point of view of a general theory. No practicable definition of negative integral orders of summability has been given.

† At the meeting of January 11th, 1912. See also Young, "On Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 167.

series of the type

$$\Sigma a_n e^{-\lambda_n s}.$$

Finally, there is the question of substituting, for conditions of the type $f = O(\phi)$, right-handed or left-handed conditions of the type

$$f = O_R(\phi), \qquad f = O_L(\phi)$$

i.e.,

$$f < K\phi, \qquad f > -K\phi.$$

Landau* has extended the general Cesàro-Tauber theorem to the case in which

$$a_n = O_L(1/n);$$

and it is fairly obvious that many of the theorems of this paper are susceptible of a similar generalisation.†

II.

Theorems concerning the Successive Derivatives of a Continuous Function of a Real Variable.

5. It may facilitate the understanding of the proofs of the theorems of this section if we begin by considering the simplest case in a geometrical and not in an analytical manner. In this case the single idea which underlies all the theorems becomes obvious intuitively. We wish to establish in this rough way the truth of the following theorem:—

If $f''(x)$ is continuous, and $f \to s$, $f'' = O(1)$, as $x \to \infty$, then $f' \to 0$.

Consider (Fig. 1) the curve

$$y = f'(x).$$

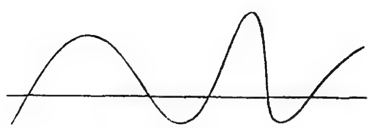

FIG. 1.

If $f'(x)$ does not tend to zero the curve has peaks whose height does not tend to zero. But since $\int^x y\,dx$ tends to a limit as $x \to \infty$, these peaks

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* *Prac Matematyczno-Fizycznych*, Vol. 21, p. 97.

† See the additional note at the end of the paper.

must each have an area which tends to zero. They must therefore tend to become (in some places) indefinitely steep, and this is incompatible with $f''(x) = O(1)$.

The theorems which follow contain a rigorous development of this idea.

6. THEOREM 1.—*If $\phi(x)$ and $\psi(x)$ are positive increasing* functions of x, and $f''(x)$ is continuous,† then*

(a) *if $f = O(\phi)$ and $f'' = O(\psi)$, then $f' = O\{\sqrt{(\phi\psi)}\}$;*

(b) *if $f = o(\phi)$, or if $\phi = 1$ and $f \to s$, and $f'' = O(\psi)$, then*
$$f' = o\{\sqrt{(\phi\psi)}\};$$

(c) *if $f = O(\phi)$ and $f'' = o(\psi)$, then $f' = o\{\sqrt{(\phi\psi)}\}$.*

We shall suppose throughout the proofs in the whole section that $f(x)$ is real. The theorems, however, may be extended at once to the case when $f(x)$ is complex. It is legitimate and convenient to suppose that $f(x)$ is defined for $x \geqslant 0$.

We shall content ourselves by giving the proof of (c), which requires a little more consideration than (a) or (b). The modifications necessary in the other cases will easily be supplied by the reader. It should be observed that the second case in (b) reduces to a special case of the first when we consider the function $f - s = o(1)$ instead of f.

We have $|f(x)| < A\phi(x)$, where A is a constant, for all values of x, and $|f''(x)| < \epsilon\psi(x)$, where ϵ is an arbitrary positive number, for every x to the right of some point C depending on ϵ.

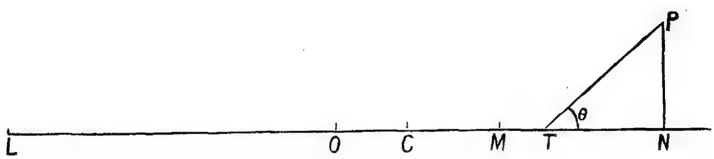

FIG. 2 (i). *Case* (i).

Consider the curve $y = f'(x)$. Let P or (x_0, y_0) be a point of it to the

* The word "increasing" is used throughout the paper to mean "not decreasing."

† The hypothesis of continuity is much more stringent than is necessary for the truth of the theorem. We adopt it partly because we are not aiming at generality of this particular kind, and partly because the more or less geometrical line of argument which the hypothesis makes legitimate seems simpler to follow than a purely analytical one.

right of C, N the foot of its ordinate, M the mid-point of ON, L the point to the left of O such that $OL = ON$. We suppose that y_0 is, say, positive. Let T or $(x_1, 0)$ be the point to the left of N, such that

$$\tan\theta = \tan PTN = \epsilon\psi(x_0).$$

Then between C and N we have

$$\frac{dy}{dx} < \epsilon\psi(x) < \epsilon\psi(x_0) = \tan\theta,$$

and hence the curve lies above the straight line PT between C and N.

We shall distinguish three cases:

(i) T lies between M and N;

(ii) T lies between M and L;

(iii) T lies to the left of L.

Case (i).—M, and therefore T, is to the right of C when x_0 is large, and so

$$2A\phi(x_0) \geqslant A\{\phi(x_0)+\phi(x_1)\}$$
$$\geqslant |f(x_0)-f(x_1)| = \left|\int_{x_1}^{x_0} y\,dx\right|$$
$$> \text{area } PTN = \tfrac{1}{2}y_0^2\cot\theta.$$

Thus

(1) $$y_0^2 < 4A\epsilon\phi(x_0)\psi(x_0).$$

Case (ii).—The curve lies above PM between M and N. Hence, arguing as above, we obtain

$$2A\phi(x_0) > \text{area } PMN = \tfrac{1}{4}\text{area } PLN > \tfrac{1}{4}\text{area } PTN,$$

and so

(2) $$y_0^2 < 16A\epsilon\phi(x_0)\psi(x_0).$$

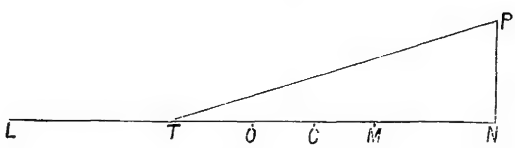

FIG. 2 (ii). *Case* (ii).

Case (iii).—If T lies to the left of L, and so of course to the left of O, it is evident that f' is positive and f monotonic for $0 < x \leqslant x_0$.

Let PT meet the ordinate at C in B. Then

$$PN/BC = TN/TC < LN/LC < LN/LO = 2.$$

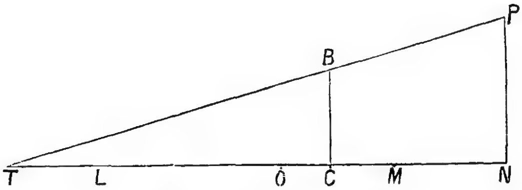

FIG. 2 (iii). *Case* (iii).

Hence y_0 is less than twice the ordinate of the curve at C, a number which depends only on ϵ. If either of the increasing functions ϕ and ψ tends to infinity, their product tends to infinity; and so evidently

$$y_0^2 < \epsilon\phi(x_0)\,\psi(x_0), \tag{3}$$

for sufficiently large values of x_0. If, on the other hand, both ϕ and ψ remain finite, we have

$$2A\phi(x_0) > \text{area } PBCN > \text{area } PMN = \tfrac{1}{4}x_0 y_0.$$

Hence $y_0^2 < K/x_0^2$, and so (3) again holds for sufficiently large values of x_0.

It follows that, for *all* sufficiently large values of x_0, one or other of (1), (2), (3) must hold. Hence

$$y = o\{\sqrt{(\phi\psi)}\},$$

and the theorem is established.

7. It should be observed that the theorem ceases to be always true if the condition that ϕ and ψ are increasing functions is removed. Thus if

$$f = x\log x - x, \quad f' = \log x, \quad f'' = 1/x,$$

and $$\phi = x\log x, \quad \psi = 1/x,$$

we have $$f = O(\phi), \quad f'' = O(\psi),$$

but it is not true that $$f' = O\{\sqrt{(\phi\psi)}\}.$$

It is instructive to consider Theorem 1 in the light of known results in the theory of "logarithmico-exponential" functions (L-functions).* If

* Hardy, "Orders of Infinity," *Camb. Tracts*, No. 12, pp. 39 *et seq.*; *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 54 *et seq.*

f is an L-function it is *generally* true that

$$f' \asymp \sqrt{(\pm ff'')}$$

(the sign being chosen so as to make the contents of the bracket positive). There are exceptional cases when the increase or decrease of f or f' is less than that of any power of x: thus the result ceases to be true if f is any one of the functions

$$\log x, \quad 1/(\log x), \quad x \log x, \quad x/(\log x).$$

Similar results can be obtained when $f = \lambda \cos \mu$ or $\lambda \sin \mu$, where λ and μ are L-functions.

8. THEOREM 2.—*If $\phi(x)$ and $\psi(x)$ are increasing functions of x and $\psi \to \infty$; and if $f = O(\phi)$, and $f^{(r)} = O(\phi\psi)$ for $r \geqslant 1$;* then $f^{(r)} = O(\phi\psi^{\epsilon})$ for any arbitrarily small ϵ.*

By Theorem 1 we have

(1) $$f' = O(\phi\psi^{1/2}).$$

From (1) and $f''' = O(\phi\psi)$, we have similarly

(2) $$f'' = O(\phi\psi^{3/4}).$$

Similarly,

(3) $$f''' = O(\phi\psi^{7/8}),$$

and so on.

If now we apply Theorem 1 successively to $f, f', \ldots$, using (2), (3), ..., we obtain a new set of equations similar to (1), (2), (3), in which the indices are reduced. This process may evidently be repeated indefinitely: suppose that after n repetitions we obtain

$$f' = O\{\phi\psi^{\alpha_n^{(1)}}\},$$
$$f'' = O\{\phi\psi^{\alpha_n^{(2)}}\},$$
$$\ldots \quad \ldots \quad \ldots\ldots$$

The law of formation of the indices is

(4) $$\alpha_n^{(r)} = \tfrac{1}{2}\{\alpha_{n-1}^{(r-1)} + \alpha_{n-1}^{(r+1)}\}, \quad \alpha_0^{(r)} = 1.$$

* The O's and o's occurring in the theorems of this section are not, in general, uniform with respect to r, that is, the K in the inequality $|f^{(r)}| < K\chi$, the existence of which is asserted by the equation $f^{(r)} = O(\chi)$, is not necessarily independent of r.

Since $f = O(\phi)$, we may suppose that $a_n^{(0)} = 0$. It is then clear from a little consideration of (4) that if we introduce the convention that $a_n^{(r)}$ is zero when $r \leqslant 0$, we have, for all values of r,

(5) $$a_n^{(r)} \leqslant \tfrac{1}{2}\{a_{n-1}^{(r-1)} + a_{n-1}^{(r+1)}\}.$$

We shall now show that

(6) $$a_n^{(r)} \leqslant a_{n-1}^{(r)}.$$

This is certainly the case if

(7) $$a_{n-1}^{(r)} \geqslant \tfrac{1}{2}\{a_{n-1}^{(r-1)} + a_{n-1}^{(r+1)}\},$$

i.e., if $$\Delta_r^2 a_{n-1}^{(r-1)} \leqslant 0,$$

where $$\Delta_r u_r \equiv u_r - u_{r+1}.$$

That this is the case is easily established by induction. If $n=1$, we have

$$a_0^{(0)} = 0, \quad a_0^{(1)} = 1, \quad \ldots, \quad a_0^{(r-1)} = 1, \quad \ldots;$$

and if $n = 2$, we have

$$a_1^{(0)} = 0, \quad a_1^{(1)} = \tfrac{1}{2}, \quad \ldots, \quad a_1^{(r-1)} = 1 - (\tfrac{1}{2})^{r-1}, \quad \ldots;$$

and so $$\Delta_r^2 a_0^{(r-1)} = 0, \quad \Delta_r^2 a_1^{(r-1)} = -(\tfrac{1}{2})^{r+1} < 0.$$

Also
$$\begin{aligned} \Delta_r^2 a_{n-1}^{(r-1)} &= a_{n-1}^{(r-1)} - 2a_{n-1}^{(r)} + a_{n-1}^{(r+1)} \\ &\leqslant \tfrac{1}{2}[a_{n-2}^{(r-2)} + a_{n-2}^{(r)} - 2\{a_{n-2}^{(r-1)} + a_{n-2}^{(r+1)}\} + a_{n-2}^{(r)} + a_{n-2}^{(r+2)}] \\ &= \tfrac{1}{2}\{\Delta_r^2 a_{n-2}^{(r-2)} + \Delta_r^2 a_{n-2}^{(r)}\}. \end{aligned}$$

Hence (7) being true for $n = 1, 2$, is true for all values of n. Hence (6) is also true, and so, for any fixed value of r, $a_n^{(r)}$ tends to a limit $a_\infty^{(r)}$ as $n \to \infty$. Hence, from (5),

$$a_\infty^{(r)} \leqslant \tfrac{1}{2}\{a_\infty^{(r-1)} + a_\infty^{(r+1)}\},$$

or $$a_\infty^{(r+1)} - a_\infty^{(r)} \geqslant a_\infty^{(r)} - a_\infty^{(r-1)}.$$

Therefore $$a_\infty^{(r+1)} - a_\infty^{(r)} \geqslant a_\infty^{(s)} - a_\infty^{(s-1)} \quad (s \leqslant r).$$

Hence

(8) $$a_\infty^{(r+1)} \geqslant a_\infty^{(s)} + (r-s+1)[a_\infty^{(s)} - a_\infty^{(s-1)}].$$

Now $0 \leqslant a_n^{(r)} \leqslant 1$, and so $0 \leqslant a_\infty^{(r)} \leqslant 1$. Hence, making $r \to \infty$ in (8), we see that we must have $a_\infty^{(s)} = a_\infty^{(s-1)}$; and therefore

$$a_\infty^{(s)} = a_\infty^{(0)} = \lim_{n\to\infty} a_n^{(0)} = \lim 0 = 0.$$

This is true for all values of s; for any fixed s, then, we can choose n so that $\alpha_n^{(s)} < \epsilon$, and our theorem is established.

The foregoing proof is the precise analytical expression of the following rough argument. If the order of $f^{(r)}$ increases as r increases, it does so by equal or increasing steps. The magnitude of the step must be less than any power of ψ; otherwise a finite number of steps would give an increase in order greater than ψ, which is contrary to the hypotheses of the theorem.

9. Theorem 1 may be generalised as follows :—

THEOREM 3.—*If ϕ and ψ are increasing functions, and $f^{(r)}$ is continuous, then if $0 < s < r$,*

(a) *if $f = O(\phi)$ and $f^{(r)} = O(\psi)$, then $f^{(s)} = O\{\phi^{(r-s)/r}\psi^{s/r}\}$;*

(b) *if $f = o(\phi)$, or if $\phi = 1$ and $f \to s$, and $f^{(r)} = O(\psi)$, then $f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}$;*

(c) *if $f = O(\phi)$ and $f^{(r)} = o(\psi)$, then $f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}$.*

Let
$$\chi_s(x) = \max_{y \leqslant x} |f^{(s)}(y)| / [\{\phi(y)\}^{(r-s)/r}\{\psi(y)\}^{s/r}],$$
and let
$$\chi(x) = \max_{0<s<r} \chi_s(x).$$

We shall first consider (a). The conclusion of (a) is certainly true if
$$\chi(x) = O(1).$$
Let us suppose that this last equation is false, so that χ, which is an increasing function, tends to infinity. We have
$$|f^{(s)}/\{\phi^{(r-s)/r}\psi^{s/r}\}| \leqslant \max_{y \leqslant x} |f^{(s)}(y)| / [\{\phi(y)\}^{(r-s)/r}\{\psi(y)\}^{s/r}] \leqslant \chi_s \leqslant \chi,$$
or
$$f^{(s)} = O\{\phi^{(r-s)/r}\psi^{s/r}\chi\}.$$

If now we apply Theorem 1*a* to f, using the relation $f = O(\phi)$, and that obtained by taking $s = 2$ in the formula just found, we obtain

(1) $$f' = O\{\phi^{(r-1)/r}\psi^{1/r}\chi^{1/2}\}.$$

Repeating the argument with f', and using (1),
$$f'' = O\{\phi^{(r-2)/r}\psi^{2/r}\chi^{3/4}\}.$$

Similarly, $$f''' = O\{\phi^{(r-3)/r}\psi^{3/r}\chi^{7/8}\},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

(2) $$f^{(r-2)} = O\{\phi^{2/r}\psi^{(r-2)/r}\chi^{(2^{r-1}-1)/2^{r-1}}\}.$$

Now we have

(3) $$f^{(r)} = O(\psi) = O(\psi\chi).^{*}$$

Hence, applying Theorem $1a$ to $f^{(r-2)}$ and using (2) and (3),

(4) $$f^{(r-1)} = O\{\phi^{1/r}\psi^{(r-1)/r}\chi^{(2^r-1)/2^r}\}.$$

Consequently we have from (1), &c., (2), (3), and (4),

$$f^{(s)}/\{\phi^{(r-s)/r}\psi^{s/r}\} = O(\chi^{a}), \quad a = (2^r-1)/2^r,$$

for $0 < s < r$. Hence $\chi_s = O(\chi^a)$ for $0 < s < r$, and so

$$\chi = \max_{0<s<r} \chi_s = O(\chi^a).$$

Since $a < 1$, this is incompatible with $\chi \to \infty$. Hence

$$\chi = O(1),$$

and the proof of (a) is completed. To prove (b) and (c) we observe that in each case we can assume the result of (a). Hence, in the case of (c), we have $$f^{(r-2)} = O\{\phi^{2/r}\psi^{(r-2)/r}\}, \quad f^{(r)} = o(\psi).$$

Applying Theorem $1c$, we have

$$f^{(r-1)} = o\{\phi^{1/r}\psi^{(r-1)/r}\}.$$

Using this equation instead of $f^{(r)} = o(\psi)$, we now obtain the result of (c) for $s = r-2$, then for $s = r-3$, and so on. Similarly, starting at the other end of the series $f, f', \ldots, f^{(r)}$, we obtain (b).

A particularly interesting case of the theorem occurs when $\phi = 1$, $\psi = 1$, $f \to s$. We then have the theorem :—

THEOREM $3a$.—*If f has derivates of all orders and if $f \to s$, then, if any derivate is bounded, all preceding derivates tend to zero, or, to state the matter in another way, if any derivate does not tend to zero, no subsequent derivate is bounded.*

This theorem contains in its turn as a particular case that quoted in § 1 of the introduction.†

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* This step is required to make the argument applicable in the proof of Theorem 6.

† Littlewood, *l.c.*, p. 437.

10. The corresponding extension of Theorem 2 is an immediate corollary of the last theorem.

THEOREM 4.—*If ϕ and ψ are increasing functions, and $\psi \to \infty$, and if*

$$f = O(\phi), \quad f^{(r)} = O(\phi\psi) \quad (r \geqslant r_0),$$

then

$$f^{(r)} = O(\phi\psi^{\epsilon}),$$

for all values of r.

For, by Theorem 3, we have for $r < r_0$,

$$f^{(r)} = O(\phi\psi^{r/r_0}) = O(\phi\psi).$$

Hence

$$f^{(r)} = O(\phi\psi)$$

for all values of r, and the theorem follows by Theorem 2.

11. The foregoing theorems give no information concerning functions for which we are given relations of the form $f^{(r)} = O(\psi)$, where $\psi \to 0$, beyond, of course, what can be deduced from $f^{(r)} = o(1)$. Subject, however, to certain conditions, Theorems 3 and 4 remain true when $\psi \to 0$. The most important form of ψ is x^{-a}, where $a \leqslant r$. The case in which $a > r$ is of less interest. Suppose, for example, $a > r$, and $\phi = 1$. Then the result of Theorem 3 is trivial, since $f^{(s)}$, the $(r-s)$-th integral of $f^{(r)}$, is plainly of the form $O(x^{r-s-a})$, and $r-s-a$ is less than $-sa/r$, which is the index assigned by the theorem.

On the other hand, if $a \leqslant r$, $x^r\psi$ is an increasing function. This suggests that we shall find it convenient, instead of using relations of the type

$$f^{(r)} = O(\psi),$$

where ψ is decreasing, to use relations of the type

$$x^r f^{(r)} = O(\psi),$$

where ψ is increasing. The preceding remarks show that, in the cases of greatest interest, the two points of view are equivalent.*

12. We shall begin by establishing the result corresponding to Theorem 1, viz.,

THEOREM 5.—*If ϕ and ψ are increasing functions, and $\phi = O(\psi)$,*

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* There are cases which are thus excluded, *e.g.*, that in which

$$f^{(r)} = O(1/x^r \log x).$$

But such cases do not seem to have applications of any particular interest.

and $f^{(r+2)}$ *is continuous, then*

(*a*) *if* $x^r f^{(r)} = O(\phi)$, $x^{r+2} f^{(r+2)} = O(\psi)$, *then* $x^{r+1} f^{(r+1)} = O\{\sqrt{(\phi\psi)}\}$;

(*b*) *if* $x^r f^{(r)} = o(\phi)$, $x^{r+2} f^{(r+2)} = O(\psi)$, *then* $x^{r+1} f^{(r+1)} = o\{\sqrt{(\phi\psi)}\}$.

Further, if $\phi = o(\psi)$, *then*

(*c*) *if* $x^r f^{(r)} = O(\phi)$, $x^{r+2} f^{(r+2)} = o(\psi)$, *then* $x^{r+1} f^{(r+1)} = o\{\sqrt{(\phi\psi)}\}$.

As in Theorem 1, we shall consider only the case (*c*) in detail. The argument is very similar to that of § 6.

We have
$$|f^{(r+2)}(x)| < \epsilon\psi(x)/(2x)^{r+2}$$
for all values of x greater than some value ξ. Consider the curve $y = f^{(r+1)}(x)$. Let P or (x_0, y_0) be a point on it: we suppose $y_0 > 0$. Let N be the foot of the ordinate of P, M the mid-point of ON, C the point $(\xi, 0)$, and T the point to the left of N, such that
$$\tan\theta = \tan PTN = \epsilon\psi(x_0)/x_0^{r+2}.$$
We can suppose $x_0 > 2\xi$, so that M lies to the right of C.

When x_0 is sufficiently large, we have, for all values of x between M and N,

(1) $$|f^{(r)}(x)| < A\phi(x)/(2x)^r \leqslant A\phi(x_0)/x_0^r,$$

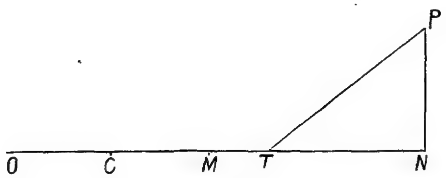

FIG. 3.

where A is a constant, and also

(2) $$\frac{dy}{dx} \leqslant |f^{(r+2)}(x)| < \epsilon\psi(x)/(2x)^{r+2} < \epsilon\psi(x_0)/x_0^{r+2} = \tan\theta.$$

Let x_1 be the abscissa of T. There are two cases to consider, according as T is to the right or to the left of M.

Case (i).—T is to the right of M. Then, by (2), the curve is above TP between T and N. Hence
$$2A\phi(x_0)/x_0^r \geqslant |f^{(r)}(x_0) - f^{(r)}(x_1)| = \left|\int_{x_1}^{x_0} y\,dx\right| > \text{area } PTN = \tfrac{1}{2}y_0^2\cot\theta.$$

Hence

(3) $$y_0^2 < 4A\epsilon\phi(x_0)\,\psi(x_0)/x_0^{2r+2}.$$

Case (ii).—T is to the left of M. Then the curve is above PM between M and N, and so

$$2A\phi(x_0)/x_0^r > \text{area } PMN = \tfrac{1}{4}x_0 y_0,$$

$$y_0 < 8A\phi(x_0)/x_0^{r+1},$$

$$y_0^2 < \frac{\phi(x_0)\,\psi(x_0)}{x_0^{2r+2}}\,\frac{64A^2\phi(x_0)}{\psi(x_0)}.$$

Since $\phi = o(\psi)$ it follows that, for sufficiently large values of x_0, we have

(4) $$y_0^2 < \epsilon\phi(x_0)\,\psi(x_0)/x_0^{2r+2},$$

and from (3) and (4) the result of the theorem follows.*

13. The analogues of Theorems 3 and 4 are as follows. That of Theorem 2 is, of course, included in that of Theorem 4.

THEOREM 6.—*If ϕ and ψ are increasing functions, and $\phi = O(\psi)$, and $f^{(r)}$ is continuous, and $0 < s < r$, then*

(*a*) *if $f = O(\phi)$ and $x^r f^{(r)} = O(\psi)$, then*

$$x^s f^{(s)} = O\{\phi^{(r-s)/r}\psi^{s/r}\};$$

(*b*) *if $f = o(\phi)$, or if $\phi = 1$, $f \to s$, and $x^r f^{(r)} = O(\psi)$, then*

$$x^s f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}.$$

Further, if ϕ satisfies the more stringent condition $\phi = o(\psi)$, then

(*c*) *if $f = O(\phi)$, $x^r f^{(r)} = o(\psi)$, then $x^s f^{(s)} = o\{\phi^{(r-s)/r}\psi^{s/r}\}$.*

THEOREM 7.—*If ϕ and ψ are increasing functions, and $\psi \to \infty$, and if*

$$f = O(\phi), \quad x^r f^{(r)} = O(\phi\psi),$$

for $r > r_0$, then $x^r f^{(r)} = O(\phi\psi^\epsilon)$ for $r > 0$ and for any position ϵ.

The analogue of Theorem 2 (*i.e.* Theorem 7 with $r_0 = 0$) follows from Theorem 5 precisely as Theorem 2 follows from Theorem 1; and Theorem 7

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* The proof is simpler than that of Theorem 1, because we have an additional condition, $\phi = o(\psi)$ in the above argument, $\phi = O(\psi)$ in the corresponding proofs of (*a*) and (*b*). We are thus able to avoid the complications of Case (iii) of Theorem 1.

follows from Theorem 6 as Theorem 4 follows from Theorem 3. Theorem 6 itself may be proved by writing " $x^r f^{(r)}$ " in place of " $f^{(r)}$," and " Theorem 5 " in place of " Theorem 1 " throughout § 9. The work of § 9 is so arranged that (as is easily verified) the additional condition

$$\phi = O(\psi) \quad or \quad \phi = o(\psi),$$

with which Theorem 5 is burdened, is satisfied whenever we have to appeal to the latter theorem.

14. There are theorems, similar to those we have been discussing, concerning the relative orders of the derivates of a function of x as x tends to a finite limit, which we may suppose to be zero. We shall, however, content ourselves by giving one pair of such theorems. The enunciations of these are the same, word for word, as those of Theorems 5 and 6, but it is to be understood that x tends to zero instead of to infinity, that ϕ and ψ increase steadily as x tends to zero, and that the hypothesis of continuity, where it occurs, refers to points other than the origin.

THEOREM 8.—*The result of Theorem* 6 *holds also when* x *tends to zero instead of to infinity.*

THEOREM 9.—*The result of Theorem* 7 *holds also with the same modification.*

The proofs of these theorems are almost exactly the same as those of Theorems 6 and 7. It is a sufficient indication of the necessary modifications to remark that the *rôle* of the point M, or $(\frac{1}{2}x_0, 0)$ in § 12, is now played by the point $(2x_0, 0)$.

15. We shall illustrate the range and limitations of Theorems 1–9 by a few simple applications.

(*a*) Let $f = x^n$. If $n \geqslant 2$, we may take $\phi = x^n$, $\psi = x^{n-2}$ in Theorem 1 and obtain

$$f' = O\{\sqrt{(x^{2n-2})}\} = O(x^{n-1}).$$

If $n \geqslant r$ we may take $\phi = x^n$, $\psi = x^{n-r}$ in Theorem 3, and obtain $f^{(s)} = O(x^k)$, where

$$k = \frac{r-s}{r}n + \frac{s}{r}(n-r) = n-s.$$

These results are evidently correct.

(*b*) Any number of illustrations may (as was pointed out, in the case of Theorem 1, in § 7) be obtained by considering functions of the form $\lambda \cos \mu$, where λ and μ are L-functions. Thus, if

$$f = x^a \cos(x^b) \quad (a > 0,\ b > 1),$$

the order of $f^{(r)}$ is $x^{a+r(b-1)}$: each differentiation increases the order by a factor x^{b-1}. This is

an illustration of the first group of theorems. A good illustration of the second is obtained by considering the function

$$f = x^a \cos x^b \quad (a > 0,\ 0 < b < 1).$$

Here the order of $x^r f^{(r)}(x)$ is x^{a+rb}.

(c) It was pointed out in § 7 that the result of Theorem 1 fails when $f = x \log x$. This illustrates the importance, in the first group of theorems, of the condition that ϕ and ψ are increasing functions. The same example may be used to show the necessity some such condition as $\phi = O(\psi)$ in the second group of theorems. Here we have

$$f \sim x \log x, \quad xf' \sim x \log x, \quad x^2 f'' \sim x,$$

and the result of Theorem 5 does not apply.

(d) The most subtle point which requires illustration is the necessity for the condition $\phi = o(\psi)$, in the third case of Theorem 5: we might well expect $\phi = O(\psi)$ to be a sufficient condition. But consider the function

$$f = x/(\log x).$$

Here

$$xf' \sim x/(\log x), \quad x^2 f'' \sim -x/(\log x)^2.$$

If $\phi = x/(\log x)$, $\psi = x/(\log x)$, we have $f = O(\phi)$, $x^2 f'' = o(\psi)$; but it is not true that

$$xf' = o\{\sqrt{(\phi\psi)}\}.$$

III.

Applications to the Theory of Dirichlet's Series.

16. THEOREM 10.—*Let r and k be integers and $0 < k < r$. Then (a) if $a \geqslant 1$ and*

(i) $a_n = O(n^a)$,

(ii) Σa_n *is summable (Cr) to sum s,*

then

$$s_n^k / n^k = o\{n^{(1+a)(r-k)/(r+1)}\},$$

if $a > -1$, and

$$s_n^k / n^k = s/k! + o(1),$$

if $a = -1$. Further (b) if $a > 1$ and

(i) $a_n = o(n^a)$,

(ii) Σa_n *is finite (Cr),*

then

$$s_n^k / n^k = o\{n^{(1+a)(r-k)/(r+1)}\}.$$

The first part of the theorem is a generalisation of the general Cesàro-Tauber theorem,* to which it reduces when $a = -1$.

If $a > r$, the theorem is trivial in either case in virtue of (ii), which gives

$$s_n^k = O(n^r) \quad (0 < k < r).$$

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* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 307.

For a similar reason the case $a = r$ is trivial in case (a). We may therefore suppose that $a < r$ in case (a), and $a \leqslant r$ in case (b).

Let $f(x)$ be a function whose $(r+1)$-th derivative is defined by the equation

$$f^{(r+1)}(x) = a_n + (x-n)(a_{n+1}-a_n) \quad (n \leqslant x \leqslant n+1),$$

so that $f^{(r+1)}(x)$ is continuous and assumes the value a_n for $x = n$. Then

$$\begin{aligned} f^{(r)}(x) &= C + \sum_0^{n-1} \int_\nu^{\nu+1} \{a_\nu + (t-\nu)(a_{\nu+1}-a_\nu)\}\, dt \\ &\qquad + \int_n^x \{a_n + (t-n)(a_{n+1}-a_n)\}\, dt \\ &= C + s_{n-1} + \tfrac{1}{2} \sum_0^{n-1} (a_{\nu+1}-a_\nu) \\ &\qquad + (x-n)\, a_n + \tfrac{1}{2}(x-n)^2 (a_{n+1}-a_n) \\ &= A + s_{n-1} + (x-n+\tfrac{1}{2})\, a_n + \tfrac{1}{2}(x-n)^2 (a_{n+1}-a_n) \\ &= A + s_n + O(a_n) + O(a_{n+1}-a_n). \end{aligned}$$

Here A and C are constants (independent of n and x), and $O(a_n)$ denotes a function of n and x the ratio of whose absolute value to that of a_n is less than a constant.*

It is easy to see that, if we repeat the process of integration, we obtain

$$f^{(r-1)}(x) = Ax + O(1) + s_n^1 + O(s_n) + O(a_n) + O(a_{n+1}-a_n),$$

$$f^{(r-2)}(x) = \frac{Ax^2}{2!} + O(x) + s_n^2 + O(s_n^1) + O(s_n) + O(a_n) + O(a_{n+1}-a_n),$$

and, generally,

(1) $$f^{(r-k)}(x) = \frac{Ax^k}{k!} + O(x^{k-1}) + s_n^k + O(s_n^{k-1}) + \ldots + O(a_n) + O(a_{n+1}-a_n),$$

for $0 \leqslant k \leqslant r$.

So far the proofs of cases (a) and (b) of the theorem are identical;

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* Our use of O here involves an extension of our ordinary notation. Generally, in such a formula as $f = O(\phi)$, ϕ is an essentially positive function; but there is clearly no reason why we should not write, *e.g.*,

$$\tfrac{1}{2}\sin x = O(\sin x), \qquad \sin^2 x = O(\sin x).$$

We may observe that the constants implied by the O's which occur in this proof may depend upon r and k, but are essentially independent of n and x.

From this point they differ slightly. We shall consider only case (*a*); the proof of case (*b*) may be left to the ingenuity of the reader.

The constant A is at our disposal: we take $A = -s$. Let us consider in particular the case $k = r$. Then, since Σa_n is summable (Cr) to sum s, we have

$$\frac{Ax^r}{r!} + s_n^r = -\frac{s}{r!}\{n^r + o(n^r)\} + \frac{sn^r}{r!} + o(n^r) = o(n^r) = o(x^r).$$

Also $s_n^{r-1}, s_n^{r-2}, \ldots, s_n, a_n$, and $a_{n+1} - a_n$ are all of the form $o(x^r)$. Thus, finally,

$$(2) \qquad f(x) = o(x^r).$$

But
$$f^{(r+1)}(x) = O(a_n) + O(a_{n+1} - a_n) = O(x^a),$$

$$(3) \qquad x^{r+1} f^{(r+1)}(x) = O(x^{r+1+a}).$$

Hence, by Theorem 6,* we have

$$(4) \qquad x^{r-k} f^{(r-k)}(x) = o(x^{r+\rho_k}),$$

where
$$r + \rho_k = \{r(k+1) + (r-k)(r+1+a)\}/(r+1),$$

or
$$\rho_k = \{(1+a)(r-k)/(r+1)\}.$$

We can now combine equations (1) and (4) in such a way as to obtain the result of the theorem. Dividing (1) by x^k, and remembering that $A = -s$, we find

$$(5) \qquad s_n^k/n^k - s/k! = o(x^{\rho_k}) + x^{-k}\{O(s_n^{k-1}) + \ldots + O(a_{n+1} - a_n)\}.$$

Suppose first $k = 0$. Then

$$\begin{aligned} s_n - s &= o(x^{\rho_0}) + O(a_n) + O(a_{n+1} - a_n) \\ &= o(x^{\rho_0}) + O(x^a) = o(x^{\rho_0}), \end{aligned}$$

since
$$a < \rho_0 = (1+a)\, r/(r+1),$$

if
$$a/(1+a) < r/(r+1);$$

a condition which is satisfied, since $a < r$.

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* It is important to observe that, as $r+1+a \geqslant r$, the condition $\phi = O(\psi)$ is satisfied. It should also be observed that, at the corresponding point of the proof of case (*b*), we require the more stringent condition $\phi = o(\psi)$. This is satisfied if $a > -1$, but not if $a = -1$. As a matter of fact the result of case (*b*) is *not* true when $a = -1$. It is not true, for example, that a series Σa_n, for which $a_n = o(1/n)$, cannot be finite $(C1)$ without being convergent. The series

$$\Sigma \frac{1}{n(\log n)^{1+ai}} \quad (a \neq 0),$$

for example, is finite $(C1)$, but not convergent, or even summable $(C1)$.

Next, suppose the result of the theorem established for $0, 1, \ldots, k-1$. Then, for $0 \leqslant p < k$, we have

$$x^{-k} O(s_n^p) = o(x^{\rho_p - k}) = o(x^{\rho_k}),$$

since $$\rho_p - \rho_k = (1+a)(k-p)/(r+1) < k-p \leqslant k.$$

Also $$x^{-k}\{O(a_n)+O(a_{n+1}-a_n)\} = O(x^{a-k}) = o(x^{\rho_k}),$$

since $$(a-k)/(a+1) < (r-k)/(r+1),$$

and so $a-k < \rho_k$. Hence, finally,

$$s_n^k/n^k - s/k! = o(x^{\rho_k}), \tag{6}$$

and the theorem is established.

A simple and interesting case of Theorem 10 is that in which $a = 0$, $r = 1$. The form of the theorem is then as follows.

THEOREM 11.—*If* $a_n = O(1)$, *and* Σa_n *is summable* $(C1)$, *then*

$$s_n = o(\sqrt{n}).*$$

17. The result of Theorem 10 gains in interest when translated into the language of the theory of Dirichlet's series.

We shall, however, find it convenient, before we proceed to this application, to state, and in part prove, a number of theorems concerning Dirichlet's series which are logically independent of the previous results of the paper. These theorems are of considerable intrinsic interest, and will enable us to present our applications of the theorems of Section II in a more subtle and precise form than would otherwise be possible.

We shall first prove the following preliminary theorem. There is nothing new in the theorem, which is contained in Bohr's investigations on the summability of Dirichlet's series.† We have thought it well to include a proof (in a somewhat condensed form), partly on account of the language and comparative inaccessibility of Bohr's dissertation, and partly because we shall have occasion to refer later on to formulæ which occur in it.

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* If a_n is the Fourier's cosine coefficient of a function $f(\theta)$ continuous for $\theta = 0$, then the conditions are certainly satisfied. In this case, however, a good deal more can be said of s_n; for, by a theorem of Riesz and Chapman (Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 391; Riesz, *Comptes Rendus*, November 22, 1909), Σa_n is summable $(C\delta)$ for any positive δ, and so $s_n = o(n^\delta)$: and indeed it may easily be proved directly that $s_n = O(\log n)$.

This example might suggest that Theorem 10 does not give the best result of its kind that can be obtained concerning s_n. This, however, is untrue. We shall see later (cf. § 25) that for no $\beta < \frac{1}{2}$ is it true that $a_n = O(1)$, together with summability $(C1)$, implies $s_n = o(n^\beta)$.

† *Bidrag til de Dirichlet'ske Raekkers Theori*, Copenhagen, 1910.

THEOREM 12.—*If* $s_n^k = O(n^{k+\beta})$, *where* $\beta > 0$, *then* $\Sigma n^{-\gamma} a_n$ *is summable* (Ck) *for any* $\gamma > \beta$.

Let $$n^{-\gamma} a_n = b_n \ (n > 0), \quad a_0 = b_0 = 0,$$

and denote the Cesàro's sums, formed from b_n instead of a_n, by t_n, t_n^1, ... Then

(1) $$t_n^k = \sum_{j=0}^{n} s_j^k \sum_{i=0}^{k} \binom{k+1}{i} \binom{n-j-i+k}{k-i} \Delta^{k+1-i} (j+i)^{-\gamma}.*$$

In this sum a finite number† of terms occur for which $j+i > n$. These terms are to be replaced by zero.‡ But it is easy to see that this convention may be abandoned without affecting the limit of t_n^k / A_n^k.§ For the terms in question are finite in number, and for all of them $n-j < k$. Further, we may write in all of them

$$s_j^k = O(n^{k+\beta}), \quad \Delta^{k+1-i} (j+i)^{-\gamma} = O(n^{-\gamma}),$$

and so their sum is of the form $O(n^{k+\beta-\gamma})$ or $o(n^k)$.

We now consider the terms of (1) for which $i = 0$, which we can write in the form

$$_0t_n^k = \sum_{0}^{n} \binom{n-j+k}{k} s_j^k \Delta^{k+1} (j^{-\gamma}).$$

We have‖ $$-Kn^k < \binom{n-j+k}{k} - A_n^k < 0 \quad (0 \leqslant j \leqslant n),$$

$$-K\nu n^{k-1} < \binom{n-j+k}{k} - A_n^k < 0 \quad (0 \leqslant j \leqslant \nu < n).$$

Hence $$_0t_n^k / A_n^k = \sum_{j=0}^{n} s_j^k \Delta^{k+1} j^{-\gamma} + R_0,$$

where $$R_0 = O\left(\frac{\nu}{n} \sum_{1}^{\nu} j^{k+\beta} \Delta^{k+1} j^{-\gamma}\right) + O\left(\sum_{\nu+1}^{n} j^{k+\beta} \Delta^{k+1} j^{-\gamma}\right).$$

The series $\Sigma j^{k+\beta} \Delta^{k+1} j^{-\gamma}$ converges (like $\Sigma j^{-1+\beta-\gamma}$). Choosing first ν and then n sufficiently large, we see that $R_0 \to 0$, and

(2) $$_0t_n^k / A_n^k \to \sum_{j=0}^{\infty} s_j^k \Delta^{k+1} j^{-\gamma}.$$

Consider next the terms in (1) for which i has a fixed value other than zero. Writing

$$_it_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} s_j^k \Delta^{k+1-i} (j+i)^{-\gamma},$$

and observing that $$\binom{n-j-i+k}{k-i} = O(n^{k-i}),$$

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 260, and Vol. 8, p. 279. In a footnote to the second paper a number of mistakes in the first are corrected. It should be observed that the notation of these papers differs slightly from that which we have since adopted, the series being there taken to be $a_0 + a_1 + \ldots$ instead of $a_1 + a_2 + \ldots$. As the argument which follows is very similar to that of the second paper quoted, we have thought it best to adhere to its notation for the purposes of this proof. The equation (1) contains one term for which $i = j = 0$, so that $(j+i)^{-\gamma}$ is meaningless, but as $s_0^k = 0$, this is immaterial.

† That is to say, a number independent of n.

‡ *E.g.*, if $i = k$, we must replace $\Delta (j+k)^{-\gamma}$ by $n^{-\gamma}$, if $j = n-k$, and by 0 if $j > n-k$.

§ A_n^k has the same meaning as in the papers cited: thus $A_n^k \sim n^k / k!$.

‖ *L.c.*, p. 279.

we find

$${}_it_n^k/A_n^k = O\left\{n^{-i}\sum_0^n j^{k+\beta}\Delta^{k+1-i}(j+i)^{-\gamma}\right\} = O\left\{\sum_0^n\left(\frac{j}{n}\right)^i j^{k-i+\beta}\Delta^{k+1-i}(j+i)^{-\gamma}\right\}.$$

We can now show that

(3) $${}_it_n^k/A_n^k \to 0,$$

by what is practically a repetition of the final argument which led to (2). From (1) and (2) the theorem follows.

It is important to observe that $s_n^k = o(n^{k+\beta})$ does not (as might perhaps be expected) involve the summability (Ck) of $\Sigma n^{-\beta}a_n$. If, *e.g.*, $k = 0$, the equation

$$\sum_1^n \nu^{-\beta}a_\nu = \sum_1^{n-1} s_\nu \Delta\nu^{-\beta} + n^{-\beta}s_n$$

shows that, when $s_n = o(n^\beta)$, the series

$$\Sigma n^{-\beta}a_n, \qquad \Sigma n^{-\beta-1}s_n$$

converge together. But the second series is divergent if, *e.g.*, $s_n = n^\beta/(\log n)$.

It is evident that, in the enunciation of Theorem 12, we need only have supposed that

$$s_n^k = O(n^{k+\beta+\epsilon})$$

for all positive values of ϵ.

18. Our next theorem involves an argument of a more delicate character.

THEOREM 13.—*If* $s_n^k = o(n^{k+\beta})$, *where* $\beta > 0$, *then* $\Sigma n^{-\beta}a_n$ *is either summable* (Ck) *or summable by none of Cesàro's means. Further, the same is true of* $\Sigma n^{-\beta-ti}a_n$, *for any value of* t.*

Suppose that $\Sigma n^{-\beta}a_n$ is summable $(C, k+1)$. Then the necessary and sufficient condition that it should be summable (Ck) is that

(1) $$\tau_n^k = o(n^{1+k}),$$

where τ_n^k is the k-th Cesàro's sum formed from

$$\alpha_n = n^{1-\beta}a_n.\dagger$$

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* The arguments of this paragraph are in detail so similar to those of § 17 that we have not thought it worth while to give more than an exceedingly condensed account of the proof. Any attempt to extend the result of Theorem 12 to the case in which $\gamma = \beta$ would, of course, have been foredoomed to failure. If, in § 17, we take $\gamma = \beta$, we are led to the divergent series

$$\Sigma j^{k+\beta}\Delta^{k+1}j^{-\beta}.$$

In this paragraph the fact that we already know our series to be summable $(C, k+1)$ enables us to use condition (1), and so argue in terms of sums τ_n^k formed from $n^{1-\beta}a_n$ instead of sums t_n^k formed from $n^{-\beta}a_n$. We thus avoid the logarithm associated with a series of the type $\Sigma(1/j)$.

† Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8. p. 304.

Suppose first that $k=0$. Then

$$\tau_n = \sum_0^n \nu^{1-\beta} a_\nu = \sum_0^{n-1} s_\nu \Delta \nu^{1-\beta} + s_n n^{1-\beta} = O\left\{\sum_1^{n-1} o(\nu^\beta)\nu^{-\beta}\right\} + o(n) = o(n).$$

Thus (1) is true for $k=0$.

When $k>0$, we use the formula

$$\tau_n^k = \sum_{j=0}^n s_j^k \sum_{i=0}^k \binom{k+1}{i}\binom{n-j-i+k}{k-i} \Delta^{k+1-i}(j+i)^{1-\beta}, \tag{2}$$

analogous to (1) of § 17. In the first place, we have

$$_0\tau_n^k = \sum_{j=0}^n s_j^k \binom{n-j+k}{k} \Delta^{k+1} j^{1-\beta} = A_n^k \left\{\sum_{j=0}^n s_j^k \Delta^{k+1} j^{1-\beta} + R_0\right\},$$

where R_0 differs from the R_0 of § 17 only in the replacing of $-\gamma$ by $1-\beta$. The first term on the right-hand side is

$$O\left\{n^k \sum_1^n o(j^{k+\beta}) j^{-\beta-k}\right\} = O\left\{n^k \sum_1^n o(1)\right\} = o(n^{k+1}).$$

And that

$$R_0 = o(n)$$

is easily shown by an argument in essentials the same as that of § 17. Finally,

$$_i\tau_n^k = \binom{k+1}{i} \sum_{j=0}^n \binom{n-j-i+k}{k-i} s_j^k \Delta^{k+1-i}(j+i)^{1-\beta}$$

$$= O\left\{n^{k-i} \sum_{j=0}^n o(j^{k+\beta}) j^{-\beta-k+i}\right\}$$

$$= O\left\{n^{k-i} \sum_{j=0}^n o(j^i)\right\} = o(n^{k+1}).$$

Combining these results, we see that (1) is satisfied, and the proof of Theorem 13, in so far as it is concerned with the point

$$s = \beta,$$

is completed.

The extension to complex values of s presents no difficulties. When, *e.g.*, $k=0$, we have

$$\tau_n = \sum_0^n \nu^{1-\beta-ti} a_\nu = \sum_0^{n-1} s_\nu \Delta \nu^{1-\beta-ti} + s_n n^{1-\beta-ti} = O\left\{\sum_0^{n-1} \nu^{-\beta} o(\nu_\beta)\right\} + o(n) = o(n),$$

as before. Similarly in the general case.

The most interesting case is, of course, that in which $k=0$.

With Theorem 13 may be associated the following theorem, the proof of which presents no difficulty.

THEOREM 14.—*If* $\Sigma n^{-\beta} a_n$ *is summable* (Ck), *and* $\beta > 0$, *then*

$$s_n^k = o(n^{k+\beta}).$$

The result of this theorem is a straightforward deduction from the formula

$$s_n^k = \sum_{j=0}^{n} t_j^k \sum_{i=0}^{k} \binom{k+1}{i} \binom{n-j-i+k}{k-i} \Delta^{k+1-i}(j+i)^{\beta},$$

analogous to (1) of § 17. The result, although useful, is much less subtle and interesting than that of the companion Theorem 13, and it seems hardly worth while to set out the details of the proof. The result obviously breaks down when $\beta = 0$.

19. Some very interesting results can be deduced from a mere combination of Theorems 13 and 14. Suppose that $\Sigma n^{-\beta} a_n$ is summable (Ck). Then, by Theorem 14, we have $s_n^k = o(n^{k+\beta})$. Hence, using Theorem 13, we obtain

THEOREM 15.—*If a Dirichlet's series is summable* (Ck) *for* $s = \beta$, *it is, at any point of the line* $s = \beta + ti$, *either summable* (Ck), *or summable by none of Cesàro's means.*

It will be observed that the condition $\beta > 0$, which is plainly irrelevant to the truth of the theorem, has now disappeared.

THEOREM 16.—*If a Dirichlet's series is summable* (Ck) *for* $s = \beta$, *and the function represented by the sum of the series can be continued, by the use of Cesàro's methods of summation, across the line* $s = \beta + ti$, *then it is summable* (Ck) *at all points of the line. Moreover, it is uniformly summable along any finite stretch of the line.*

This theorem is an immediate corollary of Theorem 15, except for the assertion of uniformity. The truth of this last assertion follows from the fact that the O's and o's which occur in the proof of Theorem 13 are uniform with respect to t.

Theorem 16 remains true if the words "by the use of Cesàro's methods of summation" are omitted. In fact, if the series is summable (Ck) for $s = \beta$, it is summable (Ck) at all points of the line at which it is regular. This is a fairly straightforward deduction from Riesz's analogue of Fatou's theorem,* viz., that *if* $a_n = o(1/n)$, *the series* $\Sigma a_n n^{-s}$ *is convergent at all regular points of the line* $\sigma = 0$.

* Riesz, *Comptes Rendus*, Nov. 22nd, 1909; see also Landau, *Prac Matematyczno-Fizycznych*, Vol. 21, p. 151.

In this connection it is interesting to give an example of a series, which represents an integral function, but is summable only in a half-plane. The series

$$\Sigma\, e^{i(\log n)^a} n^{-s} \quad (a \geqslant 2)$$

is such a series. Its lines of summability all coincide in the line $\sigma = 1$, at every point at which it is convergent.

20. We return now to Theorem 10. From Theorems 10 and 12 we can at once deduce

THEOREM 17.—*If* $0 \leqslant k \leqslant r$, *and* λ_k *is the abscissa of the line of summability* (Ck) *of the Dirichlet's series* $\Sigma a_n n^{-s}$, *and if*

(i) $a_n = O(n^{a+\epsilon})$, *where* $a > -1$, *for every positive* ϵ,

(ii) $\lambda_r \leqslant 0$,

then
$$\lambda_k \leqslant \frac{(a+1)(r-k)}{r+1}.$$

We have, however, not yet extracted from Theorem 10 the maximum of information which it may be made to yield concerning the convergence and summability of a Dirichlet's series. This is contained in the following more precise theorem, which contains Theorem 17, and is an immediate corollary of Theorems 10, 13, and 16.

THEOREM 18.—*If* $a_n = O(n^a)$, *where* $a > -1$, *and* Σa_n *is summable* (Cr), *then* $\Sigma a_n n^{-s}$ *is summable* (Ck), *where* $0 \leqslant k < r$, *for*

$$s = \frac{(a+1)(r-k)}{r+1}.$$

The same result follows from the assumptions that $a_n = o(n^a)$ *and* Σa_n *is finite* (Cr). *Further, the series is, in either case, uniformly summable on any finite stretch of the straight line*

$$s = \frac{(a+1)(r-k)}{r+1} + ti.$$

In fact we have, by Theorem 10,

$$s_n^k = o(n^{k+\beta}),$$

where
$$\beta = \frac{(a+1)(r-k)}{r+1} > 0.$$

Hence, by Theorem 13, $\Sigma a_n n^{-s}$ is summable (Ck) or by none of Cesàro's means for $s = \beta + ti$. Therefore, as it is certainly summable (Cr), it is summable (Ck). Finally, the uniformity follows from Theorem 16.

We add a few remarks as to the relations between Theorems 10 and 18. Theorem 10 included two cases, both of which were trivial when $\alpha > r$, while the first, but not the second, was also trivial when $\alpha = r$. Theorem 18 includes two corresponding cases, but neither of these is trivial for $\alpha = r$. Suppose, for simplicity, that $k = 0$. If Σa_n is summable (Cr), we must have $a_n = o(n^r)$: thus the first condition is included in the second, and the first case of the theorem reduces to a known theorem,* viz. that the summability (Cr) of Σa_n involves the convergence of $\Sigma a_n n^{-r}$. The second case gives us a new result, viz., that if the series Σa_n is finite (Cr), and $a_n = o(n^r)$—a condition no longer included in the first condition—then $\Sigma a_n n^{-r}$ is convergent.

It is possible to generalise Theorem 10 by adopting hypotheses of the type
$$a_n = O\{n^\alpha (\log n)^\beta (\log\log n)^\gamma \ldots\},$$
instead of $a_n = O(n^\alpha)$. Such generalisations have no particular interest for our present purpose. The more special hypothesis is more than sufficient for the application of the theorem to the proof of Theorem 17, and is *necessary* in the proof of Theorem 18.

21. We shall now state and prove the second principal theorem of this part of the paper.

THEOREM 19.—*If $\Sigma a_n n^{-s}$ is summable (Cr) for $s = 0$, and summable $(C, r-k)$ for $s = \beta$, where $0 < \beta \leqslant k \leqslant r$, then it is summable $(C, r-l)$, where $0 < l < k$, for*
$$s = l\beta/k.$$

The conclusion still holds if either "summable" in the hypothesis is replaced by "finite": while, if both are so replaced, the "summable" in the conclusion must also be replaced by "finite."

We consider the case in which Σa_n is finite (Cr) and $\Sigma a_n n^{-\beta}$ summable $(C, r-k)$. The other cases may be settled similarly.

We define a function $f(x)$ as in § 16. As in § 16, we find

$$(1) \quad f^{(l)}(x) = O(x^{r-l-1}) + s_n^{r-l} + O(s_n^{r-l-1}) + \ldots + O(s_n) + O(a_n) + O(\Delta a_n),$$

for $0 \leqslant l \leqslant r$. We have chosen zero as the value of the constant A of § 16. Taking first $l = 0$, we have

$$(2) \qquad f(x) = O(x^r).$$

* This result is a corollary of a theorem given by Riesz, *Comptes Rendus*, June 21st, 1909.

Now take $l = k$. Since $\Sigma a_n n^{-\beta}$ is summable $(C, r-k)$, we have, by Theorem 14,

$$s_n^{r-k} = o(n^{r-k+\beta});$$

and it is clear that every other term on the right-hand side of (1) is also of this form. Hence

$$f^{(k)}(x) = o(x^{r-k+\beta}),$$

$$x^k f^{(k)}(x) = o(x^{r+\beta}), \tag{3}$$

We now apply Theorem 5 to equations (2) and (3), observing that, as $\beta > 0$, the condition $\phi = o(\psi)$ is certainly satisfied. We obtain

$$x^l f^{(l)}(x) = o\left\{x^{r+(l\beta/k)}\right\}.$$

From this we can deduce

$$s_n^{r-l} = o\left\{n^{r-l+(l\beta/k)}\right\}, \tag{4}$$

for $0 \leqslant l \leqslant k$. For this is certainly true when $l = k$. Suppose it true for $l = k, k-1, \ldots, \lambda+1$; and consider equation (1), with $l = \lambda$. It is easy to see that every term of the equation, save $s_n^{r-\lambda}$, is of the form $o\left\{n^{r-\lambda+(\lambda\beta/k)}\right\}$. For example,

$$O(s_n^{r-\lambda-1}) = o\left\{n^{r-\lambda-1+[(\lambda+1)\beta/k]}\right\} = o\left\{n^{r-\lambda+(\lambda\beta/k)}\right\},$$

since (4) has *ex hypothesi* been proved for $l = \lambda+1$, and since $0 < \beta \leqslant k$. Hence $s_n^{r-\lambda}$ is of this form, and so the truth of (4) is established by induction.

Theorem 19 follows at once from (4) and Theorem 13, since $\Sigma a_n n^{-s}$ is certainly summable by *some* of Cesàro's means for any positive s.

Taking $k = 2$, $l = 1$ in Theorem 19, and effecting a suitable linear transformation on s, we deduce

Theorem 20.—*If $\Sigma a_n n^{-s}$ is summable (Cr) for $s = \alpha$, and summable $(C, r-2)$ for $s = \beta$, it is summable $(C, r-1)$ for $s = \frac{1}{2}(\alpha+\beta)$.*

And from this we can at once deduce a very interesting theorem of Bohr,* viz.,

Theorem 21.—*If λ_r is the abscissa of summability (Cr) of a Dirichlet's*

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* Bohr, *Nachrichten u.s.w. zu Göttingen*, 1909, p. 252; *Bidrag til de Dirichlet'ske Raekkers Theorie*, p. 101.

series $\Sigma a_n n^{-s}$, *then*

$$\lambda_{r-2}-\lambda_{r-1} \geqslant \lambda_{r-1}-\lambda_r,$$

or

$$\Delta^2\lambda_r \geqslant 0.$$

For, if we had $\lambda_{r-2}-\lambda_{r-1} < \lambda_{r-1}-\lambda_r,$

we could obviously choose values of α and β in such a way as to violate Theorem 20.

This theorem may be employed to simplify the proof of Theorem 17. For suppose this theorem established in the case $k = 0$. Then it follows from Theorem 21 that

$$(\lambda_0-\lambda_k)/k \geqslant (\lambda_k-\lambda_r)/(r-k),$$

$$\lambda_k \leqslant \frac{r-k}{r}\lambda_0+\frac{k}{r}\lambda_r \leqslant \frac{(\alpha+1)(r-k)}{r+1}.$$

It should also be observed that, *unless*

$$\lambda_0-\lambda_1 = \lambda_1-\lambda_2 = \dots = \lambda_{k-1}-\lambda_k,$$

we must have $\lambda_k < (\alpha+1)(r-k)/(r+1):$

so that the series is summable (Ck) for $s = (\alpha+1)(r-k)/(r+1)$, and even for smaller values of s: it is only in the extreme case (which is, however, that of most frequent occurrence) that we require the more delicate Theorem 18.

22. Now let us suppose that $a_n = o(n^\delta),$

for every positive δ, so that the series is certainly absolutely convergent for $s > 1$. And let us suppose further that

$$\Lambda = \lim_{r\to\infty} \lambda_r < 1;$$

i.e., that the series is summable by some of Cesàro's means for some distance to the left of the line

$$\sigma = 1.$$

If η is positive and $s = \Lambda+\eta < 1$, the series is summable (Cr); here r is a function of η, which may tend to infinity as $\eta \to 0$. Also

$$b_n = a_n n^{-\Lambda-\eta} = o(n^{-\Lambda-\eta+\delta}).$$

Applying Theorem 17 to the series $\Sigma b_n n^{-s}$, with $k = 0$, we see that the abscissa of convergence of $\Sigma a_n n^{-s}$ is not greater than

$$\Lambda+\eta+\frac{r}{r+1}(1+\delta-\Lambda-\eta) = 1-\frac{1-\Lambda-\eta}{r+1}+\frac{r\delta}{r+1}.$$

This is true for all positive values of δ: hence the abscissa of convergence is not greater than

$$1-\frac{1-\Lambda-\eta}{r+1} < 1.$$

Hence we obtain

THEOREM 22.—*If* $a_n = o(n^\delta)$, *and the series* $\Sigma a_n n^{-s}$ *is summable by any of Cesàro's means for some points to the left of* $\sigma = 1$, *then it is convergent for some points to the left of* $\sigma = 1$.

The interesting case is, of course, that in which the abscissa $\bar{\lambda}$ of absolute convergence is 1. Let us consider this case more closely. The series is summable $(C1)$ for some points to

the left of $\sigma = 1$; let $\lambda_1 = 1 - \zeta < 1$ be the abscissa of summability. The series

$$\Sigma b_n = \Sigma a_n n^{-1+\zeta-\rho}$$

is summable $(C1)$ for any positive ρ. Also

$$b_n = o(n^{-1+\zeta-\rho+\delta}).$$

Applying Theorem 17, we find $\lambda_0 \leqslant 1 - \zeta + \rho + \frac{1}{2}(\zeta - \rho + \delta)$,

or $\lambda_0 \leqslant 1 - \frac{1}{2}\zeta$,

since both δ and ρ are arbitrarily small. Our result is equivalent to the following theorem.

THEOREM 23.—*If a Dirichlet's series is such that* $a_n = o(n^{\bar{\lambda}+\delta-1})$, *where* $\bar{\lambda}$ *is the abscissa of absolute convergence, then*

$$\bar{\lambda} - \lambda_0 \geqslant \lambda_0 - \lambda_1 \geqslant \lambda_1 - \lambda_2 \geqslant \dots .$$

The point of this theorem is to complete Bohr's theorem by showing that, when

$$a_n = o(n^{\bar{\lambda}+\delta-1}),$$

the breadth of the strip of conditional convergence cannot be less than that of any strip of Cesàro summability. The condition

$$a_n = o(n^{\bar{\lambda}+\delta-1})$$

is one satisfied by the most obvious series, such as

$$\Sigma n^{-s}, \quad \Sigma(-1)^n n^{-s}, \quad \Sigma e^{in^a} n^{-s}.$$

If β is the lower limit of the values of s such that $a_n n^{-s} = o(1)$, it is evident that $\bar{\lambda} \leqslant \beta + 1$. The series to which Theorem 23 is applicable are those for which $\bar{\lambda} = \beta + 1$.

It is, of course, possible that $\bar{\lambda} < \beta + 1$; indeed we may have $\bar{\lambda} = \beta$. This is the case, for example, if

$$a_n = (-1)^i \, (n = n_i), \quad a_n = 0 \, (n \neq n_i),$$

where (n_i) is such a sequence that Σn_i^{-s} is convergent for every positive s. A particularly interesting example of such a series has been given by Bohr.* This series is

$$1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + 27^{-s} - 28^{-s} + \dots,$$

where $a_n = 1$ if $n = m^m$, $a_n = -1$ if $n = m^m + 1$, and $a_n = 0$ otherwise.

For this series $\bar{\lambda} = \lambda_0 = 0, \quad \lambda_1 = -1;$

there is a strip (of breadth 1) of summability $(C1)$, but no strip of conditional convergence. This example shows that no such result as that of Theorem 23 holds unless some restriction is placed upon a_n.

23. It may be deduced from Theorem 22, in conjunction with known theorems, that if the Riemann hypothesis as regards the roots of the ζ function is true (or indeed if Θ, the upper limit of their real parts, is less than 1), then the series

$$\frac{1}{\zeta(s)} = \Sigma \frac{\mu(n)}{n^s}$$

is convergent to the left of the line $\sigma = 1$. For it has been shown by Bohr that, for this function, $\Lambda = \Theta$. As a matter of fact, more than this is already known: it is known in fact

* *Nachrichten u.s.w.*, *l.c.*, p. 261.

that the series is convergent for $s > \Theta$.* It should, moreover, be observed that the equation $\Lambda = \Theta$ is deduced by Bohr from theorems of the nature of the "Landau-Schnee" theorem mentioned in the next paragraph, and that the final result can be deduced from this theorem directly. Thus the result cannot be obtained as a *bona-fide* deduction from Theorem 22.†

24. The nature of the theorems which we have been discussing suggests that they may have some connection with an important group of theorems due to Landau and Schnee.‡ Suppose that the conditions of Theorem 17 are satisfied. Then $\Sigma a_n n^{-s}$ is summable (Cr) for $\sigma > 0$ and absolutely convergent for $\sigma > a+1$.

It follows first, by a theorem of Bohr and Riesz,§ that if $f(s)$ is the function represented by the sum of the series, then $f(s)$ is regular for $\sigma > 0$, and

$$f(s) = O(|t|^{r+1}),$$

for $\sigma \geqslant \delta > 0$.

Now Landau and Schnee have proved the following theorem :—

If $a_n = o(n^\delta)$, so that $\Sigma a_n n^{-s}$ is absolutely convergent for $\sigma > 1$; if further $f(s)$ is regular for $\sigma > \eta$, where $\eta < 1$, and

$$f(s) = O(|t|^\lambda,$$

for $\sigma \geqslant \eta$, then $\Sigma a_n n^{-s}$ is convergent for

$$\sigma > (\eta+\lambda)/(1+\lambda).$$

Write $n^{-a} a_n = b_n$, and apply this theorem to $\Sigma b_n n^{-s}$, taking

$$\eta = -a+\delta, \quad \lambda = r+1.$$

We find at once that $\Sigma a_n n^{-s}$ is convergent for

$$\sigma > a + \frac{r+1-a}{r+2} = \frac{(a+1)(r+1)}{(r+2)}.$$

This is a result very similar to that of Theorem 17, but, since

$$(r+1)/(r+2) > r/(r+1),$$

not so good. Thus the greater depth of the considerations appealed to in this argument does not compensate for its lack of directness.

25. We conclude this section with a few miscellaneous remarks concerning the theorems we have proved. First, considering Theorem 17, we will show by an example that the information which it gives is the utmost which we can hope to obtain from the data.

The series $$\Sigma n^{-b} e^{in^a} \quad (0 < a < 1)$$

(to which we have so frequently to appeal) is summable (Cr) if $(r+1)a+b > 1$.‖ In particular, if $b = 0$, the series is summable (Cr) if $(r+1)a > 1$ and finite (Cr) if $(r+1)a = 1$.

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* Littlewood, *Comptes Rendus*, January 29th, 1912; the best previous result in this direction was due to Landau, *Handbuch*, p. 871.

† That $\Lambda = \Theta$ can, however, be proved otherwise, by making use of a theorem of Riesz (*Comptes Rendus*, June 21st, 1909; cf. Bohr, *l.c.*, p. 259).

‡ Landau, *Handbuch*, pp. 853 *et seq.* See also Landau, *Rendiconti di Palermo*, Vol. 28, p. 113; Schnee, *Math. Annalen*, Vol. 66, p. 337.

§ Bohr, *Comptes Rendus*, January 11th, 1909; *Bidrag etc.*, p. 114; *Nachrichten u.s.w.*, p. 252; Riesz, *Comptes Rendus*, *l.c. supra*.

‖ Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 142.

Suppose then
$$a = \frac{1}{r+1}.$$

Then the series $\Sigma n^{-s}a_n = \Sigma n^{-s}e^{in^a}$ is summable (Cr) for $s > 0$, and $a_n = O(1)$. Hence Theorem 17 asserts that $\Sigma a_n n^{-s}$ is convergent for
$$s > r/(r+1).$$

The series is, in fact, convergent, if, and only if,
$$s > 1-a > r/(r+1),$$
which is the limit assigned by the theorem.

Similarly it may be shown that the number $(\alpha+1)(r-k)/(r+1)$ of the theorem cannot be improved upon.

26. Our other remarks have reference to Theorems 19 *et seq.* It follows from Theorems 19 and 20 that if a Dirichlet's series has three lines of summability λ_r, λ_{r+1}, λ_{r+2} at equal distances, the summability of the series at the foot of the two extreme lines involves its summability at the foot of the middle line: in fact, we need only assume summability $(C, r+2)$ for, *e.g.*, $s = \lambda_{r+2}$, and finitude (Cr) for $s = \lambda_r$. It should be observed that this result depends essentially on the equidistance of the lines of summability. Thus for the series
$$\Sigma \left\{ \frac{(-1)^n n}{\log n} + \log n \, e^{i\sqrt{n}} + \frac{e^{in^{1/4}}}{\sqrt{n}\log n} \right\} n^{-s},$$
we have
$$\lambda_0 = 1, \quad \lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{4};$$
the series is summable $(C2)$ for $s = -\frac{1}{4}$ and convergent for $s = 1$; but is not summable $(C1)$, or even finite $(C1)$, for $s = 0$.

27. In conclusion we remark that all the theorems of this section have their analogues for integrals, which are, in fact, easier to deduce from the theorems of Section II. The reader will have no difficulty in stating these theorems for himself. We pass now to applications of a different character.

IV.

Additional Applications.

Theorems connected with the General Abel-Tauber Theorem.

28. THEOREM 24.—*If* $a_n = O(n^\alpha)$ *and* $f(x) = \Sigma a_n e^{-nx} = O(x^{-\beta})$ *as* $x \to 0$, *where* $0 < \beta < \alpha+1$, *then*
$$x^r f^{(r)} = O(x^{-\beta-\epsilon}).^*$$

We have
$$|x^r f^{(r)}| < Kx^r \Sigma n^{\alpha+r} e^{-nx} = O(x^{-\alpha-1}) = O(x^{-\beta}\psi),$$
where
$$\psi = x^{-(\alpha+1-\beta)} \to \infty.$$

* That $x^r f^{(r)} = O(x^{-\alpha-1})$ is trivial.

We may apply Theorem 9, obtaining

$$x^r f^{(r)} = O(x^{-\beta}\psi^{\epsilon}),$$

which gives the result of the theorem.

There is a similar result for the case $\alpha+1 = \beta = 0$, viz.,

THEOREM 25.—*If* $a_n = O(1/n)$, *and* $f(x) = \Sigma a_n e^{-nx} \to s$, *then*

$$x^r f^{(r)} \to 0.*$$

For we have $\quad |x^{r+1} f^{(r+1)}| < Kx^{r+1} \Sigma n^r e^{-nx} = O(1),$

and our theorem follows from Theorem 9.

29. Theorem 10, which is an extension of the general Cesàro-Tauber theorem, suggests that there is a similar extension of the general Abel-Tauber theorem. Theorem 10 gives

$$s_n = o\left\{n^{(1+\alpha)r/(r+1)}\right\}, \tag{1}$$

as a consequence of $a_n = O(n^\alpha)$ and the summability (Cr) of Σa_n. If we make r tend to infinity in (1), we obtain $s_n = o(n^{1+\alpha})$, and this equation suggests itself as the corresponding consequence when summability (Cr) is replaced by the existence of Abel's limit.

That the theorem thus suggested is true is an immediate consequence of known results. That $s_n = O(n^{1+\alpha})$ is trivial; and it has already been shown† that the hypotheses involve the convergence of $\Sigma n^{-\alpha} a_n$, and therefore, by Theorem 14, the equation $s_n = o(n^{1+\alpha})$. But the result may be made more general by assuming less than the existence of Abel's limit. All, indeed, that it is necessary to assume is that

$$f(x) = \Sigma a_n e^{-nx} = o(x^{-1-\alpha}).$$

Thus the theorem we propose to prove is as follows.

THEOREM 26.—*If* $a_n = O(n^\alpha)$, *where* $\alpha \geqslant -1$, *and if*

$$f(x) = \Sigma a_n e^{-nx} = o(x^{-1-\alpha}) \;\; (\alpha > -1), \quad = s + o(1) \;\; (\alpha = -1),$$

then $$s_n = o(n^{1+\alpha}) \;\; (\alpha > -1), \quad = s+o(1) \;\; (\alpha = -1).$$

The case $\alpha = -1$ is the general Abel-Tauber theorem; we shall therefore suppose $\alpha > -1$. The proof in this case, as also in the case $\alpha = -1$,‡ consists in dividing s_n (or $s_n - s$) into two parts, each of which

* This theorem is proved by Littlewood, *l.c.*, p. 439.

† Littlewood, *l.c.*, p. 447.

‡ Cf. Littlewood, *l.c.*, p. 440.

contains an auxiliary constant r. In each case one of these parts is shown, by an application of the theorems of Section II, to be of the desired form, while we obtain for the other part an expression whose modulus is less than ϵn^{1+a}, when r is sufficiently large. The detailed treatment of the second part we shall omit in the present case; it is not unlike the corresponding proof for the case $a=-1$, and the proof of a similar result in Theorem 27. The former of these proofs is already published, and the latter is given in detail in §§ 35, 36.

We have
$$f = o(x^{-1-a}),$$
(2)
$$x^r f^{(r)} = O(x^{-1-a});$$
the first equation being one of our hypotheses and the second being an immediate consequence of $a_n = O(n^a)$. Applying Theorem 8, with $\phi = \psi = x^{-1-a}$, so that the condition $\phi = O(\psi)$ is satisfied, we see that (2) can be replaced by the better equation

(2′)
$$x^r f^{(r)} = o(x^{-1-a}).$$

Now
$$\begin{aligned}(-x)^r \Sigma s_n n^{r-1} e^{-nx} &= -x^r \left(\frac{d}{dx}\right)^{r-1} \Sigma s_n e^{-nx} \\ &= -x^r \left(\frac{d}{dx}\right)^{r-1} \left\{\frac{f(x)}{1-e^{-x}}\right\} \\ &= \Sigma A x^r f^{(p)}(x)(1-e^{-x})^{-q} e^{-sx},\end{aligned}$$
where the A's are constants, and the summation extends to a number of sets of values of p, q, s, such that $p+q \leqslant r$. Applying (2′) we obtain
$$\Sigma o(x^{r-p-q-1-a}) = o(x^{-1-a});$$
so that

(3)
$$(-x)^r \Sigma s_n n^{r-1} e^{-nx} = o(x^{-1-a}).$$

We choose $x = r/m$, where r is fixed but at our disposal, and make $m \to \infty$. From (3), we obtain
$$\Sigma s_n n^{r-1} e^{-nr/m} = o(m^{r+a+1}),$$
and so

(4)
$$s_m \Sigma n^{r-1} e^{-nr/m} = o(m^{r+a+1}) + \Sigma (s_n - s_m) n^{r-1} e^{-nr/m}.$$

Now
$$\Sigma n^{r-1} e^{-nx} \sim r!\, x^{-r},$$
so that
$$\Sigma n^{r-1} e^{-nr/m} \sim r!\, r^{-r} m^r.$$
Hence, from (4),

(5)
$$s_m = o(m^{1+a}) + \{1+o(1)\}\, m^{-r} (r^r/r!)\, \Sigma(s_n - s_m) n^{r-1} e^{-nr/m}.$$

It is clear that our theorem is established if we can prove that

(6) $$S \equiv m^{-r-1-a}(r^r/r!) \sum_{n=1}^{\infty} | s_n - s_m | n^{r-1} e^{-nr/m} < \delta$$

when r is sufficiently large.

We divide the sum into three parts by writing

$$S = \sum_{1}^{(1-\lambda)m} + \sum_{(1-\lambda)m}^{(1+\lambda)m} + \sum_{(1-\lambda)m}^{\infty} = S_1 + S_2 + S_3,$$

where $0 < \lambda < 1$. It is easy to deduce from the equation $a_n = O(n^a)$ that

$$\begin{aligned} | s_n - s_m | &< Km^{1+a} && (in\ S_1), \\ &< Km^a | n-m | && (in\ S_2), \\ &< Kn^{1+a} && (in\ S_3), \end{aligned}$$

where the K's are independent of n, m, and r. If we substitute from these inequalities in the three sums it is not difficult to establish (6); as we have explained above, the kind of argument involved is sufficiently illustrated in § 35.

30. We pass now to a theorem of rather a different character. It is well known that the series

$$\Sigma \frac{1}{n^{1+ai}}, \quad \Sigma \frac{1}{n (\log n)^{1+ai}}, \quad \ldots,$$

are finitely oscillating (if $a \neq 0$) and that the corresponding Abel limits do not exist, and it is easy to see that, at any rate in a number of special cases, the Abel limit corresponding to the series

$$\Sigma\, n^{-s} (\log n)^{-t} (\log \log n)^{-u} \ldots,$$

where $s, t, u, \ldots$ are complex, does not exist unless the series is convergent. The most obvious common property of such series is expressed by the equation

$$\Delta a_n \equiv a_n - a_{n+1} = O(a_n/n);$$

this suggests that this property is a sufficient condition for the truth of the converse of Abel's theorem. It will be seen that we can prove a little more than this.

THEOREM 27.—*If* $\Delta a_n = O(a_n/n)$, *and if* $f(x) = \Sigma a_n e^{-nx} \to s$, *as* $x \to 0$, *then* $na_n \to 0$ *and* Σa_n *is convergent.*

We shall first establish independently the much easier theorem, deducible immediately from the above, with summability ($C1$) in the

hypothesis in place of the existence of Abel's limit. This seems to be worth while on account of the extreme difficulty of the proof of Theorem 27.

THEOREM 28.—*If $\Delta a_n = O(a_n/n)$, and if Σa_n is summable (C1), then $na_n \to 0$ and Σa_n is convergent.*

Throughout the succeeding sections we shall assume that no a_n (or b_n) vanishes. If any one does, so do all which follow: thus the limitation is trivial.

31. It is convenient to begin by establishing five lemmas, of which the first four are deductions from the equation

$$\Delta b_n = O(b_n/n),$$

which is to be understood as their hypothesis.

LEMMA 1.—*There is a constant K such that $b_n = O(n^K)$.*

LEMMA 2.—*If h and k are any constants such that $0<h<1<k<2$, there is a constant K such that*

$$|b_n - b_m| < K\,|n-m|\,|b_m|/m,$$

for $hm \leqslant n \leqslant km$.

LEMMA 3.—*There exists a constant c, such that*

$$b_{m+r} = b_m(1+\eta_r), \qquad |\eta_r| < \tfrac{1}{2},$$

for

$$|r| \leqslant cm.$$

LEMMA 4.—*We have*

$$|b_n/b_m| < K\,(n/m)^K \quad (n \geqslant m),$$

$$|b_n/b_m| < K\,(m/n)^K \quad (n \leqslant m),$$

where the K's are independent of both m and n.

We have

$$\frac{b_{n+1}}{b_n} = \left(1 - \frac{\Delta b_n}{b_n}\right) = 1 + O\left(\frac{1}{n}\right) = \exp\left[O\left(\frac{1}{n}\right)\right].$$

Hence $$\left|\frac{b_{n+1}}{b_1}\right| = \exp\left[O\left(\frac{1}{1}\right)+O\left(\frac{1}{2}\right)+\ldots+O\left(\frac{1}{n}\right)\right],^*$$

$$|b_{n+1}| < K\exp[K(1+\tfrac{1}{2}+\ldots+1/n)] < Kn^K;$$

which is the result of Lemma 1.

Again, if $m \leqslant n \leqslant km$,

$$\left|\frac{b_n-b_m}{b_m}\right| = \left|\frac{b_{m+1}}{b_m}\frac{b_{m+2}}{b_{m+1}}\cdots\frac{b_n}{b_{n-1}}-1\right|$$

$$= \exp\left[O\left(\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n-1}\right)\right]-1$$

$$= \exp\left[O\left(\frac{n-m}{m}\right)\right]-1 = O\left(\frac{n-m}{m}\right),$$

since $$(n-m)/m = O(1).$$

A similar argument gives this same equation in the case $hm \leqslant n \leqslant m$; we have thus proved Lemma 2. Lemma 3 is an immediate corollary; we have only to take $c = k$ or $c = 1/(2K)$, whichever is least, where the K is that of Lemma 2. Finally, to prove Lemma 4, we have, when $n \geqslant m$,

$$|b_n/b_m| = \exp\left[O\left(\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n-1}\right)\right]$$

$$= \exp[O\{\log(n/m)\}+K]$$

$$< K(n/m)^K;$$

which is the first inequality of Lemma 4. The second inequality follows similarly.

The last lemma is of a different character.

LEMMA 5.—*If* $t<1$, *then*

$$\sum_{-\infty}^{\infty} |\nu|\, e^{-t\nu^2} < K/t.$$

The maximum term of the series $\sum_{1}^{\infty} \nu e^{-t\nu^2}$ occurs at one of the integers p, $p+1$, separated by $\sqrt{(1/2t)}$. The terms increase up to this point

* The constants implied in the O's in this paragraph are all independent of both m and n.

and then begin to decrease. Hence we have

$$\sum_1^\infty \nu e^{-t\nu^2} < \int_0^p x e^{-tx^2} dx + 2\sqrt{(1/2t)}\, e^{-\frac{1}{2}} + \int_{p+1}^\infty x e^{-tx^2} dx$$
$$< 2\int_0^\infty x e^{-tx^2} dx + 1/t$$
$$< 2/t.$$

Thus $K = 4$ satisfies the requirements of the lemma.

32. We proceed now to the proof of Theorem 28. We shall use η to denote a number, not always the same, whose modulus is less than $\frac{1}{2}$.

By Lemma 3, we have

(1) $$a_{n+r} = a_n(1+\eta),$$

for $r \leqslant cn$. Hence

(2) $$\begin{aligned} s_{n+r} - s_n &= a_{n+1} + a_{n+2} + \ldots + a_{n+r} \\ &= a_n(1+\eta) + a_n(1+\eta) + \ldots + a_n(1+\eta) \\ &= r a_n(1+\eta). \end{aligned}$$

If now $$\sigma_n = s_1 + s_2 + \ldots + s_n,$$

$$\sigma_{n+r} - \sigma_n = \sum_{p=1}^{r} s_{n+p},$$

and so, by substitution from (2) in each s_{n+p},

$$\sigma_{n+r} - \sigma_n = r s_n + \tfrac{1}{2} r(r+1)\, a_n(1+\eta).$$

We shall suppose $r = an$, where $\frac{1}{8}c < a \leqslant c$. Then

(3) $$(\sigma_{n+r} - \sigma_n)/n = a s_n + \tfrac{1}{2} a(an+1)\, a_n(1+\eta).$$

Now, since Σa_n is summable $(C1)$,

$$\sigma_{n+r} = (n+r)\{s + o(1)\} = (1+a)n\{s+o(1)\},$$

and so $$(\sigma_{n+r} - \sigma_n)/n = as + o(1).$$

Substituting in (3) and rearranging the terms, we obtain

(4) $$s - s_n = o(1) + u n a_n,$$

where $$u = \tfrac{1}{2}\{a + 1/n\}(1+\eta).$$

We first choose $a = a_1$, where $a_1 n$ is the greatest integer contained in cn. Then when n is large

$$w_1 = |u_1| > \tfrac{1}{4}(1 + [cn])/n > \tfrac{1}{4}c.$$

Similarly we choose $a = a_2$, where $a_2 n+1$ is the greatest integer contained in $\frac{1}{4}cn$. Then

$$w_2 = |u_2| < \tfrac{3}{4}[\tfrac{1}{4}cn]/n < \tfrac{3}{16}c.$$

Now, from (4), we have

$$na_n(u_1-u_2) = o(1). \tag{5}$$

As $n \to \infty$, $a_1 \to c$, $a_2 \to \frac{1}{4}c$, and

$$|u_1-u_2| \geqslant w_1-w_2 > \tfrac{1}{4}c-\tfrac{3}{16}c = \tfrac{1}{16}c.$$

It follows from (5) that $na_n \to 0$, and the convergence of Σa_n follows from the special Cesàro-Tauber theorem.

33. There is another theorem closely allied to Theorem 28.

THEOREM 29.—*If* Σa_n *is summable* $(C1)$, *and*

$$na_n = O\left(\frac{a_1+2a_2+\ldots+na_n}{n}\right),$$

then $na_n = o(1)$, *and* Σa_n *is convergent.*

We omit the proof of this theorem, which follows without serious difficulty from (1) some elementary lemmas analogous to those of § 31, (2) the line of argument used by Landau* to prove the general Cesàro-Tauber theorem. This argument is (as Landau points out) substantially the same as that used for a special purpose by de la Vallée-Poussin in his classical memoirs on the theory of prime numbers.†

34. We shall now attack the much more difficult Theorem 27. It will be convenient to adopt certain conventions as to notation. Throughout the proof K's denote constants independent of r as well as of m. O's and o's are relative to the tending to infinity of m, but they are not necessarily uniform with respect to r; *e.g.*, the K implied in O is not necessarily independent of r. Finally, we use ϵ_r to denote any number, *independent of* m, which tends to zero as $r \to \infty$.

35. We choose $x = (r-\theta)/m$, where r is fixed, but at our disposal,

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* *Prac Matematyczno-Fizycnych*, Vol. 21, pp. 97 *et seq.*

† "Recherches Analytiques etc.," *Annales de la Société Scientifique de Bruxelles*, t. 20, pp. 247-250.

and $0 \leqslant \theta \leqslant 1$. Our first object is to establish the two following equations

(1) $$ma_m = O\{x^r f^{(r)}(x)\},$$

(2) $$x^r f^{(r)}(x) = O(ma_m),$$

for all sufficiently great values of r, and uniformly in respect to θ. These equations do not require the hypothesis of the existence of Abel's limit.

We have $$(-x)^r f^{(r)}(x) = x^r \Sigma n^r a_n e^{-nx}$$
$$= S_1+S_2+S_3+S_4,$$

where

(3) $$\begin{cases} S_1 = x^r \Sigma n^{r-1} m a_m e^{-nx}, \\ S_2 = x^r \sum\limits_{1}^{hm} n^{r-1}(na_n - ma_m)e^{-nx}, \\ S_3 = x^r \sum\limits_{hm}^{km} n^{r-1}(na_n - ma_m)e^{-nx}, \\ S_4 = x^r \sum\limits_{km}^{\infty} n^{r-1}(na_n - ma_m)e^{-nx}; \end{cases}$$

where h and k are independent of m and r, and $0 < h < 1 < k < 2$. It is convenient to suppose h and k irrational, so that hm and km cannot be integral.

We have

(4) $$S_1 = ma_m r!\{1+o(1)\}.$$

We proceed to show that, for sufficiently large values of m,

(5) $$|S_2| < \epsilon_r r! \, |ma_m|,$$

(6) $$S_4| < \epsilon_r r! \, |ma_m|.$$

Now $b_n = na_n$ is easily seen to satisfy $\Delta b_n = O(b_n/n)$, and so satisfies the results of Lemmas 1 and 4. Hence, to establish (5) and (6), it is clearly sufficient, to show that, if a is any number independent of m and r,

(5′) $$S_2' = x^r \sum_{1}^{hm} n^{r-1}(m/n)^a e^{-nx} < \epsilon_r r!,$$

(6′) $$S_4' = x^r \sum_{km}^{\infty} n^{r-1}(n/m)^a e^{-nx} < \epsilon_r r!.$$

The maximum value of $n^{r-1-a}e^{-nx}$, *qua* function of n, occurs when

$$n = (r-1-a)/x,$$

which is of the form $$m(1+\epsilon_r).$$

Hence, when r is sufficiently large, this value of n lies outside the range of summation in (5′), and the summand is an increasing function of n. Similarly, in (6′), the summand is a decreasing function of n. Consequently

$$\begin{aligned} S_2' &< x^r hm (hm)^{r-1} (m/hm)^a e^{-hmx} \\ &< h^{r-a} (r-\theta)^r e^{-h(r-\theta)} < Kr!\, r^{-\frac{1}{2}} h^r e^{r(1-h)} * \\ &< Kr! \exp[r\{(1-h)+\log h\}] \\ &< \epsilon_r r!, \end{aligned}$$

since $1-h+\log h < 0$; and

$$\begin{aligned} S_4' &< x^r \int_0^\infty t^{r-1} (t/m)^a e^{-tx} dt \\ &= (r-\theta)^r \int_0^\infty u^{r+a-1} e^{-(r-1)u} du \\ &< Kr^{-a} \Gamma(r+a) \\ &< Kr^{-1} \Gamma(r+1) \\ &= \epsilon_r r!. \end{aligned}$$

We have thus proved (5) and (6).

There remains S_3 to be considered. We have in S_3 (applying Lemma 2 to $b_n = na_n$)

$$|na_n - ma_m| < K|n-m|\,|a_m|,$$

and so

$$\begin{aligned} |S_3| &< Kx^r |a_m| \sum_{hm}^{km} n^{r-1} |n-m|\, e^{-nx} \\ &< Kx^r |a_m| \sum_{\nu=(h-1)m}^{(k-1)m} (m+\nu)^{r-1} |\nu|\, e^{-(r-\theta)(m+\nu)/m}. \end{aligned}$$

Now $|\nu| < \kappa m$, where $0 < \kappa < 1$, and

$$u - \log(1+u) > Ku^2,$$

for $-\kappa < u < \kappa$. Hence

$$\begin{aligned} \left(\frac{m+\nu}{m}\right)^{r-1} e^{-(r-\theta)\nu/m} &= \exp\left[-(r-1)\left\{\frac{\nu}{m} - \log\left(1+\frac{\nu}{m}\right)\right\} - \frac{(1-\theta)\nu}{m}\right] \\ &< Ke^{-K(r-1)(\nu/m)^2}. \end{aligned}$$

* By Stirling's theorem.

Consequently

(7) $$| S_3 | < Kx^r \, | ma_m | \, e^{-r} m^{r-2} \sum_{-\infty}^{\infty} | \nu | \, e^{-K(\nu/m)^2 (r-1)}$$
$$< Kx^r \, | ma_m | \, e^{-r} m^r/(r-1)^*$$
$$< K \, | ma_m | \, r^{r-1} e^{-r}$$
$$< | ma_m | \, \epsilon_r r!,$$

by Stirling's theorem.

From (3), (4), (5), (6), and (7), we have

$$| (-x)^r f^{(r)}(x) - r! \, \{1+o(1)\} \, ma_m | < \epsilon_r r! \, | ma_m |,$$

and from this both (1) and (2) follow immediately.

36. From (1) and (2) and the theorems of Section II it is not difficult to deduce our theorem. Let

$$g(y) = \max_{n \leqslant y} | na_n |,$$

so that $g(y)$ is an increasing function of y.

From Lemma 4, we have

(1) $$g(ay) = O\{g(y)\},$$

where a is any constant. Now, if r is sufficiently large,

$$x^r f^{(r)}(x) = O(ma_m),$$

by (2) of § 35, where $x = (r-\theta)/m$. Hence

(2) $$x^r f^{(r)}(x) = O\{g(m)\} = O[g\{(r-\theta)/x\}] = O\{g(1/x)\},$$

by (1). This equation is valid for all values of x as $x \to 0$, since the conditions $x = (r-\theta)/m$, $0 \leqslant \theta \leqslant 1$, allow x to take all small values as m takes all large integral values. If now we suppose that $g(y) \to \infty$ as $y \to \infty$, we obtain by means of Theorem 9 (since $f \to s$), the equation

(3) $$x^r f^{(r)}(x) = O[\{g(1/x)\}^{\epsilon}] = o\{g(1/x)\},$$

which holds when $x \to 0$. But we have, by (1) of § 35,

(4) $$| ma_m | = O\{x^r f^{(r)}(x)\}.$$

Hence, from (3), we have

$$| ma_m | = o[g\{m/(r-\theta)\}] = o\{g(m)\}$$
$$< \epsilon g(m),$$

* By Lemma 5.

for $m > m_0$, say. Then

(5) $$g(m) = \max_{n \leqslant m} |na_n| \leqslant \max\{\max_{n \leqslant m_0} |na_n|, \ \epsilon g(m)\}$$
$$= \max\{O(1), \ \epsilon g(m)\},$$

so that

(5) $$g(m) = o\{g(m)\} + O(1).$$

The equality (5) is incompatible with $g(y) \to \infty$; hence

$$g(y) = O(1).$$

It then follows from (2) of § 35 that

$$x^r f^{(r)}(x) = O(1),$$

as $x \to 0$, for sufficiently large values of r. But now, applying Theorem 8, we have

$$x^r f^{(r)}(x) = o(1),$$

for $r \geqslant 1$. Hence, by (1) of § 35,

$$ma_m = O\{x^r f^{(r)}(x)\} = o(1).$$

This is the first part of our theorem: the second part follows by Tauber's theorem.

37. In theorems of the "Tauberian" type, it usually happens that the "special" form, involving o in the hypothesis, is very much easier to prove than the "general" form involving O.

This is true in the present case. But the "special" theorem corresponding to Theorem 27 is trivial—it is true, but true only in that it asserts that two mutually incompatible propositions imply some other proposition; an assertion which is always true.* This is shown by

THEOREM 30.—*If* $\Delta a_n = o(a_n/n)$, *then* Σa_n *cannot converge, nor can Abel's limit exist, except in the trivial case when* $a_n = 0$ *from a certain point onwards. Also* $|na_n| \to \infty$, *and*

$$s_n \sim na_n, \quad s_n = n^{1+o(1)}.$$

We have

$$\frac{(n+1)\,a_{n+1}}{na_n} = 1 - \frac{(n+1)\,\Delta a_n - a_n}{na_n} = 1 + u_n,$$

* Any false proposition implies all other propositions, true or false: *vide* Whitehead and Russell, *Principia Mathematica*, Vol. 1, p. 108 (Prop. II, 21).

say, and
$$na_n = a_1 \prod_1^{n-1} (1+u_\nu).$$

Now
$$u_\nu = O(1/\nu),$$

so that
$$\log(1+u_\nu) = u_\nu + O(1/\nu^2),$$

$$\log(na_n) = \sum_1^{n-1} u_\nu + O(1).$$

But
$$u_\nu \sim 1/\nu,$$

and so $na_n \sim \log n$. It follows from Theorem 27 that Abel's limit cannot exist, and *a fortiori* that the series cannot converge.

Again, if $|a_n| = \alpha_n$, we have

$$s_n = \sum_1^n a_n = \sum_1^{n-1} \nu\Delta a_\nu + na_n,$$

$$\sigma_n = \sum_1^n \alpha_n = \sum_1^{n-1} \nu\Delta \alpha_\nu + n\alpha_n.$$

Now σ_n is an increasing function of n, which tends to infinity with n. Also
$$\nu\Delta a_\nu \leqslant \nu\,|\Delta a_\nu| = o(\alpha_\nu),$$

$$\sum_1^{n-1} \nu\Delta\alpha_\nu = o(\sigma_n),$$

and $\sigma_n \sim n\alpha_n$. Finally,
$$\sum_1^{n-1} \nu\Delta a_\nu = o\left(\sum_1^{n-1} \alpha_\nu\right) = o(\sigma_n) = o(n\alpha_n);$$

and therefore $s_n \sim na_n$. Also
$$na_n = e^{\log(na_n)} = e^{\log n + o(\log n)} = n^{1+o(1)}.$$

38. We proved recently* that a series Σa_n in which $a_n = o(1/\sqrt{n})$ cannot be summable by Borel's method unless it is convergent. The preceding results suggest that there should be a theorem related to this theorem as Theorem 27 is related to the general Abel-Tauber theorem.

This theorem is as follows.

THEOREM 31.—*If Σa_n is summable (B), and*
$$\Delta a_n = o(a_n/\sqrt{n}),$$
then $a_n = o(1/\sqrt{n})$ and the series is convergent.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 1.

From
$$e^{-m} \Sigma s_n \frac{m^n}{n!} \to s,$$
it follows that

(1) $$S = e^{-m} \Sigma (s_n - s_m) \frac{m^n}{n!} = s - s_m + o(1)$$

as $m \to \infty$.

We choose positive numbers H, α, such that $H < 1$, and no one of Hm, $\alpha\sqrt{m}$ $(m = 1, 2, \ldots)$ is an integer; and we write

$$S = e^{-m}\left(\sum_0^{m-Hm} + \sum_{m-Hm}^{m-\alpha\sqrt{m}} + \sum_{m-\alpha\sqrt{m}}^{m+\alpha\sqrt{m}} + \sum_{m+\alpha\sqrt{m}}^{m+Hm} + \sum_{m+Hm}^{\infty}\right) = S_1 + S_2 + S_3 + S_4 + S_5.$$

In order to discuss these sums we shall require certain deductions from the hypothesis
$$\Delta a_n = o(a_n/\sqrt{n}).$$

As in § 30, we put aside from the beginning the trivial case in which $a_n = 0$ from a certain point onwards.

39. This hypothesis involves the following consequences.

(a) We have
$$a_n = e^{o(\sqrt{n})}, \quad s_n = e^{o(\sqrt{n})}.*$$

We shall not require to use all that is implied in this assertion: it will be sufficient for our purposes that $|a_n| < Ke^{K\sqrt{n}}$ and
$$|s_n - s_m| < Ke^{K\sqrt{q}},$$
where q is the larger of m, n.

(b) We have
$$|s_n - s_m| < K\sqrt{m}\,|a_m|\,e^{Kr/\sqrt{m}},$$
where
$$r = |n - m| < Hm.$$

(c) We have
$$s_n - s_m = (n - m)\,a_m(1 + \epsilon_m),$$
if
$$r = |n - m| < \alpha\sqrt{m}.$$

Here ϵ_m is a function of m which tends to zero when once α has been fixed.

We shall first prove these results, and then apply (a) to the discussion of S_1 and S_5, (b) to that of S_2 and S_4, and (c) to that of S_3.

40. (i) *Proof of* (a).—Given ϵ we can choose p so that
$$a_{n+1} = a_n - \Delta a_n = a_n\left(1 - \frac{\eta_n}{\sqrt{n}}\right),$$
where
$$|\eta_n| < \epsilon \quad (n \geqslant p).$$

Hence, if $n = p + r$, we have

(1) $$a_{p+r} = \chi(p, r)\,a_p,$$

where
$$|\chi| < \prod_p^{p+r-1}\left(1 + \frac{\epsilon}{\sqrt{\nu}}\right) < \exp\left(\epsilon \sum_1^n \frac{1}{\sqrt{\nu}}\right) < e^{2\epsilon\sqrt{n}};$$

---

* That is to say, given $\delta > 0$ we can find n_0 so that
$$e^{-\delta\sqrt{n}} < |a^n| < e^{\delta\sqrt{n}} \quad (n > n_0).$$

and so $$|a_n| < e^{K\epsilon\sqrt{n}}\,|a_p|.$$

Similarly it may be shown that $$|a_n| > e^{-K\epsilon\sqrt{n}}\,|a_p|.*$$

Thus $a_n = e^{o(\sqrt{n})}$. That $s_n = e^{o(\sqrt{n})}$ is an immediate corollary.

(ii) *Proof of* (*b*).—Suppose, *e.g.*, $n > m$. It follows from (1) above that

$$|a_n| = |a_{m+r}| = \chi(m, r)\,|a_m|,$$

where $$|\chi| < \prod_{m}^{m+r-1}\left(1+\frac{K}{\sqrt{\nu}}\right) < \exp\left(\frac{Kr}{\sqrt{m}}\right).$$

Hence $$|s_n - s_m| < |a_{m+1}| + \ldots + |a_n| < |a_m|\,\frac{e^{K(r+1)/\sqrt{m}}}{e^{K/\sqrt{m}}-1} < K\sqrt{m}\,|a_m|\,e^{Kr/\sqrt{m}}.$$

The case in which $n < m$ may be discussed similarly.

(iii) *Proof of* (*c*).—Suppose, *e.g.*, $0 < r < \alpha\sqrt{m}$. We observe first that

$$|a_n/a_m| < e^{Kr/\sqrt{m}} < K,$$

$$a_n = O(a_m).$$

Also $$a_{n+1} = a_n + o(a_n/\sqrt{n}) = a_n + o(a_m/\sqrt{n}),$$

$$a_{m+r} = a_m\left\{1 + o\left(\sum_{m}^{m+r-1}\frac{1}{\sqrt{\nu}}\right)\right\} = a_m\{1 + o(r/\sqrt{m})\} = a_m\{1+o(1)\},$$

$$s_n - s_m = a_{m+1} + \ldots + a_{m+r} = ra_m\{1+o(1)\};$$

which establishes the result.†

41. We are now in a position to obtain upper limits for $S_1, S_2, \ldots$.

In S_1, we write $$|s_n - s_m| < Ke^{K\sqrt{m}},$$

and in S_5, $$|s_n - s_m| < Ke^{K\sqrt{n}};$$

and there is no serious difficulty in establishing the inequalities

(1) $$|S_1| < e^{-Km}, \quad |S_5| < e^{-Km}.‡$$

We next consider S_2 and S_4; let us select the latter. In this sum we use the inequality

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* We can use the inequality $1/(1-u) < e^{2u}$, which holds when u is small and positive, instead of $1+u < e^u$.

† The K's, and the additional K's and ϵ's implied in the O's and o's, in this proof, of course, depend upon α.

‡ See *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 6. The introduction of the factors $e^{K\sqrt{m}}$ or $e^{K\sqrt{n}}$, instead of m^K, n^K, will be found to make no material difference.

(*b*), which gives

$$S_4 = e^{-m}\sqrt{m}\,a_m\,O\left\{\sum_{\alpha\sqrt{m}}^{Hm} \frac{m^{m+r}}{(m+r)!}\,e^{Kr/\sqrt{m}}\right\}$$
$$= a_m\,O\left\{\sum_{\alpha\sqrt{m}}^{Hm} \exp\left(\frac{Kr}{\sqrt{m}} - \frac{r^2}{4m}\right)\right\}*$$
$$= a_m\,O\left\{\int_{\alpha\sqrt{m}}^{\infty} \exp\left(\frac{Kr}{\sqrt{m}} - \frac{r^2}{4m}\right)dr\right\}$$
$$= a_m\sqrt{m}\,O\left(\int_{\alpha}^{\infty} e^{Ku-\frac{1}{4}u^2}\,du\right).$$

It is essential to observe that the K which figures in this formula, and the further constant implied in the O, *are independent of* α. We may then suppose $\alpha > 8K > 1$, in which case

(2) $$S_4 = O\left(|a_m|\sqrt{m}\int_{\alpha}^{\infty} e^{-\frac{1}{8}u^2}\,du\right) = O(e^{-\frac{1}{8}\alpha^2}|a_m|\sqrt{m});$$

where the constant implied by O is still independent of α. In the same way we can express S_2 in the same form.

There remains S_3. To this we apply the result (*c*). We thus obtain

$$S_3 = e^{-m}a_m \sum_{-\alpha\sqrt{m}}^{\alpha\sqrt{m}} r\frac{m^{m+r}}{(m+r)!} + e^{-m}a_m\,o(1)\sum_{-\alpha\sqrt{m}}^{\alpha\sqrt{m}} |r|\frac{m^{m+r}}{(m+r)!} = S_3' + S_3'',$$

say. The work of our previous paper† shows that

(3) $$S_3'' = o(|a_m|\sqrt{m}).$$

The series which occurs in S_3' is

$$\sum_1^{\alpha\sqrt{m}} r\left\{\frac{m^{m+r}}{(m+r)!} - \frac{m^{m-r}}{(m-r)!}\right\} = \frac{m^m}{m!}\sum_1^{\alpha\sqrt{m}} r\left\{\frac{1}{\left(1+\frac{1}{m}\right)\cdots\left(1+\frac{r}{m}\right)} - \left(1-\frac{1}{m}\right)\cdots\left(1-\frac{r-1}{m}\right)\right\}$$
$$< \frac{m^m}{m!}\sum_1^{\alpha\sqrt{m}} r\left\{\frac{1-\left(1-\frac{1^2}{m^2}\right)\left(1-\frac{2^2}{m^2}\right)\cdots\left(1-\frac{r^2}{m^2}\right)}{\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\cdots\left(1+\frac{r}{m}\right)}\right\}.$$

Now, throughout the range of values considered,

$$\left(1+\frac{1}{m}\right)\cdots\left(1+\frac{r}{m}\right) > \exp\left(\frac{1}{2m}+\cdots+\frac{r}{2m}\right) > e^{r^2/4m},$$
$$1-\left(1-\frac{1^2}{m^2}\right)\cdots\left(1-\frac{r^2}{m^2}\right) < \frac{1^2}{m^2}+\cdots+\frac{r^2}{m^2} < \frac{r^3}{m^2}.$$

Hence

(4) $$S_3' = |a_m|\,O\left(\frac{m^{m-2}e^{-m}}{m!}\sum_1^{\alpha\sqrt{m}} r^4e^{-r^2/4m}\right) = |a_m|\,O\left(m^{-5/2}\int_0^{\infty} r^4e^{-r^2/4m}\,dr\right)$$
$$= O(|a_m|) = o(|a_m|\sqrt{m}).$$

From (3) and (4) it follows that

(5) $$S_3 = o(|a_m|\sqrt{m}).$$

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* Since $$\frac{m^{m+r}}{(m+r)!} < \frac{K}{\sqrt{m}}\exp\left(m - \frac{r^2}{4m}\right)$$

(cf. *Proc. London Math. Soc.*, Ser. 2, Vol., 11, p. 7).

† *L.c.*, p. 7.

From (1), (2), (5), and (1) of § 38, we deduce

(6) $$s-s_m+o(1) = O(e^{-\frac{1}{4}\alpha^2}|a_m|\sqrt{m})+o(|a_m|\sqrt{m}),$$

where the constant of the O is independent of α, and O, o refer to the tending of m to infinity after α is fixed.

As $e^{-\frac{1}{4}\alpha^2} \to 0$, as $\alpha \to \infty$, it follows that, given any positive δ, we can choose first a sufficiently large value of α and then a number m_0, so that

(7) $$|s-s_m+o(1)| < \delta |a_m| \sqrt{m},$$

for $m > m_0$.

Let n be the integer nearest to $m+\sqrt{m}$. Then $|a_n|/|a_m|$ is less than a constant.* Hence we may suppose m_0 so chosen that (7) is also satisfied when we write n for m in its left-hand member. Hence

$$|s_n-s_m+o(1)| < 2\delta |a_m| \sqrt{m}.$$

But we have already seen that

$$s_n-s_m = (n-m)\{1+o(1)\}a^m = \sqrt{m}\{1+o(1)\}a_m.$$

And the inequality $$|\sqrt{m}\{1+o(1)\}a_m+o(1)| < 2\delta|a_m|\sqrt{m}$$

is plainly contradictory, when δ is small, unless

$$a_m\sqrt{m} = o(1).$$

But then, in virtue of the Borel-Tauber theorem, Σa_n is convergent. Thus Theorem 31 is proved.†

The standard type of series for which the condition is satisfied is

$$\Sigma\, \phi(n)\, e^{i\psi(n)},$$

where $\phi(n)$ and $\psi(n)$ are L-functions, and

$$e^{-\delta\sqrt{n}} \prec \phi(n) \prec e^{\delta\sqrt{n}}, \quad 1 \prec \psi(n) \prec \sqrt{n}.$$

No such series can be summable (B) unless it is convergent; and

$$f(x) = \Sigma\, \phi(n)\, e^{i\psi(n)}\, x^n$$

cannot be regular for $x = 1$ unless the series is convergent for $x = 1$. In particular

$$\Sigma\, n^{-b} e^{in^a} \quad (0 < a < \tfrac{1}{2})$$

is summable (B) if, and only if, it is convergent. This we proved in our former paper, but by a special investigation.

42. Theorems 27 and 31 at once suggest a corresponding analogue of Fatou's theorem,‡ viz., *if* $\Delta a_n = o(a_n)$, *and* $f(x) = \Sigma a_n x^n$ *is regular for* $x = 1$, *then* $a_n = o(1)$ *and* Σa_n *is convergent*. This theorem is true, but trivial in the sense explained in § 37—the hypotheses are incompatible. In fact the conditions on a_n used in Theorems 27 and 31, and here, may be

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* See § 40, (*c*).

† The "general" form of Theorem 31 is no doubt true. But it would be futile to look for a proof until the general Borel-Tauber theorem is established.

‡ On the subject of this theorem see Hardy and Littlewood, *l.c. supra*, pp. 9–11, where references are given.

exhibited in the form

$$\frac{a_{n+1}}{a_n} = 1+O\left(\frac{1}{n}\right), \quad \frac{a_{n+1}}{a_n} = 1+o\left(\frac{1}{\sqrt{n}}\right), \quad \frac{a_{n+1}}{a_n} = 1+o(1),$$

and the last equation asserts simply that $a_{n+1}/a_n \to 1$. The point $x = 1$ is therefore a singular point of $f(x)$.*

43. While we are on the subject of these theorems involving Δa_n, it seems worth while to state the following two, which are easy deductions from known results.

THEOREM 32.—*If, for some integral* r, $\Delta^r a_n = o(1)$, *and* $f(x)$ *is regular for* $x = 1$, *then* $a_n = o(1)$ *and the series* Σa^n *is convergent.*

THEOREM 33.—*If* $\Delta a_n = O(1/n)$, *and* $(1-x)f(x) = o(1)$, *as* $x \to 1$, *then* $a_n = o(1)$.

To prove Theorem 32, we suppose first $r = 1$, and apply Fatou's theorem to

$$\Sigma \Delta a_n x^n = \{a_0-(1-x)f(x)\}/x.$$

A repetition of this argument leads to the result. Theorem 33 may be deduced similarly from the general Abel-Tauber theorem.

It follows, *e.g.*, from Theorem 32, that a series such as

$$\Sigma n^p e^{Ai(\log n)^q} x^n,$$

for which $\Delta^r a_n = o(1)$, for sufficiently large values of r, has certainly a singular point at $x = 1$.

Finally, the following theorem is interesting because of its relations to the general Abel-Tauber theorem.

THEOREM 34.—*If* $\Delta a_n = O(1/n^2)$, *and* $f(x) \to s$ *as* $x \to 1$, *then* Σa_n *is convergent.*

This is evidently only a special case of the Abel-Tauber theorem; and it is somewhat remarkable that the latter can be deduced from it in an elementary way. Suppose that Σa_n is a series for which $a_n = O(1/n)$. Then we have

$$f(x) = \Sigma a_n x^n = \Sigma \frac{t_n}{n(n+1)} x^{n+1} + (1-x)\Sigma \frac{t_n}{n} x^n,$$

where $$t_n = a_1+2a_2+\ldots+na_n;$$

or, say, $$f(x) = \Phi(x)+(1-x)\Phi'(x).$$

It can easily be shown† that $f(x) \to s$ involves

$$\Phi(x) \to s, \quad (1-x)\Phi'(x) \to 0.$$

Moreover, it will easily be verified that, if $a_n = O(1/n)$, then

$$\Delta \frac{t_n}{n(n+1)} = O\left(\frac{1}{n^2}\right), \quad \Delta \frac{t_n}{n} = O\left(\frac{1}{n}\right).$$

Applying Theorems 33 and 34, we see that $\Sigma\{t_n/n(n+1)\}$ is convergent, and $t_n/n = o(1)$. From this the convergence of Σa_n follows at once.

The form of Theorem 34 suggests that it might be easier to prove than the Abel-Tauber theorem (from the nature of the case it could not be more difficult). If so it would become of great interest. But our attempts in this direction have not been successful.

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* See § 3 of the Introduction.

† Cf. Hardy, *Quarterly Journal*, Vol. 43, p. 143.

V.

Miscellaneous Theorems.

The Multiplication of Series.

44. One of the most interesting of the properties of series whose n-th term is of order $1/n$ is their *multiplicative* character: if

$$(A)\quad a_1+a_2+\ldots, \qquad (B)\quad b_1+b_2+\ldots,$$

are two such series, and

$$(C)\quad c_1+c_2+\ldots,$$

is their product formed in accordance with Cauchy's rule, then the convergence of A and B is enough to ensure the convergence of C and the equality $AB = C$.*

The form of the theorem at once suggests that it must be connected with the Cesàro-Tauber theorem. If, indeed, it were true that the hypotheses

$$a_n = O(1/n), \qquad b_n = O(1/n),$$

implied

$$c_n = a_1 b_n + \ldots + a_n b_1 = O(1/n),$$

then, of course, the multiplication theorem would be an immediate corollary of the Cesàro-Tauber theorem—for the product of two convergent series is necessarily summable $(C1)$. But this is not the case: in fact, if

$$\Sigma a_n = \Sigma b_n = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots,$$

we find

$$(-1)^{n-1} c_n = \frac{2}{n+1}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right) \sim \frac{2\log n}{n}.$$

Thus the connection must lie somewhat deeper.

45. The clue to the connection is to be found in the negative indices of summability introduced by Chapman.†

In order to assimilate our notation to that of Cesàro and Chapman we shall, throughout this part of our work, take our series in the form $a_0+a_1+a_2+\ldots$. We shall write

$$(1-x)^{-r-1}\Sigma a_n x^n = \Sigma A_n^r x^n, \qquad (1-x)^{-r-1} = \Sigma M_n^r x^n,$$

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410, and *ibid.*, Vol. 10, p. 396. In the second of these papers the corresponding theorem is proved for the general form of Dirichlet's multiplication.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369.

so that Σa_n is summable (Cr) to sum s, if

$$A_n^r/M_n^r \to s,$$

or

$$\Gamma(r+1)\, A_n^r/n^r \to s.$$

Here r is any number greater than -1.*

46. Many of the characteristic properties of Cesàro's means still subsist when $-1 < r < 0$. In particular we may mention (1) the condition of consistency,† (2) the Cesàro multiplication theorem,‡ and (3) the theorem that, for a series summable (Cr), we must have

$$a_n = o(n^r).\S$$

But one must not assume too hastily that *all* interesting properties of summable series are capable of this generalisation. As an example let us take the following theorem, which is relevant to our present purpose.

THEOREM 35.—*If A is absolutely convergent, and B summable (Cr), where $r \geqslant 0$, then C is summable (Cr).*

This is a generalisation of Mertens' classical theorem, to which it reduces when $r = 0$.‖ The proof is very simple. We may, without real loss of generality, suppose $B = 0$, so that

$$B_n^r = o(n^r).$$

Choose n_0 so that $$|B_\nu^r| < \epsilon \nu^r \quad (\nu \geqslant \tfrac{1}{2}n_0).$$

Then $$C_n^r = a_0 B_n^r + a_1 B_{n-1}^r + \dots + a_n B_0^r,$$

$$|n^{-r} C_n^r| < \epsilon \sum_{\nu < \frac{1}{2}n} |a_\nu| + K \sum_{\frac{1}{2}n \leqslant \nu \leqslant n} |a_\nu| \quad (n \geqslant n_0),$$

which is less than $K\epsilon$, if $n \geqslant n_1 \geqslant n_0$, and n_1 is large enough.

This proof fails if $r < 0$, and the result fails too. Let us assume that (as will be proved in a moment) the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ is summable $(C, -1+\delta)$ for every positive δ. Then we have

THEOREM 36.—*It is possible to find an absolutely convergent series A and a series B summable $(C, -1+\delta)$, such that their product is not summable $(C, -\delta)$.*¶

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* It is obvious that all our results will have analogues for Riesz's methods of summation and Dirichlet's multiplication. We do not consider these here: there are certain preliminary complications which arise when we try to extend Riesz's results to negative indices, and the length of this paper makes it impossible that we should consider the matter further at present.

† Chapman, *l.c.*, p. 377.

‡ Chapman, *l.c.*, p. 378.

§ Chapman, *l.c.*, p. 379.

‖ The corresponding theorem for Dirichlet's multiplication is given by Hardy and Riesz, "The General Theory of Dirichlet's Series," one of the *Cambridge Tracts*, shortly to be published.

¶ Here, of course, δ is an arbitrarily small positive number.

We take $$A = 0+\frac{1}{1^2}+0+\frac{1}{2^2}+0+0+0+\frac{1}{3^2}+0+\ldots$$

(where $a_n = 1/\nu^2$, if $n = 2^\nu-1$, $\nu > 0$, and $a_n = 0$ otherwise), and $B = 1-\frac{1}{2}+\frac{1}{3}-\ldots$. We find

$$c_3 = 1\,\frac{1}{2^2}+\frac{1}{3}\,\frac{1}{1^2} > \frac{1}{2^2}, \qquad c_7 = 1\,\frac{1}{3^2}+\frac{1}{5}\,\frac{1}{2^2}+\frac{1}{7}\,\frac{1}{1^2} > \frac{1}{3^2},$$

$$c_{15} > \frac{1}{4^2},\ \ldots,\ c_{2^n-1} > \frac{1}{n^2},\ \ldots;$$

and plainly it is not true that $c_n = o(n^{-\delta})$ for any positive value of δ. Using the third theorem of Chapman referred to above, the theorem follows.

47. THEOREM 37.—*If $a_n = O(1/n)$ and Σa_n is summable (Cr), then Σa_n is summable $(C, -1+\delta)$ for any positive value of δ.*

By the Cesàro-Tauber theorem the series is convergent, and therefore, by Chapman's generalised condition of consistency, it is summable (C, δ). Hence the necessary and sufficient condition that it should be summable $(C, -1+\delta)$, is

(1) $$t_n^k = o(n^{k+1}),$$

where $k = -1+\delta$, and t_n^k is the Cesàro's sum of order $-1+\delta$ formed with $b_n = na_n$: that is to say,

(1′) $$t_n^k = \sum_{p=0}^{n-1} \frac{(k+1)(k+2)\ldots(k+p)}{p!}\,b_{n-p}\text{*}$$

$$= \sum_0^{\nu-1} + \sum_\nu^{n-1} = S_1+S_2,$$

say. It will make the proof clearer if we explain at once that we are going to take $\nu \sim \theta n$, where θ is a number a sufficiently small value of which will be fixed before n is made to tend to infinity.

In the first place, as $b_n = O(1)$, we have

(2) $$S_1 = O\left\{\sum_0^{\nu-1}\frac{(k+1)\ldots(k+p)}{p!}\right\} = O\left\{\frac{(k+2)\ldots(k+\nu)}{(\nu-1)!}\right\} = O(\nu^{k+1}).$$

The constant of the O is independent of θ.

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 304. In this paper only integral orders of summability are considered, but, so far as the condition quoted is concerned, the work is independent of this restriction.

Again,

(3) $\Gamma(k+1)\,S_2$

$$= \sum_1^{n-\nu} \frac{\Gamma(k+n-q+1)}{\Gamma(n-q+1)} b_q$$

$$= \sum_1^{n-\nu-1} B_q \left\{ \frac{\Gamma(k+n-q+1)}{\Gamma(n-q+1)} - \frac{\Gamma(k+n-q)}{\Gamma(n-q)} \right\} + \frac{\Gamma(k+\nu+1)}{\Gamma(\nu+1)} B_{n-\nu}$$

$$= S_2' + S_2'',$$

say. Now $B_q = o(q)$, since Σa_n is convergent. Hence, first,

(4) $$S_2'' = o(n\nu^k),$$

uniformly with respect to θ. Finally,

$$S_2' = O\left\{ \sum_1^{n-\nu-1} o(q) \frac{\Gamma(k+n-q)}{\Gamma(n-q+1)} \right\} = o\left\{ \sum_1^{n-\nu-1} q(n-q)^{k-1} \right\}.$$

But $$\sum_1^{n-\nu-1} q(n-q)^{k-1} = n^k \sum_1^{n-\nu-1} \frac{q}{n}\left(1-\frac{q}{n}\right)^{k-1}$$

$$= O\left\{ n^{k+1} \int_0^{1-\theta} u(1-u)^{k-1}\,du \right\},$$

and so

(5) $$S_2' = o\{n^{k+1} K(\theta)\},$$

where $K(\theta)$ is a number dependent on θ alone and the o is uniform in respect to θ.*

From (1′)–(5) it follows that

(6) $$t_n^k = O(\nu^{k+1}) + o(n\nu^k) + o\{n^{k+1} K(\theta)\}$$
$$= n^{k+1}\{\theta^{k+1} O(1) + o(\theta^k) + o(K(\theta))\}.$$

First choose θ so that $\theta^{k+1} O(1) < \epsilon$. When θ is fixed we can choose n_0 so that

$$|o(\theta^k)| < \epsilon, \quad |o(K(\theta))| < \epsilon,$$

$$|t_n^k| < 3\epsilon n^{k+1},$$

for $n > n_0$. Thus (1) is established and the theorem proved.

* $K(\theta) \to \infty$ as $\theta \to 0$. So does θ^k in (6) below. No difficulty is caused by this, but the uniformity of the O's and o's in respect to θ is essential to the proof.

48. It appears, then, that all Cesàro's means, of index greater than -1, are equivalent in regard to series whose general term is of order $1/n$.*

As a corollary we deduce

THEOREM 38.—*The product of any number of convergent series for which $a_n = O(1/n)$ is convergent.*

This follows at once from the fact that the index of summability of the product of two such series is $-1-1+1=-1$.† We might say, more generally, "the product of any number of such series and any number of absolutely convergent series"—for we can multiply the absolutely convergent and the non-absolutely convergent series separately, and then form the product of their products, the final result being independent of the order of multiplication.‡

49. The definition of Cesàro summability breaks down when r is a negative integer, and the concept does not seem to be a useful one for any value of r less than -1.§ The question is, however, suggested whether it is not possible to frame a satisfactory definition in the case $r=-1$. A series summable $(C, -1)$ might be expected to have the property of giving a convergent product when multiplied by *any* convergent series.

It is, however, certain that series whose general term is $O(1/n)$ do not necessarily possess this property. In fact, it is easy to see that the general term of the product of

$$1-\frac{1}{2}+\frac{1}{3}-\ldots, \qquad \frac{1}{\sqrt{\log 3}}-\frac{1}{\sqrt{\log 4}}+\frac{1}{\sqrt{\log 5}}-\ldots$$

does not tend to zero.

In this connection the following result is interesting.

THEOREM 39.—*Given any numbers α, β, such that $\alpha<1$, $\beta<1$, it is possible to find two convergent series Σa_n, Σb_n, such that*

$$a_n = o(n^{-\alpha}), \qquad b_n = o(n^{-\beta}),$$

and the product series is not convergent.‖

The example required is to be found by considering the square of the series

$$\Sigma n^{-b} e^{in^a},$$

where $0<a<1$, $b<1$. The series itself is summable (Ck) if $(k+1)a+b>1$.

---

* That a convergent series of positive decreasing terms is summable $(C, -1+\delta)$ was proved by Chapman (*l.c.*, p. 405). It was this result which suggested Theorem 37. The general term of such a series is, of course, of the form $o(1/n)$.

† Hardy (*Proc. London Math. Soc.*, Ser. 2. Vol. 6, p. 414) proved only that the product of *two* such series was convergent; it did not follow from his argument that, *e.g.*, the series formed by cubing the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$ was convergent: and as the property expressed by $a_n = O(1/n)$ is lost in the process of multiplication, his argument could not be repeated.

‡ This was pointed out to us by Mr. Chapman.

§ Cf. Chapman, *l.c.*, p. 376.

‖ This theorem answers a question raised by Hardy, *l.c.*, p. 419.

The behaviour of the general term of the squared series, viz.,

$$c_n = \sum_1^n \frac{e^{i\{\nu^a+(n+1-\nu)^a\}}}{\{\nu(n+1-\nu)\}^b},$$

may be shown to be similar to that of the integral

$$\int_0^x \frac{e^{i\{t^a+(x-t)^a\}}}{\{t(x-t)\}^b}\,dt = x^{1-2b}F(x^a),$$

where $x = n$, and

$$F(z) = \int_0^1 \frac{e^{zi\{u^a+(1-u)^a\}}}{\{u(1-u)\}^b}\,du.$$

The behaviour of this function for large positive values of z may be studied by methods depending on Cauchy's theorem.* It can be proved that

$$F(z) = Kz^{-\frac{1}{2}}e^{2^{1-a}zi}(1+o(1)) + Kz^{-(1-b)/a}e^{zi}(1+o(1)).\dagger$$

From this it is easy to deduce that the squared series is convergent or summable if, and only if, *both* of the series

$$(1) \qquad \sum n^{1-\frac{1}{2}a-2b}e^{2^{1-a}in^a}, \qquad \sum n^{-b}e^{in^a},$$

of which the second is the same as the original series, are convergent or summable. We can now distinguish two cases.

(i) If $b \geqslant 1-\frac{1}{2}a$,‡ the convergence or summability of the second of the series (1) carries with it that of the first. In this case *the index of summability of the squared series is the same as that of the original series.* Now the index k of the original series is given by

$$(k+1)a+b = 1.$$

Thus this state of affairs occurs if, and only if, $k \leqslant -\frac{1}{2}$.

(ii) If $b < 1-\frac{1}{2}a$, the first of the series (1) is the more important. Thus the index of summability l of the squared series is given by

$$(l+1)a+\tfrac{1}{2}a+2b-1 = 1,$$

so that

$$l = 2k+\tfrac{1}{2}.$$

The index of summability of the squared series has a value less by $\frac{1}{2}$ than the value $2k+1$ assigned by Cesàro's multiplication theorem.

In particular, *the squared series is convergent if $b > 1-\frac{3}{4}a$.* If

$$1-a < b < 1-\tfrac{3}{4}a,$$

---

* We are not professing to do more than sketch the general lines of the proof of Theorem 39.

† Each $o(1)$ is in reality an asymptotic series of descending powers of z. The actual values of the K's are

$$2^{2(b-1)}\sqrt{\left\{\frac{2^a\pi}{ia(1-a)}\right\}}, \qquad \frac{2}{a}\Gamma\left(\frac{1-b}{a}\right)i^{(1+a-b)/a}.$$

where the powers of i have their principal values.

‡ This condition is the same as the condition that $f(x) = \sum n^{-b}e^{in^a}x^n$ should be bounded in the unit circle.

the original series is convergent, but not the squared series. In order to obtain an example such as is required by the theorem, we have only to take a sufficiently small.

50. Before leaving the subject of multiplication we may mention one or two results whose proof is easy, and which possess no particular theoretical interest, but which are sometimes useful in practice. It is sufficient for the convergence of C that

$$a_n = O(n^{-\alpha}), \quad B_n - B = o(n^{-\beta}),$$

where $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, $\alpha + \beta \geqslant 1$. The O and the o may be interchanged. The condition $B_n - B = o(n^{-\beta})$ is certainly satisfied if $\Sigma n^{\beta} b_n$ is convergent; and, if $\beta < 1$, either of these conditions involves the summability $(C, -\beta)$ of the series B.

Tauber's Theorem for Double Series.

51. In this sub-section we propose to investigate the analogue of Tauber's theorem for double series. We shall also prove the analogue of the generalised form of the theorem, given by Tauber himself and Pringsheim:* the theorem that

$$a_1 + 2a_2 + \ldots + na_n = o(n),$$

together with the existence of Abel's limit, involves the convergence of Σa_n. We must begin by a few preliminary explanations.

When we say that a function of two variables, $a_{m,n}$ or $f(x, y)$, tends to a limit, we mean always a Pringsheim double limit; we shall not be concerned with *repeated* limits. The existence of a double limit for, *e.g.*, $a_{m,n}$ does not imply the existence of a simple limit when m or n tends separately to infinity. Nor does it imply the inequality

$$|a_{m,n}| < K$$

for all values of m and n.†

This last inequality, which plays an important part in all these questions, we call the *condition of finitude*.‡ When the double limit exists, the condition of finitude may or may not be satisfied. It is important to distinguish these two cases in our notation. We shall write the condition of finitude in the form

$$a_{m,n} = O(1).$$

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* See Pringsheim, *Münchener Sitzungsberichte*, Vol. 30, p. 37, and Vol. 31, p. 507.

† For the elements of the theory of double limits and series, see Pringsheim, *Münchener Sitzungsberichte*, Vol. 27, p. 101, and *Math. Ann.*, Vol. 53, p. 289; London, *Math. Ann.*, Vol. 53, p. 322; Bromwich, *Infinite Series*, Chap. 5.

‡ Cf. Bromwich and Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 163.

If the condition of finitude is satisfied, and $a_{m,n} \to 0$, we shall write

$$a_{m,n} = o(1).$$

Generally, by $a_{m,n} = o(\phi)$, we imply (i) that $a_{m,n} = O(\phi)$, and (ii) that $a_{m,n}/\phi \to 0$. It is plain that similar definitions may be given for functions of two continuous variables which tend to limiting values.

We shall find it convenient to assume throughout that $a_{m,n} = 0$, if either m or n is zero.

The reader will find no difficulty in verifying the following lemmas.

LEMMA 1.—*If* $a_{m,n} = o(1)$, *then*

$$\sum_1^m a_{\mu,n} = o(m), \qquad \sum_1^n a_{m,\nu} = o(n), \qquad \sum_1^m \sum_1^n a_{\mu,\nu} = o(mn).$$

LEMMA 2.—*If* $a_{m,n} = o(1)$, *then*

$$(1-x)(1-y)\,\Sigma a_{m,n} x^m y^n = o(1).$$

In the second lemma it is supposed that $0 < x < 1$, $0 < y < 1$, and $x \to 1$, $y \to 1$. Neither lemma is true if we substitute $a_{m,n} \to 0$ for $a_{m,n} = o(1)$.*

We shall write

$$\sum_1^n a_{m,\nu} = p_{m,n}, \qquad \sum_1^m a_{\mu,n} = q_{m,n}, \qquad \sum_1^m \sum_1^n a_{\mu,\nu} = s_{m,n},$$

$$\sum_1^m \mu p_{\mu,n} = \sum_1^m \sum_1^n \mu a_{\mu,\nu} = \rho_{m,n}, \qquad \sum_1^n \nu q_{m,\nu} = \sum_1^m \sum_1^n \nu a_{\mu,\nu} = \sigma_{m,n},$$

$$\sum_1^m \sum_1^n \mu\nu a_{\mu,\nu} = t_{m,n}.$$

It is obvious from Lemma 1 that

(1) $mp_{m,n} = o(1)$, (2) $nq_{m,n} = o(1)$, (3) $mna_{m,n} = o(1)$,

imply respectively

(1′) $\rho_{m,n} = o(m)$, (2′) $\sigma_{m,n} = o(n)$, (3′) $t_{m,n} = o(mn)$.

---

* Thus, if $a_{m,1} = m$, $a_{m,n} = 0\ (n > 1)$, we find

$$\frac{1}{mn} \sum_1^m \sum_1^n a_{\mu,\nu} = \frac{m+1}{2n}.$$

The condition $a_{m,n} \to 0$ does not even involve the convergence of the power series for $x < 1$, $y < 1$ (cf. Bromwich and Hardy, *l.c. supra* p. 166).

Again, it is easily verified that

$$t_{m,n} = (n+1)\rho_{m,n} - \sum_1^n \rho_{m,\nu} = (m+1)\sigma_{m,n} - \sum_1^m \sigma_{\mu,n},$$

$$\sum_1^n \nu a_{m,\nu} = (n+1)p_{m,n} - \sum_1^n p_{m,\nu}, \qquad \sum_1^m \mu a_{\mu,n} = (m+1)q_{m,n} - \sum_1^m q_{\mu,n}.$$

Hence it follows that either of (1′) or (2′) implies (3′), and *a fortiori* either of (1) or (2) implies (3′), and that (1) and (2) imply respectively

$$(1'') \quad m\sum_1^n \nu a_{m,\nu} = o(n), \qquad (2'') \quad n\sum_1^m \mu a_{\mu,n} = o(m).$$

52. We can now prove

THEOREM 40.—*If* (1) $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$, *and*

$$(2) \quad f(x, y) = \Sigma a_{m,n} x^m y^n \to A,$$

as $x \to 1$, $y \to 1$, *then* $\Sigma a_{m,n}$ *is convergent to sum* A.

We shall first show that the conclusion holds if conditions (1), (2), and (3) of § 51, are all satisfied. We shall then show that condition (3) may be dispensed with.

We suppose $$x = 1 - \frac{1}{\mu}, \quad y = 1 - \frac{1}{\nu},$$

and write $$f(x, y) = \left(\sum_1^\mu \sum_1^\nu + \sum_1^\mu \sum_{\nu+1}^\infty + \sum_{\mu+1}^\infty \sum_1^\nu + \sum_{\mu+1}^\infty \sum_{\nu+1}^\infty\right) a_{r,s} x^r y^s$$

$$= \phi_1 + \phi_2 + \phi_3 + \phi_4,$$

say. Choose μ and ν so that $|rsa_{r,s}| < \epsilon$ for $r \geqslant \mu$, $s \geqslant \nu$. Then

$$|\phi_4| < \frac{\epsilon}{\mu\nu(1-x)(1-y)} = \epsilon,$$

so that $\phi_4 \to 0$. Again,

$$\phi_2 = \sum_{\nu+1}^\infty y^s \sum_1^\mu a_{r,s} x^r = (1-x)\sum_{\nu+1}^\infty y^s \sum_1^{\mu-1} q_{r,s} x^r + x^\mu \sum_{\nu+1}^\infty q_{\mu,s} y^s$$

$$= \phi_2' + \phi_2'',$$

* This theorem seems tc be the appropriate form of the analogue of Tauber's theorem. The introduction of the conditions $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$ is due to Dr. W. H. Young, who was the first to show that a double series which satisfies them cannot be summable by Cesàro's means without being convergent (see Young, "On Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 167).

say. Choosing μ and ν so that $|sq_{r,s}| < \epsilon$, for $r \geqslant \mu$, $s \geqslant \nu$, we see, by an argument similar to that used above for ϕ_4, that $\phi_2'' \to 0$. And

$$|\phi_2'| < (1-x) \sum_{\nu+1}^{\infty} y^s \sum_{1}^{\lambda-1} |q_{r,s}| + (1-x) \sum_{\nu+1}^{\infty} y^s \sum_{\lambda}^{\mu-1} |q_{r,s}| x^r.$$

Now $|sq_{r,s}| < K$ for all values of r and s, and we can choose λ so that $|sq_{r,s}| < \epsilon$ for $r \geqslant \lambda$, $s \geqslant \nu$. Then

$$|\phi_2'| < \frac{K(1-x)}{\nu(1-y)} + \frac{\epsilon}{\nu(1-y)} = \frac{K}{\mu} + \epsilon,$$

and so $\phi_2' \to 0$. Thus $\phi_2 \to 0$, and similarly $\phi_3 \to 0$. Hence

(1) $$\phi_1 \to A.$$

But $$\sum_1^{\mu} \sum_1^{\nu} a_{r,s} - \phi_1 = \sum_1^{\mu} \sum_1^{\nu} a_{r,s} \{(1-x^r)+(1-y^s)-(1-x^r)(1-y^s)\}$$
$$= \chi_1 + \chi_2 + \chi_3,$$

say. In the first place

$$|\chi_3| < (1-x)(1-y) \sum_1^{\mu} \sum_1^{\nu} rs\, |a_{r,s}| = \frac{1}{\mu\nu} \sum_1^{\mu} \sum_1^{\nu} rs\, |a_{r,s}|,$$

and so $\chi_3 \to 0$. In the second place

$$|\chi_1| = \left| \sum_1^{\mu} p_{r,\nu} (1-x^r) \right| < (1-x) \sum_1^{\mu} r\, |p_{r,\nu}| = \frac{1}{\mu} \sum_1^{\mu} r\, |p_{r,\nu}|,$$

and so $\chi_1 \to 0$. Similarly $\chi_2 \to 0$. Hence

$$s_{\mu,\nu} \to A,$$

and the result of Theorem 40 is established if conditions (1), (2), and (3) of § 51 are all satisfied.

58. Now $$f(x, y) = \sum_1^{\infty} y^n \sum_1^{\infty} m a_{m,n} (x^m/m)$$
$$= \sum_1^{\infty} y^n \sum_1^{\infty} (a_{1,n} + 2a_{2,n} + \dots + m a_{m,n}) \,\Delta \frac{x^m}{m} = S_1 + S_2,$$

where $$S_1 = \Sigma \frac{g_{m,n}}{m(m+1)} x^m y^n, \qquad S_2 = (1-x) \Sigma \frac{g_{m,n}}{m+1} x^m y^n,$$

$$g_{m,n} = a_{1,n} + 2a_{2,n} + \dots + m a_{m,n}.$$

In the first place $$S_2 = (1-x)(1-y) \Sigma \gamma_{m,n} x^m y^n,$$

where $$\gamma_{m,n} = \frac{1}{m+1}\sum_1^m\sum_1^n r a_{r,s} = \frac{\rho_{m,n}}{m+1} = o(1).$$

Hence, by Lemma 2 of § 51, $S_2 \to 0$, and so $S_1 \to A$. We shall now show that, if $mp_{m,n} = o(1)$, $nq_{m,n} = o(1)$, the coefficients

$$a_{m,n} = \frac{g_{m,n}}{m(m+1)}$$

of S_1 satisfy the *three* conditions corresponding to (1), (2), (3) of § 51. We can then, after what was established in the last paragraph, infer the convergence of S_1 for $x = 1$, $y = 1$.

Denoting by $\varpi_{m,n}$, $\kappa_{m,n}$ the sums formed from $a_{m,n}$ as $p_{m,n}$, $q_{m,n}$ were from $a_{m,n}$, we find first that

$$m\varpi_{m,n} = \frac{1}{m+1}\sum_1^n g_{m,s} = \frac{1}{m+1}\sum_1^m\sum_1^n r a_{r,s} = \frac{\rho_{m,n}}{m+1} = o(1),$$

by (1′) of § 51. Secondly,

$$n\kappa_{m,n} = n\sum_1^m\left(\frac{1}{\mu} - \frac{1}{\mu+1}\right)(a_{1,n} + 2a_{2,n} + \ldots + \mu a_{\mu,n})$$

$$= n\left\{\left(1 - \frac{1}{m+1}\right)a_{1,n} + \left(\frac{1}{2} - \frac{1}{m+1}\right)2a_{2,n} + \ldots + \left(\frac{1}{m} - \frac{1}{m+1}\right)ma_{m,n}\right\}$$

$$= nq_{m,n} - \frac{n}{m+1}\sum_1^m r a_{r,n} = o(1),$$

by (2″) of § 51. Finally,

$$mna_{m,n} = \frac{n}{m+1}\sum_1^m r a_{r,n} = o(1).$$

Hence S_1 converges for $x = 1$, $y = 1$. But it will be found, by a transformation similar to that used immediately above, that

$$\sum_1^m\sum_1^n \frac{a_{1,s} + 2a_{2,s} + \ldots + r a_{r,s}}{r(r+1)} = \sum_1^m\sum_1^n a_{r,s} - \frac{\rho_{m,n}}{m+1}.$$

Therefore $\Sigma a_{m,n}$ is convergent, and Theorem 40 is proved.

It will be observed that (as in the ordinary proof of Tauber's theorem) we have in reality proved a good deal more than the convergence of the series. We have proved, *e.g.*, that if $f(x, y) \to A$, when $x, y \to 1$ through a particular sequence of values $1 - 1/m$, $1 - 1/n$, then $s_{m,n} \to A$ when m and n tend to infinity through the corresponding sequence.

54. We can easily establish a more general theorem.

THEOREM 41.—*If* (1) $\rho_{m,n} = o(m)$, $\sigma_{m,n} = o(n)$, *and* (2) $f(x, y) \to A$, *then* $\Sigma a_{m,n}$ *converges to the sum* A.

If we refer back to the proof of Theorem 40, it will be seen that all that was actually assumed in § 53 is that

$$\rho_{m,n} = o(m), \quad nq_{m,n} = o(1).$$

These conditions are therefore sufficient, and so, of course, are

$$mp_{m,n} = o(1), \quad \sigma_{m,n} = o(n).$$

We shall now prove that these conditions are satisfied by the series S_1 of § 53 whenever the conditions of the present theorem are satisfied by the original series. It will then follow that S_1 is convergent for $x = 1$, $y = 1$, and the proof may be completed as before.

The verification of the first test is precisely the same as before. Hence all that we have to show is that

$$\sum_1^m \sum_1^n \frac{s(a_{1,s}+2a_{2,s}+\ldots+ra_{r,s})}{r(r+1)} = o(n).$$

But it will easily be verified that this expression is equal to

$$\sigma_{m,n} - t_{m,n}/(m+1) = o(n),$$

by (3′) of § 51.

THEOREM 42.—*If* $(m+n)^2 a_{m,n} = o(1)$, *and* $f(x, y) \to A$, *then* $\Sigma a_{m,n}$ *converges to the sum* A.

This theorem is merely a special case of Theorem 40.* It is interesting on account of its simplicity, and also because, when this special condition is satisfied, we can prove rather more. Suppose that we are given only that

$$f(x, x) \to A.$$

Then
$$f(x, x) = \Sigma c_n x^{n+1},$$

where
$$c_n = a_{n,1} + a_{n-1,2} + \ldots + a_{1,n} = o(1/n).\dagger$$

Hence, by Tauber's theorem, the *diagonal series* of $\Sigma a_{m,n}$ converges. That the double series is convergent follows from another general theorem, whose proof we may leave to the reader.

THEOREM 43.—*If* $(m+n)^2 a_{m,n} = o(1)$, *the convergence of the diagonal series involves that of the double series, and conversely.*‡

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* It is the form of the theorem that we had adopted before we saw Dr. Young's conditions.

† The proof of this is almost obvious.

‡ As all the rows and columns converge, the repeated series also converge to the same sum.

Two final remarks may be added before we leave this subject. It is not unnatural to suppose that the condition

$$(1) \qquad mna_{m,n} = o(1)$$

might prove to be sufficient for the truth of the analogue of Tauber's theorem. It is, however, possible to construct such a series, for which $f(x, x) \to A$ while the diagonal series oscillates: and it can be shown that, in virtue of (1), the double series also must oscillate. Thus the truth of the suggestion is made to appear exceedingly improbable.

Lastly, as regards the analogue for double series of the Cesàro-Tauber theorem, Dr. Young has shown that the conditions (1) and (2) of § 51, together with the summability $C(k, l)$ of the series, involve its convergence. It is easy to prove directly that the more general conditions (1′), (2′) are also sufficient when $k = l = 1$; and there is no doubt of the correctness of the extension of this result to series summable by more general Cesàro's means, though we have not troubled to work this out. No doubt also (as is suggested by Dr. Young) the corresponding theorems with O instead of o can be established with comparatively little trouble. The extension of Theorems 40 and 41 to this case, on the other hand, would in all probability be exceedingly troublesome.

It would be natural to try to exhibit the analogues of the Cesàro-Tauber theorem as corollaries of Theorems 40 and 41, using the extensions of the theorems of Abel, Frobenius, and Hölder, given by Bromwich and Hardy in these *Proceedings*.* But difficulties arise owing to the *rôle* played by the "condition of finitude" in these theorems, and the relations between them are less evident than in the case of simple series.

Additional Extensions of the Ordinary Form of Tauber's Theorem.

55. The argument by which the usual form of Tauber's theorem is established may be adapted to prove a more general theorem.

THEOREM 44.—*Suppose that ϕ is a real or complex function of its argument, and that*

$$f(x) = \Sigma a_n x^n \sim \phi\left(\frac{1}{1-x}\right),$$

as $x \to 1$ by real values. Suppose further that it is possible to find a function ψ which satisfies the conditions

(i) *$\psi(x)$ is positive and increasing, and ψ/x decreasing,*

(ii) $\psi = O(\phi)$,

(iii) $na_n = o\{\psi(n)\}$.

Then

$$a_0 + a_1 + \ldots + a_n \sim \phi(n),$$

as $n \to \infty$. In conditions (ii) *and* (iii) *the O and the o may be interchanged.*

When $\psi = \phi = 1$, this is equivalent to Tauber's theorem.

We have, when $x = 1 - 1/n$, and we take the first form of the conditions,

$$\sum_0^n a_\nu - f\left(1 - \frac{1}{n}\right) = \sum_0^n a_\nu (1 - x^\nu) - \sum_{n+1}^\infty a_\nu x^\nu = (1-x) \sum_0^n o\{\psi(\nu)\} + \frac{o\{\psi(n)\}}{n(1-x)}$$

$$= (1-x)\, o\{n\psi(n)\} + o\{\psi(n)\} = o\{\psi(n)\};$$

* Bromwich and Hardy, *l.c.*, *supra*; see also Bromwich, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 67.

and the result follows immediately. Similarly when the second form of the conditions is taken. Thus, if

$$f(x) = \sum_1^\infty \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right),$$

we could take $\psi = 1$, and deduce that

$$1+\tfrac{1}{2}+\ldots+\frac{1}{n} \sim \log n.$$

On the other hand, if $a_n = n^{-s}$, where $0 < s < 1$, we have $\phi(n) = \Gamma(1-s)\, n^{1-s}$, and it is impossible to choose ψ so as to satisfy the conditions. In this case

$$1+\frac{1}{2^s}+\ldots+\frac{1}{n^s} \sim \frac{n^{1-s}}{1-s},$$

so that the result of the theorem is untrue.

In any particular case the equation

$$s_n = f\left(1-\frac{1}{n}\right) + o\{\psi(n)\}$$

(or the corresponding equation with an O) would convey, as a rule, more precise information than is stated in the conclusion of the theorem.

From Theorem 44 we can easily deduce a more general theorem.

THEOREM 45.—*The result of Theorem* 44 *is still true if condition* (iii) *is replaced by the more general condition*

$$\text{(iii}')\quad a_1+2a_2+\ldots+na_n = o(n\psi).$$

A similar modification may be made in the second form of the theorem.

For, if

$$a_1+2a_2+\ldots+na_n = t_n,$$

we have

$$f(x)-a_0 = \sum_1^\infty a_n x^n = \sum_1^\infty t_n \Delta \frac{x^n}{n} = \sum \frac{t_n x^n}{n(n+1)} + (1-x)\sum \frac{t_n x^n}{n} = f_1+f_2,$$

say. Now

$$f_2 = o\{(1-x)\sum \psi(\nu)x^\nu\} = o\left\{(1-x)\sum_1^n \psi(\nu) + (1-x)\frac{\psi(n)}{n}\sum_{n+1}^\infty \nu x_\nu\right\}$$

$$= o\left\{(1-x)\, n\psi(n) + \frac{\psi(n)}{(1-x)\, n}\right\} = o\{\psi(n)\},$$

if $x = 1-1/n$. And so, as $x \to 1$ through these values,

$$f_1(x) = \sum \frac{t_n x^n}{n(n+1)} \sim \phi\left(\frac{1}{1-x}\right).$$

But this series satisfies the conditions of Theorem 44. Hence

$$\sum_1^n \frac{t_\nu}{\nu(\nu+1)} = \phi(n) + o\{\psi(n)\},$$

$$\sum_0^n a_\nu = a_0 - a_{n+1} + \sum_1^{n+1} a_\nu = \sum_1^n \frac{t_\nu}{\nu(\nu+1)} - \frac{t_{n+1}}{n+1} + a_0 - a_{n+1}$$

$$= \phi(n) + o\{\psi(n)\} \sim \phi(n).$$

Similarly we can prove the alternative form of the theorem.

56. Theorems 44 and 45 have, of course, analogues for the most general form of Dirichlet's series. We content ourselves with stating the analogue of Theorem 45 for ordinary Dirichlet's series.

THEOREM 46.—*Suppose that*

$$f(s) = \Sigma a_n n^{-s} \sim \phi(1/s),$$

as $s \to 0$ *by real values. Suppose also that we can find a function* ψ *such that*

(i) ψ *increases and* $\psi/\log n$ *decreases,*

(ii) $\psi(n) = O\{\phi(\log n)\}$,

(iii) $n \log n \, a_n = o(\psi)$,

or, more generally,

(iii') $$\sum_1^n \log \nu \, a_\nu = o(\log n \, \psi).$$

Then $$\sum_1^n a_\nu \sim \phi(\log n).$$

In conditions (ii) *and* (iii), *or* (iii'), *the O and o can be interchanged.*

Suppose, for example, that $a_n = 1/n$, if n is a prime p, and $a_n = 0$ otherwise. Then

$$f(s) = \Sigma \frac{1}{p^{1+s}} \sim \log\left(\frac{1}{s}\right),$$

$$\sum_1^n \log \nu \, a_\nu = \sum_{p \leqslant n} \frac{\log p}{p} = O(\log n).$$

We take $\psi = 1$, using conditions (ii) (with o) and (iii') (with O), and deduce

$$\sum_{p \leqslant n} \frac{1}{p} \sim \log \log n.$$

A more precise discussion would lead to the well known result

$$\sum_{p \leqslant n} \frac{1}{p} = \log \log n + O(1).$$

By a similar argument we can deduce the more accurate equation*

$$\sum_{p \leqslant n} \frac{1}{p} = \log \log n + A + o(1),$$

* Landau, *Handbuch*, p. 102. The result can also be deduced in a very elegant manner from Landau's form of Tauber's theorem for Dirichlet's series (*Monatshefte für Math. u. Phys.*, Vol. 18, p. 12), to which Theorem 46 reduces for $\psi = \phi = 1$. We take

$$a_n = \frac{1}{n} - \frac{1}{nln} \quad (n = \varpi), \qquad a_n = -\frac{1}{nln} \quad (otherwise),$$

so that $$\frac{l1\, a_1 + l2\, a_2 + \ldots + ln\, a_n}{ln} = \frac{1}{ln} \sum_{p < n} \frac{lp}{p} - \frac{1}{ln} \sum_1^n \frac{1}{\nu} = o(1).$$

Also $$\Sigma a^n n^{-s} = \Sigma \frac{1}{p^{1+s}} - \Sigma \frac{1}{n^{1+s} ln} \to A,$$

as $s \to 0$; whence the result.

where A is a constant, from the equations

$$f(s) = \log\left(\frac{1}{s}\right) + A + o(1), \qquad \sum_{p \leqslant n} \frac{\log p}{p} \sim \log n.$$

57. We shall conclude this paper with some remarks relating to the extension of Abel's and Tauber's theorems to the case in which $x \to 1$ along a path which *touches* the unit circle.

Let r, θ be polar coordinates. We shall call a curve C, passing from the inside of the circle to the point 1, *a regular path of order* a, if (i) r, regarded as a function of θ, has a continuous first derivative,*

$$\text{(ii)} \quad 1-r \sim H\theta^a,$$

as $r \to 1$, $\theta \to 0$. Thus a "Stolz-path," *i.e.*, a path such as is contemplated in Stolz's extension of Abel's theorem, is of order 1. The circle described on (0, 1) as diameter is of order 2, and so on. A path such as $r = 1-e^{-1/\theta}$ or $r = 1-e^{-(1/\theta)^2}$ may be described as *of infinite order*.

So long as we are concerned with paths of zero order (*i.e.*, paths which do not touch the circle) the consideration of complex paths introduces no new difficulty of a serious nature.† But as soon as we consider paths tangential to the circle, Abel's theorem ceases to hold. In this connection we have proved the following theorems.

THEOREM 47.—*It is possible to find a convergent series Σa_n such that $f(x) = \Sigma a_n x^n$ does not tend to a limit when $x \to 1$ along any circle interior to and touching the unit circle.*

An example is afforded by the series $\Sigma n^{-b} e^{in^a}$, where $0 < a < \frac{1}{2}$. In this case $f(x)$ tends to a limit when x approaches 1 along the upper half of the smaller circle, but oscillates when it approaches along the lower half.

THEOREM 48.—*If $s_n = A + o(n^{-\beta})$, then Abel's limit exists along any regular path of order a, where $a \leqslant 1/(1-\beta)$. In particular it exists if*

* This condition is, of course, quite unnecessarily restrictive. But the adoption of a more general definition would confuse rather than help to elucidate the results we wish to discuss.

† For proofs of the generalised forms of Abel's and Tauber's theorems see Bromwich, *Infinite Series*, p. 211 and p. 251. We have proved that the summability of Σa_n by Borel's method is enough to ensure the existence of Abel's limit along any "Stolz-path."

*$\Sigma n^{\beta} a_n$ is convergent. If $s_n = A + o(1/n)$, and in particular if $\Sigma n a_n$ is convergent, it exists along any regular path of finite order.**

We do not propose to write out proofs of these theorems. That of Theorem 47 would carry us too far from our subject, while that of Theorem 48 should be easily supplied by the reader. We mention them because they indicate a state of affairs in view of which it is rather remarkable that Tauber's theorem should hold in the form in which we propose now to prove it.

We suppose now that C is any regular path of finite or infinite order. That is to say, we suppose only that it is defined by a relation between r and θ, in virtue of which r has a continuous derivative, and we place no limitation on the closeness of its contact with the unit circle. We suppose, however, that it has only the one point $x = 1$ in common with the circle.

The curve C is a rectifiable curve of finite length, and the integral

$$\int_x^1 f(t)\,dt,$$

taken along the curve, has certainly a definite meaning, whenever f is a function continuous along the curve. And if $f(x) \to A$, when $x \to 1$ along C, it is easy to see that

$$\frac{1}{1-x}\int_x^1 f(t)\,dt \to A. \tag{1}$$

THEOREM 49.—*Suppose that $na_n = o(1)$ and $f(x) = \Sigma a_n x^n \to A$ as $x \to 1$ along C. Then Σa_n converges to the sum A.*

Obviously we may suppose $A = 0$. If $1-\xi$ is any point on C, other than 1, $\Sigma a_n t^n$ is uniformly convergent on the arc of C which joins x and $1-\xi$, and

$$\int_x^{1-\xi} f(t)\,dt = \Sigma \frac{a_n}{n+1}\{(1-\xi)^{n+1} - x^{n+1}\}.$$

Since $\Sigma \dfrac{a_n x^n}{n+1}$ is uniformly convergent for $|x| \leqslant 1$, it follows that we may replace ξ in this equation by 0. Hence, using (1), we see that

$$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = o(1-x), \tag{2}$$

as $x \to 1$ along C.

* The closeness of the contact may be as great as that of the path $r = 1 - e^{-1/\theta}$. It is not difficult to prove the more general result in which we start from $k!\, s_n^k/n^k = A + o(n^{-\beta})$.

Let
$$\nu = \left[\frac{1}{|1-x|}\right]$$
and
$$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = \sum_0^{\nu} + \sum_{\nu+1}^{\infty} = \phi_1+\phi_2.$$
Then
$$\phi_2 = o\left(\sum_{\nu+1}^{\infty} \frac{1}{n^2}\right) = o(1/\nu).$$
Also
$$1-x^{n+1} = (n+1)(1-x)+O\{n^2(1-x)^2\},$$
and so
$$\phi_1 = (1-x)\sum_0^{\nu} a_n+O\left\{(1-x)^2 \sum_0^{\nu} n\,|a_n|\right\}$$
$$= (1-x)\,s_n+o\{\nu(1-x)^2\}.$$

Substituting these expressions for ϕ_1 and ϕ_2 in (2), and dividing by $1-x$, we obtain
$$s_n = o\{\nu(1-x)\}+o\left\{\frac{1}{\nu(1-x)}\right\} = o(1),$$
and the theorem is established.

58. The result of Theorem 49 may be extended to include cases in which the path C has points (other than $x = 1$) in common with the unit circle.* In particular it holds *when C is the unit circle itself*. It is to observed that the condition $f(x) \to A$ was only used as a special hypothesis which ensures the truth of the relation (1); a more general theorem is obtained by taking that relation as a hypothesis. When stated in this form, and when C is the circle itself, Theorem 49 reduces to a known theorem in the theory of Fourier series, and it appears that our argument does not differ in principle from an argument used by Riemann in his classical memoir on trigonometrical series.†

Theorem 49 is a "Tauberian" theorem; it is natural to try to state a corresponding "Abelian" theorem. This theorem is

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* It has not seemed to us worth while at the present moment to go deeply into the question as to the most general hypotheses possible in regard to the nature of C. The essence of the matter lies in the fact that the restrictions which have to be imposed on C *have nothing to do with the closeness of its contact with the circle*: they are conditions relating to continuity, rectifiability, and so forth.

† See Riemann, *Gesammelte Werke*, p. 213; Fatou, *Thèse* (Stockholm, 1906), and *Acta Mathematica*, Vol. 30,

THEOREM 50.—*If* $a_n = o(1/n)$, *and* Σa_n *is convergent to sum* A, *then*

$$\frac{1}{1-x}\int_x^1 f(t)\,dt \to A,$$

as $x \to 1$ *along any regular path* C.

If C is of order 1 (a Stolz-path) this conclusion can be replaced by $f(x) \to A$; but that this is so in general appears to us to be improbable.

[*Added November 2nd*, 1912.—We add a few remarks in reference to the questions mentioned at the end of § 4. We understand from Dr. Marcel Riesz, to whom we communicated the proof-sheets of this paper, that he has succeeded in proving the analogue of Theorem 19 for the general Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$ and for all orders of summation integral or non-integral, and in effecting similar generalisations of some of the theorems of Section II.

The supposition that the theorems of § 2, and some, at any rate, of the "Tauberian" theorems, may be generalised by the use of right- or left-handed conditions, proves to be well founded. As an example, we may quote the theorem: *if* $f(x) = \Sigma a_n x^n \to s$, *and* $a_n = O_L(1/n)$, *then* Σa_n *is convergent to sum* s. This is the generalisation of the general Abel-Tauber theorem.]

CORRECTIONS

Section II

p. 421, *line* 15. For $\Delta_r^2 a_0^{(r-1)} = 0$ read $\Delta_r^2 a_0^{(r-1)} \leqslant 0$; since $\Delta_r^2 a_0^{(0)} = -1$.

p. 422, *line* 7 *up*. For $\leqslant \chi_s$ read $= \chi_s$.

p. 427, *line* 7 *up*. For $x \geqslant r$ read $n \geqslant r$.

p. 428, *line* 7. For 'necessity some' read 'necessity for some'.

Section III

p. 428, *line* 13 *up*. For $\alpha \geqslant 1$ read $\alpha \geqslant -1$.

—— *line* 8 *up*. For $\alpha > 1$ read $\alpha > -1$.

p. 433, *line* 2. In the first sum read Δ^{k+1-i}; in the second read $j^{k-i+\beta}$.

p. 434, *line* 3 *up*. In the last sum read ν^{β}.

p. 436, *line* 4. For 'point at which' read 'point of which'.

p. 439, *line* 9. In the first bracket read λ_k.

—— *line* 4 *up*, and *p.* 440, *line* 8. Insert 'for every positive δ'.

p. 442, *line* 6. For the second $>$ read $=$.

Section IV

p. 443, *line* 12 *up*. For $n^{-\alpha}$ read $n^{-1-\alpha}$.

p. 444, *lines* 1 *and* 5 *up*. For $+$ read $-$.

—— *lines* 1, 3, 4 *up*, and *p.* 445, *line* 2. For $r!$ read $\Gamma(r)$.

p. 445, *line* 5. For $(1-\lambda)m$ in the 3rd sum read $(1+\lambda)m$.

p. 447, *line* 11. For '$c = k$ or' read '$c = k-1$, $c = 1-h$ or'.

p. 450, *lines* 4–12 *up* (5 times), *p.* 451, *lines* 7 and 13, *p.* 452, *lines* 5–8 (3 times). For $r!$ read $\Gamma(r)$.

p. 451, *line* 6 *up*. Read $e^{-(r-\theta)(m+\nu)/m}$.

p. 455, *line* 15 *up*. Read $Ke^{K\sqrt{q}}$.

—— *footnote*, *p.* 458, *line* 14, *p.* 459, *line* 8. For a^n read a_n.

p. 459, *line* 6 *up*. For t^n read t_n.

Section V

p. 462, *line* 6. Read $o(n^{-\delta})$.

p. 472, *line* 12 *up*. Read $f(x)$.

p. 477, *lines* 6 *and* 9. For s_n read s_ν.

COMMENTS

In this long paper, Hardy and Littlewood not only solve a number of new problems, but leave over many questions for further discussion, for example, in 1914, 4; 1915, 11; 1916, 8; 1920, 7; 1924, 7; 1943, 4, and other questions which have inspired later writers.

Section II

In 1914, 11 (p. 147, footnote), Hardy and Littlewood point out that the theorem of § 5, found by Littlewood† in 1910, had been given in 1897, both by Hadamard‡ and by Kneser;§ see the Comments on 1914, 11.

Theorem 1, § 6, was extended by Mordell|| to the case where ϕ and ψ are both decreasing. The geometrical argument (i) in § 6 may also be adapted to this case, the line PT having negative gradient $-\epsilon\psi(x_0)$.

In the proof of Theorem 2, § 8, the law of formation (4) is only used for $n \geqslant 2, r \geqslant 1$. For $n = 1, r \geqslant 1$, the law is

$$a_1^{(r)} = \tfrac{1}{2}\{a_1^{(r-1)}+a_0^{(r+1)}\}.$$

The inequality

$$\Delta_r^2 a_{n-1}^{(r-1)} \leqslant \tfrac{1}{2}\{\Delta_r^2 a_{n-2}^{(r-2)}+\Delta_r^2 a_{n-2}^{(r)}\}$$

follows from (4) and (5), when $n \geqslant 3, r \geqslant 1$. It is false for $n = 2, r = 2$, since

$$\Delta_r^2 a_1^{(1)} = -1/8, \quad \Delta_r^2 a_0^{(0)} = -1, \quad \Delta_r^2 a_0^{(2)} = 0,$$

but is only required for $n \geqslant 3$.

An analogue of Theorem 3, § 9, for integrals of fractional order, was obtained by Riesz.†† Other *convexity theorems* have been given by later writers.

Section III

Theorems 10, 13, 14, and 18 were extended by Ananda-Rau‡‡ to Dirichlet series of the form $\sum a_n l_n^{-s}$, with Riesz summability of type l_n and fractional order. Theorems 16 and 19 were extended by Riesz.§§

In the proof of Theorem 13, § 18, the argument shows that $\sum n^{-\beta}a_n$ is either summable (C, k) or not $(C, k+1)$. The conclusion is obtained by repeating the argument, with k replaced by $k+1$, $k+2,\ldots$. The result quoted is in § 4 of 1910, 3.

In order to deduce the more general version of Theorem 16, § 19, from the analogue of Fatou's theorem,|||| we may observe that, if

$$\tau_n = \sum_1^n \nu^{1-\beta}a_\nu \quad \text{and} \quad \sigma > -1,$$

then the series

$$\sum a_n n^{-\beta-s} = \sum n^{1-\beta}a_n \,.\, n^{-s-1}$$

† *Proc. London Math. Soc.* (2), 9 (1911), 434–48 (437–8).

‡ *J. de math. pures et appl.* (5), 3, 331–87 (334–5).

§ *J. für die reine u. angew Math.* 118, 186–223 (199–200).

|| *J. London Math. Soc.* 3 (1928), 119–21.

†† Riesz (1), *Acta litt. ac. sci. Univ. Hungaricae* 1 (1923), 114–26. It has been stated that substantially the same result was obtained by Ananda-Rau in a Smith's Prize Essay (Cambridge, 1918); see his obituary notice, *Pub. Ramanujan Institute* 1 (1968–9), 1–10.

‡‡ *Proc. London Math. Soc.* (2), 34 (1932), 414–40. See also Iyer, *Ann. of Math.* (2), 36 (1935), 100–16.

§§ Riesz (1), loc. cit., for Theorem 19; (2) ibid., 2 (1924), 18–31, for Theorem 16.

|||| Riesz's analogue was stated in Riesz (3), *Comptes rendus* 149 (1909), 909–12, and proved in Riesz (4), *Acta Math.* 40 (1916), 349–61. The version quoted here was given by Landau, *Prace Mat.-Fiz.* 21 (1910), 97–177.

is summable (C, k) if and only if $\sum \tau_n^k \Delta^{k+1} n^{-s-1}$ is summable $(C, -1)$, and then

$$\sum_{(C,k)} a_n n^{-\beta-s} = \sum_{(C,-1)} \tau_n^k \Delta^{k+1} n^{-s-1}.$$

Since $\tau_n^k = o(n^{k+1})$, by Theorem 13, the series

$$\sum \tau_n^k(\Delta^{k+1} n^{-s-1} - (d^{k+1}/dn^{k+1}) n^{-s-1})$$

is absolutely convergent and summable $(C, -1)$, with its sum regular, for $\sigma > -1$; cf. 1910, 3, Bohr,††† Andersen‡‡‡, Bosanquet and Chow.§§§

The details for the example at the end of § 19 are given in 1915, 11. For a remark about the series used in § 25, see the Comments on 1911, 1.

Section IV

In the proof of Theorem 28, § 32, the Tauberian condition actually used is that, for n in the interval $m \leqslant n \leqslant m+cm$ $(c > 0)$,

$$|a_n - a_m| < \tfrac{1}{2}|a_m|,$$

i.e. that a_n lies in a variable disc, of radius $\frac{1}{2}|a_m|$, about a_m. This implies that, for $m \leqslant n \leqslant m+cm$, a_n is in a variable sector, with vertex at the origin and angle $\frac{1}{3}\pi$, symmetrical about the segment $(0, a_m)$, and further that $s_q - s_p$ lies in the same sector whenever $m \leqslant p < q \leqslant m+cm$. Agnew|||||| has shown that the last condition is included in a general Tauberian condition due to Pitt.†††† Thus the convergence of $\sum a_n$ in Theorem 28, and also in Theorem 27, §§ 34–6, follows from Pitt's Tauberian theorem for Abel summability.‡‡‡‡ The conclusion $na_n = o(1)$ may then be obtained from the formula (§ 32)

$$s_n - s_m = (n-m)a_m(1+\eta) \quad (|\eta| < \tfrac{1}{2}),$$

with $n = m+[cm]$.

Similar remarks may be made about Theorem 31, §§ 38–41, where the interval $(m, m+\alpha\sqrt{m})$ $(\alpha > 0)$ plays the role of $(m, m+cm)$ $(c > 0)$. The convergence of $\sum a_n$ follows from Pitt's Tauberian theorem for Borel summability,§§§§ and the conclusion $n^{\frac{1}{2}} a_n = o(1)$ then follows from the formula of § 32, with $n = m+[\alpha\sqrt{m}]$.

Hardy and Littlewood never returned to these Tauberian theorems, and they are not mentioned in D.S.

The property of $\sum n^{-b} e^{in^a}$ obtained for $0 < a < \frac{1}{2}$, at the end of § 41, is proved for $0 < a \leqslant \frac{1}{2}$ in 1913, 1, §§ 11–12; see the Comments on 1913, 1. The non-regularity of the related power series at $x = 1$ holds for $0 < a < 1$; see 1913, 6 (in Vol. IV).

Section V

Theorem 35, § 46 is contained in a result published in 1911 by Fekete,|||||||| that *if A is absolutely summable (C, r) and B is summable (C, s) $(r, s$ non-negative integers$)$, then C is summable $(C, r+s)$*; see D.S., p. 245, where further references are given.

The lemma required for Theorem 37 is proved in D.S., Theorem 65; cf. 1931, 8, Lemma 1. For an alternative proof of Theorem 37, see D.S., Theorem 45.

††† *Bidrag* . . ., p. 112; English translation, p. 98.
‡‡‡ *Proc. London Math. Soc.* (2), 27 (1928), 39–71.
§§§ *J. London Math. Soc.* 16 (1941), 42–8.
|||||| *Ann. of Math.* (2), 42 (1941), 293–308.
†††† *Proc. London Math. Soc.* (2), 44 (1938), 243–88.
‡‡‡‡ Loc. cit., Theorem 13.
§§§§ Loc. cit., Theorem 16.
|||||||| *Math. és Termés. Ért.* 29 (1911), 719–26.

A definition of summability $(C, -1)$ (see § 49) was given by Young;††††† see D.S., p. 98. An example in D.S., Theorem 166, shows that the product series for a pair of series, one convergent and the other summable $(C, -1)$, need not be convergent.

In (i) and (ii) of § 49, the index of summability of $\sum n^{-b}e^{in^a}$ is assumed for *fractional orders*. For details see D.S., pp. 141–5. The property stated in the footnote to (i) is proved in 1913, 6 (in Vol. IV), § 8.

The condition $0 < a < \frac{1}{2}$ in the example after Theorem 47, § 57, is corrected in 1920, 7, pp. 207 and 224. If $0 < a < 1$, it is necessary and sufficient to assume that $b > 1-a$ (for convergence) and $b \leqslant 1-\frac{1}{2}a$ (for oscillation along the lower arc). See 1911, 1, § 6, and 1913, 6, § 8.

The proof of Theorem 49, § 57, shows that *if* (o): $na_n = o(1)$, *and* $\nu = [1/|1-x|]$, *then*

$$\Phi(x) = \frac{1}{1-x}\sum \frac{a_n}{n+1}(1-x^{n+1}) = s_\nu + o(1)$$

as x *tends to* 1 *in any manner from within the unit circle* (compare 1920, 7, § 2.2, Lemma α). This implies (see 1920, 7, Theorem O)

(Ω). *If* (o) *holds, then* $\sum a_n$ *converges to* A *if and only if* (Λ): $\Phi(x) \to A$ *as* $x \to 1$ *along a single arbitrary path* (*or along every path*) *in* $|x| < 1$.

Theorems 49 and 50 both follow from (Ω). For if C is a regular path, then

$$\Phi(x) = \frac{1}{1-x}\int\limits_x^1 f(t)\,dt,$$

where the integral is taken along C. In Theorem 50 the integral may be taken along the segment $(x, 1)$, and then the result holds for all paths (whether regular or not).

In 1920, 7 (Theorem U), Hardy and Littlewood prove a more general form of (Ω), with (O): $na_n = O(1)$ in place of (o), which shows that the same is possible in Theorems 49 and 50 (see 1920, 7, Theorems Q, R, and T). Finally, in 1924, 7 (Theorem R′) Hardy and Littlewood show that Theorem 49, with (O), remains true if C is a single arbitrary path.

The conjecture after Theorem 50, that the conclusion in that theorem cannot be replaced by $f(x) \to A$, if $x \to 1$ along a tangential path, is proved in 1920, 7 (Theorem V).

If C is taken along the circumference of the circle, and $f(e^{i\theta})$ is the boundary function, Theorems 49, 50, and (Ω) become results concerning a conjugate pair of Fourier series; see § 58 and 1920, 7, § 1.5. These extend to results for a single Fourier series; see 1920, 7, § 4 and in Vol. III, 1917, 10, 1924, 3, and other papers. For related results about Cesàro summability of power series see 1924, 1 (in Vol. III).

The one-sided Tauberian theorem, stated in the addendum, is proved in 1914, 4 (Theorem 11).

††††† *Proc. London Math. Soc.* (2), 17 (1918), 195–236 (209–10).

AN EXTENSION OF A THEOREM ON OSCILLATING SERIES

By G. H. HARDY.

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1. Suppose that $$\lambda_1,\ \lambda_2,\ \lambda_3,\ \dots$$
is an ascending sequence of positive* numbers increasing beyond all limit with n. Let
$$C(\omega) = \sum_{\lambda_n \leqslant \omega} c_n,$$
so that
$$C(\omega) = c_1 + c_2 + \dots + c_p,$$
if $\lambda_p \leqslant \omega < \lambda_{p+1}$; and let
$$C^\kappa(\omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^\kappa c_n,$$
κ being any positive number. It will easily be verified that
$$C^\kappa(\omega) = \kappa \int_0^\omega C(\tau)(\omega-\tau)^{\kappa-1}\, d\tau.$$
Then, if
$$\omega^{-\kappa} C^\kappa(\omega) \to C,$$
as $\omega \to \infty$, we shall say that the series Σc_n is *summable* (R, λ, κ)† *to sum* C.

If $\lambda_n = n$, this definition of the sum of an oscillating series is equivalent to Cesàro's.‡

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* λ_1 may be zero.

† That is to say, *by Riesz's means of type* λ *and order* κ. These methods of summation were introduced by M. Riesz in a note in the *Comptes Rendus* of 5 July, 1909. A more systematic account of them, and of their applications to the theory of Dirichlet's series, will be given in the *Cambridge Tract* on "The General Theory of Dirichlet's Series" that Dr. Riesz and I are now preparing in collaboration.

‡ Riesz, *Comptes Rendus*, 12 June, 1911. When I speak of Cesàro's methods of summation I include the methods of non-integral order whose theory has been developed by Knopp and Chapman (Knopp, *Inaugural Dissertation*, Berlin, 1907; Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369).

1913, 3 Proceedings of the London Mathematical Society (2), 12, 174–80.

If, in the general expression for $C^\kappa(\omega)$, we take $\kappa = 1$, $\omega = \lambda_p$, we obtain

$$\frac{C^\kappa(\omega)}{\omega} = \frac{(\lambda_2-\lambda_1)C_1+(\lambda_3-\lambda_2)C_2+\ldots+(\lambda_p-\lambda_{p-1})C_{p-1}}{\lambda_p},$$

where $$C_n = c_1+c_2+\ldots+c_n.$$

Thus, when $\kappa = 1$, Riesz's definition is the natural generalisation of Cesàro's which arises when we attach *weights* to the successive partial sums C_n.

2. My principal object in this note is to prove the following theorem.

THEOREM 1.—*If* Σc_n *is summable* (R, λ, κ), *and*

(1) $$c_n = O\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right),$$

then Σc_n *is convergent. In other words, no series which satisfies the condition* (1) *can be summable by Riesz's methods without being convergent.*

This theorem is the generalisation for Riesz's methods of summation of what Mr. Littlewood and I have called *the general Cesàro-Tauber theorem,* the theorem* which asserts that a series for which $c_n = O(1/n)$ cannot be summable by Cesàro's means without being convergent.

I have already published a proof† of the special case of this theorem in which $\kappa = 1$, assuming, however, that the λ's are subject to the restriction

(2) $$\lambda_{n+1}/\lambda_n \to 1.$$

When this last condition is satisfied my theorem, and its extension to general values of κ, may be deduced as a corollary from Mr. Littlewood's extension of Tauber's theorem.‡ This method of procedure is, however, open to several objections. In the first place, Mr. Littlewood's theorem is a more difficult theorem than that which we are using it to prove. Secondly, the proof also depends on another theorem of which no proof has yet been published, viz., the theorem that, *if* Σc_n *is summable* (R, λ, κ), *to sum* C, *and* $\Sigma c_n e^{-\lambda_n x}$ *is convergent for* $x > 0$, *then*

$$\Sigma c_n e^{-\lambda_n x} \to C,$$

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 307.

† *L.c.*, p. 313.

‡ Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 434.

as $x \to 0$.* Finally, until a proof is given of Mr. Littlewood's theorem, which shall be free from restrictions on the λ's, we are compelled to adhere to the assumption (2). The presence of this unnecessary restriction is at any rate an æsthetic blemish in the theorem.

The idea of obtaining a direct and general proof of the theorem, which should be free from any restriction on the λ's, was suggested to me by Dr. Riesz. Dr. Riesz himself indicated to me the general lines of a proof in the case $\kappa = 1$. The form of this proof was different from that which I adopted in my previous paper. I find, however, that the line of argument which I then followed can be adapted so as to lead to the desired result, and I have followed it here, as the preliminary transformations on which it is based seem to be of some interest in themselves, and can be applied for other purposes.

3. I proceed now to the proof of the theorem. I write

$$b_n = \lambda_n c_n,$$

and use $B(\omega)$, $B^\kappa(\omega)$ to denote the sums formed from the b's in the way in which $C(\omega)$, $C^\kappa(\omega)$ were formed from the c's. It is evident that there is no loss of generality in supposing κ to be integral. Further, we may suppose our series to be *real*.†

We have

$$\frac{C^\kappa(\omega)}{\omega^\kappa} - \frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}} = \sum_{\lambda_n \leqslant \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^\kappa \left\{1 - \left(1 - \frac{\lambda_n}{\omega}\right)\right\} a_n$$

$$= \frac{B^\kappa(\omega)}{\omega^{\kappa+1}}.$$

Hence (*a*) *the necessary and sufficient condition that the series* Σc_n, *if known to be summable* $(R, \lambda, \kappa+1)$, *should be summable* (R, λ, κ), *is that*

$$B^\kappa(\omega) = o(\omega^{\kappa+1}).$$

* For the case $\kappa = 1$, see Hardy, *l.c.*, pp. 311 *et seq.* Dr. Riesz has found a proof of a theorem a good deal more general than that stated in the text, which will be published in the *Cambridge Tract* already referred to.

The steps of the deduction referred to in the text are as follows. From (1) we infer the convergence of $\Sigma c_n e^{-\lambda_n x}$ for $\kappa > 0$. From the summability of Σc_n, and the theorem stated in the text, we infer the existence of the limit as $x \to 0$. Finally, from (1), (2) and Mr. Littlewood's theorem, we infer the convergence of Σc_n.

† Cf. Hardy, *l.c.*, p. 303.

Again, $$\frac{d}{d\omega}\left\{\frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}}\right\} = \frac{\kappa+1}{\omega^2}\sum_{\lambda_n\leqslant\omega}\left(1-\frac{\lambda_n}{\omega}\right)^{\kappa}\lambda_n a_n$$

$$= (\kappa+1)\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}.$$

Hence (β) *the necessary and sufficient condition that the series* Σc_n *should be summable* $(R, \lambda, \kappa+1)$ *is that the integral*

$$\int^{\infty}\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}\,d\omega$$

should be convergent.

I shall now show that, when the c's satisfy condition (1) of the theorem, (β) implies (α). From this it will obviously follow that the summability of the series implies its convergence.

We have, in the first place,

(1) $$|c_n| < K(\lambda_n-\lambda_{n-1})/\lambda_n.$$

Now, let us suppose that (α) is not true, and that *e.g.*, the upper limit of $\omega^{-\kappa-1}B^{\kappa}(\omega)$ is positive. Then there is a positive constant H, such that

(2) $$B^{\kappa}(\omega) > H\omega^{\kappa+1},$$

for values of ω exceeding all limit.

Suppose $\xi > \omega$. Then

(3) $$B^{\kappa}(\xi)-B^{\kappa}(\omega) = \kappa\int_{\omega}^{\xi}B^{\kappa-1}(u)\,du.$$

Also $$|B^{\kappa-1}(u)| = \Big|\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}b_n\Big|$$

$$< K\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}(\lambda_n-\lambda_{n-1})$$

$$< K\int_0^u (u-w)^{\kappa-1}\,dw = \frac{K}{\kappa}u^{\kappa}.$$

Hence

(4) $$|B^{\kappa}(\xi)-B^{\kappa}(\omega)| < K\int_{\omega}^{\xi}u^{\kappa}\,du = \frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < \frac{K}{\kappa+1}(\xi-\omega)\xi^{\kappa}.$$

Let $$\Omega = (1+\rho)\,\omega,$$

where ρ is a positive constant to be chosen later. Then, for $\omega < \xi < \Omega$,

we have, by (2) and (4),

$$B^\kappa(\xi) \geqslant B^\kappa(\omega) - |B^\kappa(\xi) - B^\kappa(\omega)| > H\omega^{\kappa+1} - \frac{K}{\kappa+1}(\xi-\omega)\xi^\kappa$$

$$> \left\{ H - \frac{K\rho(1+\rho)^\kappa}{\kappa+1} \right\} \omega^{\kappa+1}.$$

But, when H and K are given, we can evidently choose ρ so that

$$\frac{\mathrm{K}\rho(1+\rho)^\kappa}{\kappa+1} < \tfrac{1}{2}H.$$

Then $$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1},$$

for $$\omega < \xi < \Omega,$$

and so $$\int_\omega^\Omega \frac{B^\kappa(\xi)}{\xi^{\kappa+2}}\,d\xi > \tfrac{1}{2}H(\Omega-\omega)\frac{\omega^{\kappa+1}}{\Omega^{\kappa+2}} = \frac{H\rho}{2(1+\rho)^{\kappa+2}}.$$

That this should hold for values of ω exceeding all limit is a contradiction of the hypothesis (β). The theorem is therefore established.

4. I add some remarks relating to the case in which the increase of the λ's is *rapid and fairly regular.*

Suppose that $$\lambda_{n+1}/\lambda_n \geqslant 1+\delta > 1,$$

where δ is a constant. Then the condition (1) of the theorem reduces to

$$c_n = O(1).$$

But, in this case, *more* is true than is asserted by the theorem. It is, in fact, true that *no* series can be summable (R, λ, κ) unless it is convergent; in other words, for such forms of λ_n, Riesz's methods are completely trivial; they sum convergent series and convergent series only. This can be deduced from another of Riesz's theorems, viz., that* *if* Σc_n *is summable* (R, λ, κ) *to sum* C, *then*

$$C_n - C = o\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^\kappa.$$

This conclusion, so far as it goes, bears out Mr. Littlewood's conjec-

* A proof of this theorem also will be given in the *Tract*.

ure* that, for such "regular high indices" λ_n, the existence of the limit

$$\lim_{\kappa \to 0} \Sigma c_n e^{-\lambda_n x},$$

always involves the convergence of Σc_n.

5. Landau has generalised the "general Cesàro-Tauber" theorem by substituting for the hypothesis

$$c_n = O(1/n),$$

the less exacting hypothesis $c_n = O_L(1/n)$,

that is to say, $c_n > -K/n$.†

There is, of course, a corresponding extension of Theorem 1, which runs as follows.

THEOREM 2.—*A series Σc_n, for which*

$$(1) \qquad c_n = O_L\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$$

cannot be summable by Riesz's means unless it is convergent.

The proof of this theorem requires merely a slight modification of the proof of Theorem 1. We have

$$(1) \qquad c_n > -K(\lambda_n - \lambda_{n-1})/\lambda_n.$$

If

$$(2) \qquad B^\kappa(\omega) > H\omega^{\kappa+1}$$

for values of ω exceeding all limit, we take

$$\Omega = (1+\rho)\omega,$$

and show, by substantially the same argument‡ as in § 3, that

$$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1} \quad (\omega < \xi < \Omega).$$

We thus prove, as in § 3, that

$$\overline{\lim_{\omega \to \infty}} \frac{B^\kappa(\omega)}{\omega^{\kappa+1}} = 0.$$

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* Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 446.

† *Prac Matematyczno-Fizycznych*, Vol. 21, p. 97. In Landau's form of the theorem c_n must obviously be supposed to be real, or it must be asserted explicitly that both the real and the imaginary parts of c_n satisfy his condition.

‡ The only difference is that we use algebraical inequalities instead of inequalities between absolute values.

In order to prove that the lower limit of indetermination is also zero, that is to say, that the inequality

$$B^{\kappa}(\omega) < -H\omega^{\kappa+1} \quad (H > 0),$$

cannot hold for values of ω exceeding all limit, we need only modify the argument to the extent of considering an interval

$$(1-\rho)\,\omega = \Omega < \xi < \omega,$$

instead of an interval $\omega < \xi < \Omega = (1+\rho)\,\omega$.*

* *Either* interval would have served our purposes in proving Theorem 1.

CORRECTIONS

p. 176, 1*st footnote, line* 5. For $\kappa > 0$, read $x > 0$.

p. 177, *line* 8. Here and for the rest of the proof, for (α) and (β) read 'the condition in (α)' and 'the condition in (β)'.

p. 177, 3 *lines up*, and *p.* 178 (3 *times*). For $\frac{K}{\kappa+1}$, after the first occurrence read K.

p. 179, *last line*, For $=$ read $\leqslant$.

COMMENTS

Riesz's theorem, stated in § 2, is included in Theorem 28 of H.R.

The proof of Theorem 1, in § 3, establishes the summability $(R, \lambda, 1)$ of $\sum c_n$, but not its convergence. This was pointed out by Ananda-Rau,† who gave a proof of the final step, which is the case $\kappa = 1$ of the theorem.‡ In fact, if $\lambda_p < \omega < \lambda_{p+1}$ and $\xi > \omega$, the Tauberian condition only gives $|B(\xi)-B(\omega)| = |B(\xi)-B(\lambda_p)| < K|\xi-\lambda_p|$, which does not imply (4) with $\kappa = 0$. On the other hand, the condition $B(\omega) = o(\omega)$ is equivalent to $B(\lambda_n) = o(\lambda_n)$, and convergence is established by replacing the exceptional intervals $\omega < \xi < (1+\rho)\omega$ by intervals $\lambda_p < \xi < (1+\rho)\lambda_p$.

The proof of Theorem 2 also establishes the summability $(R, \lambda, 1)$ of $\sum c_n$, and Ananda-Rau's method gives

$$\overline{\lim}\, B(\lambda_n)/\lambda_n \leqslant 0,$$

but not

$$\underline{\lim}\, B(\lambda_n)/\lambda_n \geqslant 0.$$

Ananda-Rau§ showed by an example that the final step cannot be taken if λ_n is unrestricted, but that it is valid if $\lambda_{n+1}/\lambda_n \to 1$. Szasz|| showed that it is sufficient to add the (necessary) condition:

$$\underline{\lim}\, c_n \geqslant 0.$$

Some related Tauberian theorems for general Dirichlet series are given by Hardy and Littlewood in 1914, 11. See also D.S., Theorems 67, 103, and 104, and the Notes there. A number of gaps in 1913, 3 and 1914, 11 were filled independently by Ingham, in an unpublished manuscript dated 1924; see his obituary notice.††

† *Proc. London Math. Soc.* (2), 17 (1918), 334–6.

‡ Hardy says (§ 2) that Riesz had indicated to him the 'general lines of a proof' in this case.

§ *Proc. London Math. Soc.* (2), 30 (1930), 367–72.

|| *Sitz. d. Bayerischen Akad. d. Wiss.* 59 (1929), 325–40 (published in 1930).

†† *Bull. London Math. Soc.* 1 (1969), 109–24 (Paper 1 (*b*)).

TAUBERIAN THEOREMS CONCERNING SERIES OF POSITIVE TERMS.

By *G. H. Hardy* and *J. E. Littlewood.*

THE theorems which we state here, and of which we hope to publish complete proofs later elsewhere, are of the same general character as those which we proved in our paper "Contributions to the Arithmetic Theory of Series" recently published in the *Proceedings of the London Mathematical Society.** There is, however, one distinctive feature in them, viz., that they are concerned with power series, or Dirichlet's series, whose coefficients are all positive.

Our analysis is based upon the following lemma, which is a particular case of a theorem due to Landau.†

LEMMA. *If* $0<x<1$, *and*

$$f(x) \sim \frac{1}{(1-x)^k} \qquad (k>0)$$

as $x \to 1$; *and* $(1-x)f'(x)$ *is an increasing function of* x; *then*

$$f'(x) \sim \frac{k}{(1-x)^{k+1}}.$$

From this lemma it is easy to deduce

THEOREM 1. *If* $f(x)=\Sigma\alpha_n x^n$ *is a power series whose coefficients are positive, and*

$$f(x) \sim \frac{1}{(1-x)^k} \qquad (k>0)$$

as $x \to 1$, *then*

$$f^{(r)}(x) \sim \frac{k(k+1)\dots(k+r-1)}{(1-x)^{k+r}}.$$

And from this we deduce‡

THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$s_n = \alpha_0 + \alpha_1 + \dots + \alpha_n \sim \frac{n^k}{\Gamma(1+k)}.$$

* Vol. xi., pp. 411–478.

† *Prac Matematyczno-Fizycznych*, vol. xxi., p. 116.

‡ This proof is far more difficult than that of Theorem 1, but is of the same general character as that of the proof of the generalised form of Tauber's theorem (Littlewood, *Proc. Lond. Math. Soc.*, vol. ix., pp. 434–448). The theorem is a particular case of a more general theorem in which the hypothesis that a_n is positive is replaced by the hypothesis that $a_n > -Kn^{k-1}$. The superior generality of the latter theorem is, however, only apparent, as it is quite easy to show that the two theorems are equivalent.

1913, 10 (with J. E. Littlewood) *Messenger of Mathematics*, 42, 191–2.

We have also proved the analogues of Theorems 1 and 2 for ordinary Dirichlet's series, viz.,

THEOREM 3. *If* $s > 1$ *and*

$$f(s) = \Sigma \frac{a_n}{n^s} \sim \frac{1}{(s-1)^k}$$

as $s \to 1$, *the coefficients* a_n *all being positive, then*

$$-f'(s) = \Sigma \frac{a_n \log n}{n^s} \sim \frac{k}{(s-1)^{k+1}}.$$

THEOREM 4. *If the conditions of Theorem* 3 *are satisfied, then*

$$s_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n} \sim \frac{(\log n)^k}{\Gamma(1+k)}.$$

We have not yet proved the analogous theorems immediately suggested for Dirichlet's series of the most general type

$$\Sigma a_n e^{-\lambda_n s}.$$

COMMENTS

This paper is an introduction to results proved in 1914, 4.

TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received October 3rd, 1913.—Read November 13th, 1913.]

1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let
$$f(x) = \Sigma a_n x^n$$
be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let
$$s_n = a_0 + a_1 + \ldots + a_n,$$
$$L(u) = (\log u)^{\alpha_1} (\log \log u)^{\alpha_2} \ldots,$$
where the α's are real. Then it is known that, *if*
$$s_n \sim A n^{\alpha} L(n),$$
where $A \neq 0$, *as* $n \to \infty$, *the indices* $\alpha, \alpha_1, \alpha_2, \ldots$ *being such that* $n^{\alpha} L(n)$ *tends to a positive limit or to infinity, then*
$$f(x) \sim A \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right),$$
as $x \to 1$.‡

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* A short abstract of some of the principal results of this paper was published under the title "Tauberian Theorems concerning Series of Positive Terms" in the *Messenger of Mathematics*, Vol. 42, pp. 191, 192.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320; Vol. 9, pp. 434–448; Vol. 11, pp. 1–16 and pp. 411–478.

‡ The first of the α's which is not zero must be positive. If $\alpha = \alpha_1 = \alpha_2 = \ldots = 0$, the theorem reduces to Abel's well known theorem. The special theorem in which $\alpha_1 = \alpha_2 = \ldots = 0$

The principal object of this paper is to prove that *the converse of the theorem is true when the coefficients a_n are positive.* We shall prove, for example, that *if* $a_n \geqslant 0$, *and*

$$f(x) = \Sigma a_n x^n \sim \frac{A}{(1-x)^a} \quad (A > 0,\ a > 0),$$

then

$$s_n \sim \frac{An^a}{\Gamma(1+a)}.$$

This result is a very curious one, largely because it lies much deeper and is much harder to prove than a first impression might tempt one to believe. Its appearance is that of a "special"* (or "o") Tauberian theorem. In reality, as will appear in the sequel, it is a theorem of the "general" (or "O") type, and left-handed"† in addition. It is, in fact, of the same order of difficulty as the theorem "*if* $a_n > -K/n$, *and* $f(x) \to A$, *then* Σa_n *converges to the sum* A."‡ The proof therefore naturally involves all the apparatus of repeated differentiation on which the proofs of such theorems ultimately depend.§

2. We begin by proving some subsidiary theorems which are interesting in themselves. It is hardly necessary to remark that all variables and functions considered in them are real. We suppose first that x is a variable which tends to infinity.

We shall begin by proving a theorem which is due to Landau, and on which nearly all our subsequent analysis depends. The theorem is of great interest in itself, inasmuch as its general character is that of an "O" Tauberian theorem, and it was the first theorem of this nature stated explicitly.

THEOREM 1.—*Suppose that* (i) $f(x)$ *is differentiable, and* (ii) $xf'(x)$

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was proved by Appell, *Comptes Rendus*, Vol. 87, p. 689. The substance of the general theorem is due to Lasker, *Phil. Trans. Roy. Soc.*, (A), Vol. 196, p. 444: Lasker does not actually state it, but it is a trivial deduction from the theorem which he proves. The theorem was first stated explicitly in the form we have adopted by Pringsheim, *Acta Mathematica*, Vol. 28, p. 29. Pringsheim, however, proves a more general theorem, inasmuch as he considers paths of approach of x to 1 other than the real axis.

* *Cf.* Hardy and Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 413.

† *L.c.*, p. 416.

‡ We stated this theorem without proof at the end of our paper already quoted (*l.c.*, p. 478).

§ Littlewood, "The Converse of Abel's Theorem," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 434 *et seq.*

steadily increases. Then

$$f(x) \sim x,$$

involves

$$f'(x) \sim 1.*$$

Suppose that the theorem is untrue. Then it must be possible to find a number h different from 1, and a sequence (x_ν) of values of x tending to infinity, such that

$$f'(x_\nu) \to h,$$

as $\nu \to \infty$. Let us suppose, for example, that $h > 1$; and let δ be a positive number. Then, as $\nu \to \infty$,

$$\frac{f(x_\nu + \delta x_\nu) - f(x_\nu)}{\delta x_\nu} = \frac{1}{\delta x_\nu} \int_{x_\nu}^{x_\nu + \delta x_\nu} f'(x)\, dx \geqslant \frac{x_\nu f'(x_\nu)}{\delta x_\nu} \int_{x_\nu}^{x_\nu + \delta x_\nu} \frac{dx}{x}$$

$$\sim \frac{h}{\delta} \int_{x_\nu}^{x_\nu + \delta x_\nu} \frac{dx}{x} = \frac{h \log(1+\delta)}{\delta}.$$

But, since $f(x) \sim x$,

$$\frac{f(x_\nu + \delta x_\nu) - f(x_\nu)}{\delta x_\nu} \sim 1;$$

and these two results are contradictory if δ is sufficiently small. The hypothesis that $h < 1$ may be disposed of similarly.

3. THEOREM 2.—*Let $\phi(x)$ be a function which tends to infinity with x and has a continuous and positive derivative, and suppose that*

(i) $\dfrac{\phi(x)}{\phi'(x)} \sim x,$

(ii) $xf'(x)$ *steadily increases.*

Then

$$f(x) \sim \phi(x)$$

involves

$$f'(x) \sim \phi'(x).$$

This theorem follows at once from Theorem 1 by means of the substitution

$$x = \phi(y).$$

---

* The converse inference may, of course, *always* be made. Theorem 1 was proved by Landau (*Rendiconti di Palermo*, Vol. 26, p. 218). We insert a proof for the sake of completeness.

By means of one or other of the substitutions

$$x = \frac{1}{y-c}, \qquad x = \frac{1}{c-y}$$

we deduce

THEOREM 2a.—*Let $\phi(x)$ be a function of x which tends to infinity as x tends to c from above or from below; and suppose that*

(i) $\dfrac{\phi(x)}{\phi'(x)} \sim -(x-c)$,

(ii) $(x-c)f'(x)$ *steadily decreases or increases.**

Then $f(x) \sim \phi(x)$ involves $f'(x) \sim \phi'(x)$.

Suppose, in particular, that

$$\phi(x) = \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, and $x \to 1$ from below. Then $1-x \sim a\phi/\phi'$. Hence we have

THEOREM 3.—*If* $$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, and $(1-x)f'(x)$ increases as $x \to 1$, then

$$f'(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

4. From Theorem 3 we can deduce a preliminary theorem concerning power series which seems of considerable interest in itself.

THEOREM 4.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive*† *coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right) \quad (A > 0, \ a > 0),$$

then $$f^{(r)}(x) \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right).$$

---

* $(x-c)f'(x)$ must be an increasing or a decreasing function according as x tends to its limit from below or above.

† We use "positive" to include "zero."

We have

$$g(x) = \Sigma s_n x^n = \frac{f(x)}{1-x} \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Also

$$(1-x)g'(x) = (1-x)\,\Sigma n s_n x^{n-1} = s_1 + (2s_2 - s_1)\,x + (3s_3 - 2s_2)x^2 + \ldots$$

has all its coefficients positive, since s_n increases steadily with n. Hence $(1-x)\,g'(x)$ increases with x, and so, by Theorem 3,

$$g'(x) \sim \frac{(a+1)A}{(1-x)^{a+2}} L\left(\frac{1}{1-x}\right),$$

$$f'(x) = (1-x)\,g'(x) - g(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

A repetition of the argument leads to a complete proof of the theorem.

It is important to observe that this process of differentiation is *not* legitimate when $a = 0$. Suppose, *e.g.*, that

$$f(x) \sim \log\left(\frac{1}{1-x}\right).$$

We cannot infer that

$$f'(x) \sim \frac{1}{1-x};$$

all that the argument leads to is

$$f'(x) = o\left\{\frac{1}{1-x}\log\left(\frac{1}{1-x}\right)\right\}.$$

We can show, moreover, by actual examples, that the suggested inference would be invalid. Suppose, for example, that

$$f(x) = \Sigma x^{a^n} \quad (a \geqslant 2).$$

Then it is easy to see that $s_n \sim \log_a n$, and so

$$f(x) \sim \frac{1}{\log a}\log\left(\frac{1}{1-x}\right).$$

But it is not true that $(1-x)f'(x)$ leads to a limit as $x \to 1$. This is most easily proved by means of Theorem 8 below. Since

$$xf'(x) = \Sigma\, a^n x^{a^n}$$

is a series of positive terms, $f'(x) \sim A/(1-x)$ would involve

$$t_\nu = \sum_{a^n \leqslant \nu} a^n \sim A\nu;$$

and this is obviously untrue, since whenever ν passes through a value equal to a power of a, a new term is introduced into t_ν which is greater than the sum of all which precede.*

* See Hardy, *Quarterly Journal*, Vol. 38, pp. 279 *et seq.*, for analytical formulæ which show in an explicit manner the behaviour as $x \to 1$ of the series Σx^{a^n}, $\Sigma(-1)^n x^{a^n}$ and their derivatives.

In the sequel we shall use not Theorem 4 itself, but the theorem into which it is transformed by the substitutions

$$x = e^{-t}, \quad f(x) = F(t).$$

THEOREM 4a.—*If* $a_n \geqslant 0$, *and*

$$F(t) = \Sigma a_n e^{-nt} \sim At^{-a} L\left(\frac{1}{t}\right),$$

as $t \to 0$, *then*

$$(-1)^{(r)} F^{(r)}(t) \sim \frac{\Gamma(a+r)}{\Gamma(a)} At^{-a-r} L\left(\frac{1}{t}\right).$$

5. Theorem 4 is capable of various interesting generalisations.

THEOREM 5.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may be replaced by the more general condition that*

$$na_n = O_L\{n^a L(n)\},$$

i.e., that

$$na_n > -Kn^a L(n).$$

Let

$$g(x) = \Sigma b_n x^n = \Sigma \{a_n + Kn^{a-1} L(n)\} x^n.$$

Then $b_n > 0$, and

$$g(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{1}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

Hence, by Theorem 4,

$$g'(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{a}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right);$$

and so

$$f'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

THEOREM 6.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may also be replaced by* $s_n \geqslant 0$, *or by* $s_n^k \geqslant 0$, *where* s_n^k *is any one of Cesàro's means formed from the series* Σa_n; *or, more generally, by* $s_n = O_L\{n^a L(n)\}$ *or*

$$s_n^k = O_L\{n^{a+k} L(n)\}.$$

In the proof of Theorem 4, the condition $a_n \geqslant 0$ is used only to justify the differentiation of the asymptotic equality

$$g(x) = \Sigma s_n x^n \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

And the result of the theorem shows that $s_n \geqslant 0$ is a sufficient condition for this. Repeating this argument we see that $s_n^k \geqslant 0$ is a sufficient condition.

The more general results may be established in the same way, if we appeal at each stage to Theorem 5 instead of to Theorem 4.

6. Suppose that the conditions of Theorem 4 (or of one of its generalisations) are satisfied. Then

$$\Sigma na_n x^n = xf'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Operating repeatedly in this manner, we see that

$$\Sigma n^r a_n x^n \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right)$$

for all positive integral values of r. We shall now show that *this result holds for all values of r greater than $-\alpha$.*

It is plainly enough to prove this when $r = -\beta$, $0 < \beta < \alpha$. We write

$$x = e^{-t}, \quad f(x) = F(t),$$

so that

$$F(t) \sim \frac{A}{t^{\alpha}} L\left(\frac{1}{t}\right),$$

as $t \to 0$. Then

$$\Sigma n^{-\beta} a_n e^{-nt} = \frac{1}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} \Sigma a_n e^{-n(t+u)} du = \frac{1}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} F(t+u) du.$$

Also

$$\int_0^\infty u^{\beta-1} F(t+u) du \sim A \int_0^\infty \frac{u^{\beta-1}}{(t+u)^{\alpha}} L\left(\frac{1}{t+u}\right) du$$

$$\sim \frac{A\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\beta-\alpha} L\left(\frac{1}{t}\right).*$$

The result is thus established for $-\alpha < r < 0$; the general result then follows by using the special result in which r is a positive integer. We have thus proved

THEOREM 7.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients (or subject to the more general conditions of Theorems 5 or 6), and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right) \quad (A > 0,\ \alpha > 0),$$

then

$$f_r(x) = \Sigma n^r a_n x^n \sim \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{A}{(1-x)^{\alpha+r}} L\left(\frac{1}{1-x}\right),$$

for any value of r, integral or not, greater than $-\alpha$.

7. We pass now to the proof of the principal theorem of the paper.

THEOREM 8.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right),$$

where $A > 0$ and the indices $\alpha, \alpha_1, \alpha_2, \ldots$ are such that $n^{\alpha} L(n)$ tends to a positive limit or to infinity as $n \to \infty$, then

$$s_n \sim \frac{A}{\Gamma(\alpha+1)} n^{\alpha} L(n).$$

We suppose $A = 1$, and write $x = e^{-t}$, so that $t \to 0$, and

(1) $$f(x) = F(t) = \Sigma a_n e^{-nt} \sim t^{-\alpha} L(1/t).$$

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* These transformations, of course, merely express the general lines of a straightforward proof, the details of which will easily be supplied by anyone accustomed to work of this character.

In the first place, we have

$$s_n \leqslant e \sum_0^n a_\nu e^{-\nu/n} \leqslant eF(1/n),$$

and so, from (1),

(2) $$s_n = O\{n^a L(n)\}.$$

Next, we have

(3) $$\Sigma s_n e^{-nt} \sim t^{-a-1} L(1/t).$$

Differentiating this relation r times, as we may do in virtue of Theorem 4, since $a+1>0$, we obtain

(4) $$\Sigma s_n n^r e^{-nt} \sim \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).$$

We shall now prove that, *if any positive ϵ is given, we can choose, first r and ζ, and then*

$$t_0 = t_0(\epsilon, r, \zeta),^*$$

in such a way that

(5) $$\sum_{(1+\zeta)(a+r)/t}^{\infty} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

(6) $$\sum_0^{(1-\zeta)(a+r)/t} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.†

We shall suppose that r and ζ are functions of one another such that $\zeta^2 r \to \infty$ and $\zeta^3 r \to 0$ as $r \to \infty$ and $\zeta \to 0$. We may suppose, for example, that $\zeta^5 r^2 = 1$. The condition that $\zeta^3 r \to 0$ will not be used until § 9.

8. It follows from (2) that the left-hand side of (5) is of the form

$$O \sum_{(1+\zeta)(a+r)/t}^{\infty} n^{a+r} L(n) e^{-nt}.‡$$

The maximum of the function under the sign of summation, considered as

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* r will be large, ζ small, and t_0 small; and r and ζ will be connected in a way which will be defined precisely in a moment.

† The limits of summation are not in general integral. The extreme terms, considered as functions of t, are not of higher order than $t^{-a-r} L(1/t)$, and the argument is in no way affected by including or excluding an additional term or two.

‡ Here and in the sequel the constant implied by the O is independent of r, ζ, and t.

a function of n, occurs for a value of n given by an equation

$$a+r+\epsilon_n = nt,$$

where ϵ_n is a function of n only which tends to zero as $n \to \infty$. Hence the function decreases steadily throughout the limits of the summation, and

$$\sum_{(1+\zeta)(a+r)/t}^{\infty} n^{a+r} L(n)\, e^{-nt} < \nu^{a+r} L(\nu)\, e^{-\nu t} + \int_{(1+\zeta)(a+r)/t}^{\infty} u^{a+r} L(u)\, e^{-ut}\, du,$$

where $n = \nu$ corresponds to the first term of the sum. The isolated term may be neglected.*

The integral we write in the form

$$\int_{(1+\zeta)(a+r)/t}^{\infty} u^{a+r} L(u)\, e^{-ut/(1+\zeta)}\, e^{-\zeta ut/(1+\zeta)}\, du.$$

The maximum of the function $u^{a+r} L(u) e^{-ut/(1+\zeta)}$ is given by an equation of the form

$$(1+\zeta)(a+r+\epsilon_u) = ut.$$

Writing $(1+\zeta)(a+r+\epsilon_u)/t$ for u in the first three factors of the subject of integration, and observing that the functions

$$\left(1+\frac{\epsilon_u}{a+r}\right)^{a+r}, \qquad L\left\{\frac{(1+\zeta)(a+r+\epsilon_u)}{t}\right\}$$

are, when r is large enough and t small enough, certainly less than constant multiples of 1 and $L(1/t)$ respectively, we see that our integral is of the form

$$O\left[\left\{\frac{(1+\zeta)(a+r)}{t}\right\}^{a+r} L\left(\frac{1}{t}\right) e^{-a-r} \int_{(1+\zeta)(a+r)/t}^{\infty} e^{-\zeta ut/(1+\zeta)}\, du\right]$$

$$= O\left[\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-(a+r)\{1+\zeta-\log(1+\zeta)\}}\, t^{-a-r-1} L\left(\frac{1}{t}\right)\right]$$

$$= O\left\{\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-a-r}\, t^{-a-r-1} L\left(\frac{1}{t}\right)\right\},$$

since $\zeta - \log(1+\zeta) > 0$ when $\zeta > 0$. Our conclusion now follows from

* See the last footnote.

the facts that

$$\Gamma(a+r+1) \sim (a+r)^{a+r+\frac{1}{2}} e^{-a-r} \sqrt{(2\pi)},$$

as $r \to \infty$, and that $\zeta^2 r \to \infty$.

The inequality (6) may be established in the same way. We write

$$e^{-ut} = e^{-ut/(1-\zeta)} e^{\zeta ut/(1-\zeta)},$$

and have finally to observe that $\zeta + \log(1-\zeta) < 0$. Otherwise the argument is practically the same.

From (4), (5), and (6) it follows that *when ϵ is given, we can choose r, ζ, and $t_0(\epsilon, r, \zeta)$ in such a way that*

$$(7)\quad (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right) < \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} s_n n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.* Hence, as s_n is an increasing function of n, we obtain

$$(8)\quad s_{(1-\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

$$(9)\quad s_{(1+\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} > (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).\dagger$$

9. Now write
$$n = \frac{a+r}{t} + \lambda,$$

so that $|\lambda| < \zeta(a+r)/t$. The next step in the proof consists in showing that we may replace $n^r e^{-nt}$, in the inequalities (8) and (9), by

$$\left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\frac{r\lambda^2 t^2}{2(a+r)^2}\right\}.$$

We have here to make use of the second relation between r and ζ, namely that $\zeta^3 r$ is small. We have

$$n^r e^{-nt} = \left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\lambda t + r\log\left(1+\frac{\lambda t}{a+r}\right)\right\}.$$

---

* We do not imply that r, ζ, t_0 have the *same* values as before.

† We may interpret s_x, when x is not integral, as meaning $s_{[x]}$. But the truth of the inequalities would not be affected by the inclusion or exclusion of an additional term or two.

Also, as $|\lambda| < \zeta(a+r)/t$, we have

$$r\log\left(1+\frac{\lambda t}{a+r}\right)-\lambda t = -\frac{r\lambda^2t^2}{2(a+r)^2}+\frac{r\lambda^3t^3}{3(a+r)^3}-\dots-\frac{a\lambda t}{a+r}$$
$$= -\frac{r\lambda^2t^2}{2(a+r)^2}+O(\zeta^3 r)+O(\zeta);$$

and the factor
$$e^{O(\zeta^3 r)+O(\zeta)}$$

tends to 1 as r tends to infinity and ζ to zero. If now we make this substitution in (8) and (9), and also substitute for $\Gamma(a+r+1)$ its asymptotic equivalent given by Stirling's theorem, we arrive at the following conclusion. *Given any positive* ϵ, *it is possible to choose first* r *and* ζ, *and then* $t_0 = t_0(\epsilon, r, \zeta)$, *in such a way that*

$$(10)\qquad s_{(1-\zeta)(a+r)/t}\sum e^{-\frac12 r\lambda^2t^2/(a+r)^2} < \frac{(1+\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac12}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

$$(11)\qquad s_{(1+\zeta)(a+r)/t}\sum e^{-\frac12 r\lambda^2t^2/(a+r)^2} > \frac{(1-\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac12}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$, *the values of* λ *included in the sums being those which differ from* $(a+r)/t$ *by an integer and are less in absolute value than* $\zeta(a+r)/t$.

10. In the inequalities (10) and (11) we may suppose that λ ranges from $-\infty$ to $+\infty$. For, when this is so,

$$(12)\qquad \sum e^{-\frac12 r\lambda^2t^2(a+r)^2} \sim \frac{a+r}{t}\sqrt{\left(\frac{2\pi}{r}\right)},$$

as $t\to 0$. On the other hand,

$$(13)\qquad \sum_{\lambda>\zeta(a+r)/t} e^{-\frac12 r\lambda^2t^2/(a+r)^2} = O(e^{-\frac12\zeta^2 r})+\int_{\zeta(a+r)/t}^{\infty} e^{-\frac12 r\lambda^2t^2/(a+r)^2}\,d\lambda.$$

The integral is

$$\frac{\zeta(a+r)}{t}\int_1^\infty e^{-\frac12\zeta^2\mu^2 r}\,d\mu = O\left\{\frac{a+r}{t\sqrt{r}}e^{-\frac12\zeta^2 r}\right\}.$$

As $\zeta^2 r$ is large, the sum (13) is small compared with the sum (12).

A similar argument may, of course, be applied to the terms for which λ is large and negative. We may therefore suppose that λ ranges from $-\infty$ to $+\infty$ in (10) and (11).

We now use the asymptotic relation (12) to transform these inequalities. Observing that $r \sim a+r$ as $r \to \infty$, and that, when r is fixed,

$$L\left(\frac{1}{t}\right) \sim L\left(\frac{a+r}{t}\right)$$

as $t \to 0$, we see that *given* ϵ *it is possible to choose* r, ζ, *and* $t_0(\epsilon, r, \zeta)$, *so that*

$$s_{(1-\zeta)(a+r)/t} < \frac{1+\epsilon}{\Gamma(a+1)}\left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{13}$$

$$s_{(1+\zeta)(a+r)/t} > \frac{1-\epsilon}{\Gamma(a+1)}\left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{14}$$

for $0 < t \leqslant t_0$. Taking $n = (1-\zeta)(a+r)/t$ and $n = (1+\zeta)(a+r)/t$ in turn, and remembering that ζ is small, we see that *when* ϵ *is given it is possible to choose* n_0 *so that*

$$(1-\epsilon)\, n^a L(n) < \Gamma(1+a)\, s_n < (1+\epsilon)\, n^a L(n),$$

for $n \geqslant n_0$. Thus Theorem 8 is proved.

11. Theorem 8 has, as we remarked, in § 1, the appearance of a "special" (or "o") Tauberian theorem.* But we can at once deduce from it a theorem of an obviously "general" (or "O") character.

THEOREM 9.—*If we suppose, in Theorem* 8, *that* $a > 0$, *then the condition that* $a_n \geqslant 0$ *may be replaced by the condition that*

$$na_n = O_L\{n^a L(n)\};$$

i.e., that

$$na_n > -Kn^a L(n).$$

For let $$g(x) = \Sigma\{a_n + Kn^{a-1}L(n)\}\, x^n = \Sigma b_n x^n.\dagger$$

Then $b_n > 0$, and

$$g(x) \sim \frac{A+K\Gamma(a)}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

* When all the a's are zero, the theorem really is an "o" theorem. This may account for its having this appearance in general.

† We may suppress enough terms at the beginning to ensure that $L(n)$ is defined for all values of n in question.

Hence, by Theorem 8,

$$\sum^{n} \{a_\nu + K\nu^{a-1}L(\nu)\} \sim \frac{A+K\Gamma(a)}{\Gamma(a+1)} n^a L(n),$$

and so

$$s_n \sim \frac{A}{\Gamma(a+1)} n^a L(n).$$

It is also easy to see that, in all the theorems which we have been discussing, the function $L(u)$, instead of having the special form

$$(\log u)^{a_1} (\log \log u)^{a_2} \ldots,$$

may be any logarithmico-exponential function,* such that

$$u^{-\delta} \prec L(u) \prec u^{\delta}.$$

Hence we deduce

THEOREM 10.—*If* $a_n \geqslant 0$, *and*

$$f(x) \sim \mathfrak{L}\left(\frac{1}{1-x}\right),$$

where $\mathfrak{L}(u)$ *is any logarithmico-exponential function such that*

$$1 \prec \mathfrak{L}(u) \prec u^{\Delta},$$

so that $\mathfrak{L}(u) = u^a L(u)$, *where* $a \geqslant 0$ *and* $u^{-\delta} \prec L(u) \prec u^{\delta}$, *then*

$$s_n \sim \frac{1}{\Gamma(a+1)} \mathfrak{L}(n).$$

12. Theorem 9 may be proved by another method which possesses considerable interest. It is less direct than that which we have followed, and involves an appeal to a theorem of which we have not published any proof. But it exhibits the relations between Theorem 9 and some of our former theorems in a very interesting light.

We suppose, for simplicity, that

$$a_1 = a_2 = \ldots = 0.$$

It is easy to see that it is enough to prove that *if*

$$f(x) = o(1-x)^{-a},$$

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* For an explanation of the terminology and notation of the next few lines see Hardy, "Orders of infinity," *Camb. Math. Tracts*, No. 12.

and $$a_n = O_L(n^{a-1}),$$

then $$s_n = o(n^a).$$

We write $$g(x) = \Sigma(a_n + Kn^{a-1})x^n = \Sigma b_n x^n.$$

Then $b_n > 0$ and $g(x) = O(1-x)^{-a}$; and it follows, as at the beginning of § 7, that

$$\sum_0^n b_\nu = O(n^a),$$

and so that $s_n = O(n^a)$. But, from the equations

$$s_n = O(n^a), \quad \Sigma s_n x^n = o(1-x)^{-a-1},$$

it follows, by Theorem 26 of our last paper,* that

$$\sigma_n = s_0 + s_1 + \ldots + s_n = o(n^{a+1}).$$

Now we know that if $\phi(x)$ is a function of x which has a continuous second derivative $\phi''(x)$ for all sufficiently large values of x, and

$$\phi(x) = o(x^{a+1}), \quad \phi''(x) = O(x^{a-1}),$$

then $$\phi'(x) = o(x^a).\dagger$$

It is possible to generalise this result by writing

$$\phi''(x) = O_L(x^{a-1})$$

for $$\phi''(x) = O(x^{a+1}).$$

And this generalised result possesses an analogue for series, namely: *if*

$$s_0 + s_1 + \ldots + s_n = o(n^{a+1}), \quad a_n = O_L(n^{a-1}),$$

then $$s_n = o(n^a).$$

From this it is clear that we can deduce the result required. It must not be supposed, however, that in this proof we really dispense with the process of r-fold differentiation. This is involved in our former Theorem 26, by an appeal to which we covered the most difficult transition of our proof.

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* *L.c.*, p. 443.

† This follows from Theorem 1 of our last paper if $a \geqslant 1$, and from Theorem 5 in any case.

13. The analogue of Theorem 9, in the case in which

$$a = a_1 = a_2 = \ldots = 0,$$

is as follows.

THEOREM 11.—*If $f(x) \to A$ as $x \to 1$, and $a_n > -K/n$, then Σa_n converges to the sum A.*

This theorem is true, and constitutes a very interesting extension of Littlewood's generalisation of Tauber's theorem. But a special proof is required.*

We have, as $x \to 1$, $$f(x) = A + o(1),$$

and $$f''(x) = \Sigma n(n-1) a_n x^{n-2} > -K \Sigma (n-1) x^{n-2} = O_L (1-x)^{-2},$$

From this it may be deduced that $f' = o(1-x)^{-1}$.

We write y for $1-x$, so that $y \to 0$, by positive values. The theorem we wish to prove is that *the equations*

$$F(y) = A + o(1), \quad F''(y) = O_L(1/y)^2$$

imply $$F'(y) = o(1/y).$$

Suppose first that, if possible $$F'(y_s) > H/y_s \quad (H > 0),$$

for an infinity of values y_s of y whose limit is zero. We have also $F''(y) > -K/y^2$, and so, if $y > y_s$,

$$F'(y) = F'(y_s) + \int_{y_s}^{y} F''(u)\, du > \frac{H}{y_s} - \frac{K(y-y_s)}{y_s^2}.$$

It is clearly possible to choose a positive number δ, so that

$$F'(y) > \tfrac{1}{2} H/y_s,$$

for

$$y_s \leqslant y \leqslant \eta_s = (1+\delta) y_s.$$

And then $$F(\eta_s) - F(y_s) = \int_{y_s}^{\eta_s} F'(y)\, dy > \tfrac{1}{2} H \frac{\eta_s - y_s}{y_s} = \tfrac{1}{2}\delta H,$$

which contradicts $$F(y) = A + o(1).$$

Similarly we can show that it is impossible that

$$F'(y_s) < -H/y_s \quad (H > 0).$$

In this case we start from the fact that, if $0 < y < y_s$,

$$F'(y) = F'(y_s) - \int_{y}^{y_s} F''(u)\, du < -\frac{H}{y_s} + \frac{K(y_s - y)}{y^2},$$

and argue in the same way. Hence $F'(y) = o(1/y)$.

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* If we write $$g(x) = \Sigma \left(a_n + \frac{K}{n}\right) x^n = \Sigma b_n x^n,$$

$g(x)$ is of higher order than $f(x)$. Hence we cannot prove that $s_n = O(1)$ in the way in which we proved $s_n = O(n^a)$ in § 12.

Hence $$\Sigma n a_n x^n = o\left(\frac{1}{1-x}\right),$$

and $$n a_n = O_L(1).$$

Hence, by Theorem 9, $$a_1 + 2a_2 + \ldots + n a_n = o(n);$$

and the convergence of Σa_n now follows from Pringsheim's generalisation of Tauber's theorem.*

14. In Theorem 9 we supposed that $\alpha > 0$. An argument similar to that of the last section enables us to remove this restriction.

THEOREM 12.—*The result of Theorem* 9 *holds even when* $\alpha = 0$.†

We have $$f(x) \sim L\left(\frac{1}{1-x}\right),$$

and $$f''(x) > -K\Sigma(n-1)L(n)x^{n-2} = O_L\left\{\frac{1}{(1-x)^2}L\left(\frac{1}{1-x}\right)\right\}.$$

From this we deduce‡ $$f'(x) = o\left\{\frac{1}{1-x}L\left(\frac{1}{1-x}\right)\right\}.$$

Hence $$\Sigma n a_n x^n = o\left\{\frac{1}{1-x}L\left(\frac{1}{1-x}\right)\right\};$$

and therefore, by Theorem 9, $a_1 + 2a_2 + \ldots + n a_n = o\{nL(n)\}$.

That $s_n \sim L(n)$ now follows from Theorem 45 of our last paper.§

15. Before leaving power series and passing on to Dirichlet's series we may add one further remark. The theorems which we have proved are all of what we have called an "Abel-Tauber" type; in all of them we start from (i) a hypothesis as to the behaviour of $f(x) = \Sigma a_n x^n$ as $x \to 1$, (ii) an inequality satisfied by a_n, and deduce information as to the behaviour of s_n. There are, of course, corresponding theorems of a "Cesàro-Tauber" type, in which the hypothesis (i) is replaced by a hypothesis as to the behaviour of one of Cesàro's means formed from Σa_n. These theorems are naturally easier to prove. We may content ourselves with enunciating the simplest analogue of Theorem 9, viz.,

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* See Bromwich, *Infinite Series*, p. 251, Ex. 28.

† This theorem contains Theorem 11 as a particular case.

‡ The proof is similar to that of § 13.

§ The argument by which Theorem 9 itself was proved would only lead to the result with an unnecessarily severe restriction on a_n, viz., that

$$a_n = O_L\{\psi(n)\},$$

where $$\int^n \psi(u)\,du \sim L(n).$$

Thus, if $L(u) = \log u$, this argument would require $a_n = O_L(1/n)$, whereas the real condition is $a_n = O_L(\log n/n)$. The reason why a more elaborate argument is needed when $\alpha = 0$ than when $\alpha > 0$ lies in the fact that

$$\int^n n^\alpha L(u)\,\frac{du}{u}$$

is of order $n^\alpha L(n)$ when $\alpha > 0$, but of higher order when $\alpha = 0$.

THEOREM 13.—*If* $(s_0+s_1+\ldots+s_n)/(n+1) \sim An^a \quad (A>0,\ a>0),$

and $na_n = O_L(n^a),$

then $s_n \sim An^a.$

It was substantially this theorem which was assumed at the end of § 12. The reader will have no difficulty in framing further theorems of this type and of a more general character.

16. We conclude by a brief statement of the analogues of the most important of the preceding theorems for ordinary Dirichlet's series. There are corresponding theorems, for Dirichlet's series of the general type $\Sigma a_n e^{-\lambda_n s}$, which it is our intention to publish elsewhere, and we shall therefore not enter into the details of the proofs.

THEOREM 14.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with positive coefficients, convergent for* $s>1$, *and*

$$f(s) \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right) \quad (a>0),$$

as $s \to 1$, *then*

$$(-1)^r f^{(r)}(s) = \Sigma a_n (\log n)^r n^{-s} \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(s-1)^{a+r}} L\left(\frac{1}{s-1}\right).$$

The argument by which we prove this theorem is substantially the same as that which we used in proving Theorem 4. We have only to observe that, if $g(s) = f(s)/(s-1)$, then

$$(s-1)\,g'(s) = f'(s) - \frac{f(s)}{s-1} = -\Sigma \log n\, a_n n^{-s} - \frac{1}{s-1}\Sigma a_n n^{-s}$$

steadily decreases as $s \to 0$.

Theorem 14, though interesting in itself, does not give us precisely what is required for the proof of the analogue of Theorem 8. This is contained in

THEOREM 15.—*A result similar to that of Theorem* 14 *holds for series of the form*

$$\Sigma a_n \left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} \quad (a_n \geqslant 0).$$

The proof of this theorem is very much the same as that of Theorem 14.

17. From Theorem 15 we can deduce the analogue of Theorem 8, viz.,

THEOREM 16.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with*

positive coefficients, convergent for $s > 1$, *and*

$$f(s) \sim \frac{A}{(s-1)^{a}} L\left(\frac{1}{s-1}\right),$$

the indices $a, a_1, \ldots$ *being such that* $(\log n)^a L(\log n)$ *tends to a positive limit or to infinity as* $n \to \infty$, *then*

$$s_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n} \sim \frac{A}{\Gamma(a+1)} (\log n)^a L(\log n).$$

We have first

$$s_n \leqslant e \sum_{1}^{n} \frac{a_\nu}{\nu} e^{-s \log \nu / \log n} < ef\left(\frac{1}{\log n}\right),$$

and so

$$s_n = O\left\{(\log n)^a L(\log n)\right\}.$$

Next

$$f(s) = \Sigma s_n \left\{\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\right\} \sim \frac{A}{(s-1)^{a}} L\left(\frac{1}{s-1}\right),$$

a relation which takes the place of (3) of § 7. We differentiate r times, as we are entitled to do in virtue of Theorem 15; and the rest of the argument follows the lines of the proof of Theorem 8, no new difficulty of principle occurring.

17. We do not propose to write this argument out at length, nor to discuss in detail the analogues of Theorems 9 *et seq.* It may, however, be observed that the left-handed condition which now occurs instead of

$$na_n = O_L\left\{n^a L(n)\right\}$$

is

$$\log n \, a_n = O_L\left\{(\log n)^a L(\log n)\right\}.$$

Further, the analogue of Theorem 11 deserves a separate statement. It is

THEOREM 17.—*If* $f(s) = \Sigma a_n n^{-s} \to A$ *as* $s \to 1$, *and* $a_n > -K/\log n$, *then* $\Sigma(a_n/n)$ *converges to the sum* A.

This is the "left-handed" form of Littlewood's* generalisation of Landau's† analogue of Tauber's theorem for ordinary Dirichlet's series.

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* Littlewood, *l.c.*, p. 438. As we are supposing s to tend to unity instead of to zero, our condition is $a_n > -K/\log n$ instead of $a_n > -K/n \log n$.

† Landau, *Monatshefte für Math.*, Vol. 18, p. 8.

CORRECTIONS

p. 179, *line* 7. For $(-1)^{(r)}$ read $(-1)^r$.

—— *lines* 14–15. For $\dfrac{K}{\Gamma(\alpha)}$ read $K\Gamma(\alpha)$ (twice).

—— *line* 18. For 'Cesàro's means' read 'Cesàro's sums'; compare line 20.

—— *line* 19. Read s_n.

p. 180, *line* 7. For $\sum$ read $\sum\limits_{1}^{\infty}$.

p. 182, *line* 1 *up*. For 'Our conclusion' read 'The inequality (5)'.

—— *footnote*. For 'the last footnote' read 'footnote † on p. 181'.

p. 184, *line* 9 *up*. The series in (12) should be the same as that in (10) and (11).

p. 186, *line* 8. The line should read '$u^{-\delta} \prec L(u) \prec u^{\delta}$, if $\alpha > 0$; or $1 \prec L(u) \prec u^{\delta}$, if $\alpha = 0$'.

p. 187, *line* 9 *up*. For $x^{\alpha+1}$ read $x^{\alpha-1}$.

p. 189, *last footnote, line* 2 *up*. For n^{α} read u^{α}.

p. 191, *line* 7. For $e^{-s\log\nu/\log n}$ read $e^{-\log\nu/\log n}$. For $f\left(\dfrac{1}{\log n}\right)$ read $f\left(1+\dfrac{1}{\log n}\right)$.

COMMENTS

There is an abstract of the main results of this paper in 1913, 10.

Theorem 1 seems to need some amendment before it includes Theorem 2. If we transform Theorem 2, by putting $y = \phi(x)$, $x = \psi(y)$, $f(x) = F(y)$, it becomes: *Theorem* 1′. *If* $\psi'(y)$ *is positive and continuous, and* $\psi(y) \to \infty$ *as* $y \to \infty$, *and if* (i) $\psi(y)/\psi'(y) \sim y$, (ii) $F'(y)\psi(y)/\psi'(y)$ *increases, then* $F(y) \sim y$ *implies* $F'(y) \sim 1$ *as* $y \to \infty$. Theorem 2 is obtained by reversing the transformation. The proof of Theorem 1′ is similar to that of Theorem 1.

To obtain Theorem 3, a positive factor α should be inserted on the right in condition (i) of Theorem 1′, and on the left in condition (i) of Theorems 2 and 2(a).

The formula in § 6, for the fractional integral of a sum of powers of exponentials, is due to Liouville.†

Theorem 11, § 13, is the O_L-Tauberian theorem for Abel summability; stated in an addendum to 1913, 2. It includes Landau's O_L-theorem for Cesàro summability‡ and Littlewood's O-theorem for Abel summability,§ which both include Hardy's O-theorem for Cesàro summability (1910, 3). The O_L-Tauberian condition was extended by Schmidt|| to: s_n *slowly decreasing*, i.e. $\underline{\lim}(s_m - s_n) \geqslant 0$ as $n \to \infty$, $m > n$, $m/n \to 1$. Theorem 11 is deduced from Theorem 8, the 'positive' Tauberian theorem. In D.S., pp. 154–5 and 162–3, Hardy shows, conversely, that Theorem 8 (with $\alpha = 1$, $L(u) = 1$) may be deduced from Theorem 11.

The one-sided result for sequences, stated at the end of § 12, needs to be proved independently of the first proof of Theorem 9, in order to complete the second proof. For $\alpha = 0$, the result is Landau's O_L-theorem for $(C, 1)$ summability (loc. cit.); the general case is included in a theorem of Mordell.††

† *J. de l'École polytech.* 13, cah. 21 (1832), 1–67.
‡ *Prace Mat.-Fiz.* 21 (1910), 97–177.
§ *Proc. London Math. Soc.* (2), 9 (1911), 434–48.
|| *Math. Zeit* 22 (1925), 89–152.
†† *J. London Math. Soc.* 3 (1928), 86–9.

The corresponding one-sided result for derivatives, also stated, is not used in § 12. A proof, for $\alpha = -1$, is contained (effectively) in the proof of Theorem 11, since it makes no difference whether $y \to 0$ or $y \to \infty$. Mordell‡‡ gave another proof for $\alpha = -1$, and a proof in Landau's *Darstellung* . . ., §§ for $\alpha \leqslant -1$, is substantially unaltered if $\alpha > -1$. These proofs depend on Taylor's theorem, and do not require the continuity of f''; compare the footnote at the end of 1914. 11.

In §§ 12 and 14, references to 'our last paper' are to 1913, 2. The theorems for Dirichlet's series of general type, mentioned at the beginning of § 16, are proved by Hardy and Littlewood in 1914, 11.

‡‡ *J. London Math. Soc.* 3 (1928), 119–21.

§§ *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie,* 2nd edn. (1929), p. 58. Springer, Berlin.

NOTE ON LAMBERT'S SERIES

By G. H. Hardy.

[Received September 25th, 1913.—Read November 13th, 1913.]

1. In a very interesting memoir "Über Lambertsche Reihen," published recently in *Crelle's Journal*,* Herr K. Knopp proves the following theorem :—

Suppose that the series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l} \quad (l = 0,\ 1,\ 2,\ \ldots,\ k-1),$$

are convergent, so that the series

$$\sum_{1}^{\infty} a_n \frac{x^n}{1-x^n}$$

is certainly absolutely convergent for $r = |x| < 1$. *Let* $f(x)$ *denote the sum of the latter series for* $r < 1$, *and let*

$$x = rx_0 = re^{2\kappa\pi i/k},$$

where κ *is prime to* k. *Then*

$$\lim_{r \to 1} (1-r) f(x) = \sum_{1}^{\infty} \frac{a_{k\nu}}{k\nu}.$$

In its simplest form, when $k = 1$, this theorem is the analogue for Lambert's series, that is to say, series of the form

$$\Sigma a_n \frac{x^n}{1-x^n},$$

of Abel's theorem on the continuity of power series. I gave a proof of

* *Journal für Math.*, Vol. 142, p. 283. A less general theorem of the same character was proved by Franel, *Math. Annalen*, Vol. 52, pp. 543 *et seq.*

1914, 5 Proceedings of the London Mathematical Society (2), 13, 192–8.

this special form of the theorem in a paper published some years ago in these *Proceedings*;* and in a later paper† I stated without proof a generalised form of the result, in which the hypothesis of convergence is replaced by that of summability by some one of Cesàro's means. It is naturally suggested that Knopp's theorem should be capable of a similar generalisation, in which the hypothesis is that of the summability of the various series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l}.$$

That the theorem thus suggested is, in fact, true will appear in the sequel. It is not, however, precisely that which I propose to prove. A little reflection, in fact, shows that Knopp's hypothesis may be replaced by another which is more natural and also slightly more general. His theorem consists in reality of two parts. Let us write

$$f(x) = \sum_1^\infty a_{k\nu} \frac{x^{k\nu}}{1-x^{k\nu}} + \sum_{l=1}^{k-1} \sum_0^\infty a_{k\nu+l} \frac{x^{k\nu+l}}{1-x^{k\nu+l}} = f_0(x) + \sum_{l=1}^{k-1} f_l(x),$$

the first series containing those terms which become infinite as $x \to x_0$. It is clear that the limit assigned by the theorem arises solely from $f_0(x)$: the theorem is, in fact, equivalent to two theorems, expressed respectively by the equations

$$\lim (1-r) f_0(x) = \Sigma \frac{a_{k\nu}}{k\nu}, \tag{1}$$

$$\lim (1-r) f_l(x) = 0. \tag{2}$$

Conditions for the truth of (1) are naturally expressed in terms of the series on the right-hand side; but there is nothing in (2) to suggest the introduction of the series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l}.‡$$

I propose therefore to modify Knopp's condition. The condition

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 253.

† *Math. Annalen*, Vol. 64, p. 91.

‡ The denominator $k\nu$ arises from the fact that

$$\lim \frac{1-r}{1-x^{k\nu}} = \lim \frac{1-r}{1-r^{k\nu}} = \frac{1}{k\nu}.$$

But

$$\lim \frac{1-r}{1-x^{k\nu+l}} = 0.$$

which I shall suppose satisfied is that

$$\sum_{\nu=0}^{m} a_{k\nu+l} = o(m).$$

This is more appropriate and also more general:* we shall find, moreover, that the adoption of the more general hypothesis leads to a simplification of the proof.

This modified form of Knopp's theorem is a particular case of the theorem which follows, and the proof of which is the object of this note.

2. THEOREM 1.—*Let* $a_{k\nu+l} = b_{\nu,\,l}$, *and suppose that an integer* p *exists, such that* (i) *the series*

$$\Sigma \frac{b_{\nu,\,0}}{k\nu} = \Sigma \frac{a_{k\nu}}{k\nu}$$

is summable (C, p), (ii) *the* p-*th Cesàro sum* $B^{p}_{\nu,\,l}$, *formed from the series* $\Sigma\, b_{\nu,\,l}$, *satisfies the relation*

$$B^{p}_{\nu,\,l} = o(\nu^{p+1}).$$

Then

$$\lim_{r\to 1} (1-r) f(x) = \Sigma \frac{a_{k\nu}}{k\nu}.\dagger$$

3. This theorem, like Knopp's theorem, is really equivalent to two. We first consider $f_0(x)$, and write $y = x^k$ (so that y is real), and $c_\nu = b_{\nu,\,0}/\nu$. Then it is clear that what we have to prove is

THEOREM 2.—*If* Σc_ν *is summable* (C, p), *then*

$$\lim_{y\to 1} \Sigma c_\nu \frac{\nu y^\nu (1-y)}{1-y^\nu} = \Sigma c_\nu.$$

---

* If s_n is the sum of the first n terms of a series Σu_n the convergence of $\Sigma (u_n/n)$ involves $s_n = o(n)$, whereas the converse is not true.

† That the series which represents $f(x)$ still converges absolutely for $r < 1$ is trivial. That the hypothesis (ii) is more general than the hypothesis that $\Sigma \frac{a_{k\nu+l}}{k\nu+l}$ is summable (C, p), follows from Theorem 14 of Mr. Littlewood's and my paper "Contributions to the Arithmetic Theory of Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 435.

The *least* values of p for which (i) or (ii) is satisfied may differ according to the value of l. If one of them is satisfied for any special p, it is satisfied for any greater p; there is therefore no objection to supposing all the p's the same.

This is the theorem which I stated in my paper in the *Math. Annalen* already referred to. It may be proved as follows.

Write $$y = e^{-u}, \quad \phi_\nu(u) = \frac{\nu e^{-\nu u}(1-e^{-u})}{1-e^{-\nu u}}.$$

The result will follow from Theorem 3 of the paper quoted, if we can show that

(1) the differences $$\Delta^\lambda \phi_\nu(u) \quad (0 \leqslant \lambda \leqslant p+1)$$

can be divided into q_λ groups of successive terms of the same sign, where q_λ is a number which may depend upon u, but remains less than a constant as $u \to 0$;

(2) the absolute value of

$$|\nu^\lambda \Delta^\lambda \phi_\nu(u)| \quad (0 \leqslant \lambda \leqslant p)$$

is less than a constant for all values of ν and u in question.

Let $$\Phi(\xi) = \frac{\xi e^{-\xi u}}{1-e^{-\xi u}} = \frac{\xi}{e^{\xi u}-1} = \xi \Psi(\xi).$$

Then it is easily verified that

$$\Psi^{(\lambda)}(\xi) = (-u)^\lambda \left\{ \frac{A_{\lambda,1}}{e^{\xi u}-1} + \frac{A_{\lambda,2}}{(e^{\xi u}-1)^2} + \ldots + \frac{A_{\lambda,\lambda+1}}{(e^{\xi u}-1)^{\lambda+1}} \right\},$$

$$\Phi^{(\lambda)}(\xi) = (-u)^{\lambda-1} \left\{ \frac{\lambda A_{\lambda-1,1} - A_{\lambda,1}\xi u}{e^{\xi u}-1} + \frac{\lambda A_{\lambda-1,2} - A_{\lambda,2}\xi u}{(e^{\xi u}-1)^2} \right.$$

$$\left. + \frac{\lambda A_{\lambda-1,\lambda} - A_{\lambda,\lambda}\xi u}{(e^{\xi u}-1)^\lambda} - \frac{A_{\lambda,\lambda+1}\,\xi u}{(e^{\xi u}-1)^{\lambda+1}} \right\},$$

where the A's are positive constants and $A_{\lambda,1} = 1$, $A_{\lambda,\lambda+1} = \lambda!$ for all values of λ. If $\Phi^{(\lambda)}(\xi) = 0$, we obtain

$$(\lambda - \xi u)\, e^{\lambda \xi u} + \ldots + \ldots = 0,$$

an equation in ξu whose remaining terms contain powers of $e^{\xi u}$ lower than the λ-th. It is plain that the number s of positive roots of this equation depends only on λ. Let us denote these roots by $\eta_1, \eta_2, \ldots, \eta_s$. Then the roots of $\Phi^{(\lambda)}(\xi) = 0$ are

$$\xi = \eta_1/u, \quad \eta_2/u, \quad \ldots, \quad \eta_s/u.$$

Now $$\phi_\nu(u) = (1-e^{-u})\,\Phi(\nu),$$

$$\Delta^\lambda \phi_\nu(u) = (1-e^{-u})\Delta^\lambda \Phi(\nu) = (-1)^\lambda (1-e^{-u})\Phi^{(\lambda)}(\xi),$$

where $\nu \leqslant \xi \leqslant \nu+\lambda$. Choose u small enough to ensure that no two of $\eta_1/u,\ \eta_2/u,\ \ldots,\ \eta_s/u$ differ by less than λ, so that the interval $(\nu,\ \nu+\lambda)$ can include at most one root of $\Phi^{(\lambda)}(\xi)=0$. As ν increases from 1 onwards, $\Delta^\lambda\phi_\nu(u)$ remains of fixed sign until $\nu+\lambda > \xi_1/u$. After this alternations may occur, but they must cease as soon as $\nu > \xi_1/u$. The total number of possible changes of sign associated with the root ξ_1/u is at most $\lambda+1$. Proceeding in this way we see that $\Delta^\lambda\phi_\nu(u)$ cannot change sign more than $(\lambda+1)\,s$ times in all. This proves the proposition (1).

In order to prove (2) we must show that

$$|(1-e^{-u})\,\nu^\lambda\,\Delta^\lambda\Phi(\nu)|$$

is less than a constant; and this will be so if the same assertion is true of

$$|u\xi^\lambda\Phi^{(\lambda)}(\xi)|.$$

Referring back to the explicit formula for $\Phi^{(\lambda)}(\xi)$, we see that what we have to prove is that the functions

$$\frac{(\xi u)^\lambda}{(e^{\xi u}-1)^\mu}\ (\mu \leqslant \lambda), \qquad \frac{(\xi u)^{\lambda+1}}{(e^{\xi u}-1)^\mu}\ (\mu \leqslant \lambda+1)$$

are less than constants, and this is obvious. Thus (2) is true, and the proof of Theorem 2 is accordingly completed.

4. In order to complete the proof of Theorem 1, we have to show that

$$f_l(x) = \Sigma b_{\nu,\,l}\,\frac{x^{k\nu+l}}{1-x^{k\nu+l}} = o\left(\frac{1}{1-r}\right).$$

The series may be written in the form

$$\Sigma b_{\nu,\,l}\,\frac{ar^{k\nu+l}}{1-ar^{k\nu+l}},$$

where $a = e^{2l\kappa\pi i/k}$. Hence our theorem will follow as a corollary of

THEOREM 3.—*If*

$$g(r) = \Sigma c_\nu\,\frac{r^{k\nu+l}}{1-ar^{k\nu+l}},$$

where a is any number other than a positive number not less than 1, *and the p-th Cesàro sum C_ν^p formed from Σc_ν satisfies*

$$C_\nu^p = o(\nu^{p+1}),$$

then

$$g(r) = o\left(\frac{1}{1-r}\right),$$

as $r \to 1$.

The proof of this is simple. Let $\rho = r^k$ and $\beta = ar^l$. Then

$$\Delta \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \Delta \frac{\rho^\nu}{1-\beta\rho^\nu} = \frac{(1-\rho)\,\rho^\nu}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})},$$

$$\Delta^2 \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \frac{(1-\rho)^2\,\rho^\nu(1+\beta\rho^{\nu+1})}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})(1-\beta\rho^{\nu+2})},$$

and generally

$$\Delta^{p+1} \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \frac{(1-\rho)^{p+1}\,\rho^\nu\chi_{p+1}}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})\dots(1-\beta\rho^{\nu+p+1})},$$

where χ_{p+1} is a polynomial in ρ and β whose coefficients depend only on p. As β satisfies the same condition as a, the factors in the denominator are all greater in absolute value than a constant. Hence

$$\Delta^{p+1} \frac{r^{k\nu}}{1-ar^{k\nu+l}} = (1-\rho)^{p+1}\rho^\nu O(1);$$

and so $\Sigma c_\nu \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \Sigma C_\nu \Delta \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \dots = \Sigma C_\nu^p \Delta^{p+1} \frac{r^{k\nu}}{1-ar^{k\nu+l}}$

$$= O(1-r)^{p+1}\,\Sigma o(\nu^{p+1})\, r^{k\nu} = o\left(\frac{1}{1-r}\right).$$

5. It is easy to verify that all our conditions are satisfied if

$$a_n = (-1)^n n^s,$$

where s is any number real or complex, and k is odd. Hence, when x approaches the point $e^{2\kappa\pi i/k}$ along a radius vector,

$$f(x) = \Sigma \frac{(-1)^n n^s x^n}{1-x^n} \sim \frac{k^{s-1}}{1-r}\,\Sigma(-1)^{k\nu}\nu^{s-1}.$$

If k is even, the series on the right is no longer summable; and $f(x)$ is, in fact, of higher order than $1/(1-r)$. Suppose, *e.g.*, that s is positive, and $k=2$, $\kappa=1$, so that $x \to -1$. Then

$$f(x) = \Sigma \frac{n^s y^n}{1-y^n} \sim \frac{\Gamma(s+1)\,\zeta(s+1)}{(1-y)^{s+1}},$$

as

$$y = -x \to 1.\text{*}$$

6. It is natural to suppose that Theorems 1–3 retain their validity

* See Knopp, *Dissertation*, Berlin, 1907, p. 34.

when $x \to x_0$ along any "Stolz-path," *i.e.*, any curve which has a continuous tangent and does not touch the unit circle. Knopp* has extended his theorem to this case, but only under a narrower hypothesis, viz., that the series $\Sigma \left| \frac{a_n}{n} \right|$ is convergent. It is quite easy to see that Theorem 3 is still true under the more general hypothesis; but to make the corresponding extension of Theorem 2 (and so of Theorem 1) appears to be a less simple matter. The proof would presumably be based upon a theorem of Dr. Bromwich† which includes as a special case the theorem of mine used in § 3.

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* *L.c.*, § 1, p. 300.

† *Math. Annalen*, Vol. 65, p. 359.

COMMENTS

In D.S., Appendix IV, p. 373, footnote, Hardy remarks that Theorem 2, § 3, is more simply deduced from the theorem of Bromwich† mentioned in § 6 (cf. 1936, 1, p. 1, footnote) that: *if* (i) $\sum n^p |\Delta^{p+1} f_n(y)| < K$ $(0 < y < 1)$, (ii) $f_n = o(n^{-p})$ $(0 < y < 1)$, *and* (iii) $f_n(y) \to 1$ *as* $y \to 1$, *and if* $\sum c_n$ *is summable* (C, p) *to* s, *then* $\sum c_n f_n(y)$ *converges for* $0 < y < 1$ *and tends to* s *as* $y \to 1$.

Theorem 3 is included in a similar theorem, in which n^p is replaced by n^{p+1} in (i) and (ii), and 1 replaced by 0 in (iii).

In 1921, 6 (in Vol. II) Hardy and Littlewood prove the deeper theorem that: *if* $\sum c_n$ *is summable by Lambert's method*‡ (*i.e. the conclusion in Theorem* 2 *holds*), *then* $\sum c_n$ *is summable by Abel's method.* See also D.S., pp. 373–4, and the Comments on 1921, 6. Some related Tauberian theorems are given in 1936, 1.

† *Math. Annalen* 65 (1908), 350–69; see the Comments on 1907, 6.
‡ D.S., p. 372.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXVII.

On the region of convergence of Borel's integral.

1. BOREL'S integral, associated with a power series

(1) $$\Sigma a_n x^n,$$

is

(2) $$f(x) = \int_0^\infty e^{-t} u(tx)\, dt,$$

where

(3) $$u(x) = \Sigma \frac{a_n x^n}{n!}.$$

If the series (1) has a positive radius of convergence, the region of convergence of the integral (2) is Borel's "polygon of summability"; the integral is convergent everywhere inside, and nowhere outside, the polygon, and represents the analytic function $f(x)$ defined in the ordinary way by the series (1).

Let us suppose now that the radius of convergence of (1) is zero. If (2) converges for $x = x_0$, it converges uniformly along the straight line $(0, x_0)$.* And if it represents an analytic function $f(x)$ in a region D, that region must extend up to the origin, and the origin must be a singular point of $f(x)$.

My object in this note is to show by examples how Borel's integral may converge in two different regions of the plane, having only the origin as a common boundary point, and represent, in these two regions, two different analytic functions.

2. I consider first the series

$$1 + 0 - \frac{2!}{1!} + 0 + \frac{4!}{2!} + 0 - \ldots,$$

in which $$a_{2\nu} = (-1)^\nu \frac{2\nu!}{\nu!}, \quad a_{2\nu+1} = 0.$$

Here Borel's integral is

$$f(x) = \int_0^\infty e^{-t-x^2t^2}\, dt,$$

* See Note XXXI., vol. xl., p. 161.

which is plainly convergent if

$$-\tfrac{1}{4}\pi \leqq \operatorname{am} x \leqq \tfrac{1}{4}\pi$$

or

$$\tfrac{3}{4}\pi \leqq \operatorname{am} x \leqq \tfrac{5}{4}\pi,$$

i.e., in two quadrants abutting at the origin.

Suppose x real and positive. Then

$$f(x) = \frac{1}{x}\int_0^\infty e^{-(u/x)-u^2}\,du,$$

or, if $x = 1/y$,

$$f(x) = y\int_0^\infty e^{-yu-u^2}\,du = ye^{\frac{1}{4}y^2}\int_{\frac{1}{2}y}^\infty e^{-v^2}\,dv$$

$$= ye^{\frac{1}{4}y^2}\left\{\tfrac{1}{2}\sqrt{\pi} - \int_0^{\frac{1}{2}y} e^{-v^2}\,dv\right\}$$

$$= F(y)$$

say. The function $F(y)$ is an integral function of y. Thus, in the quadrant which includes the positive real axis,

$$f(x) = F(1/x).$$

In the other quadrant it is plain that

$$f(x) = F(-1/x),$$

which differs from $F(1/x)$ by

$$\frac{\sqrt{\pi}}{x}e^{-\frac{1}{4}x^2}.$$

Thus $f(x)$ is equal to different analytic functions in the two regions.

3. As a second example I shall consider the series in which

$$a_n = \sum_0^\infty \frac{(-1)^\nu \nu^n}{\nu!}.$$

Here

$$u(x) = \sum_0^\infty \frac{x^n}{n!}\sum_0^\infty \frac{(-1)^\nu \nu^n}{\nu!} = \sum_0^\infty \frac{(-1)^\nu}{\nu!}e^{\nu x}$$

$$= e^{-e^x}.$$

Thus Borel's integral is

$$f(x) = \int_0^\infty e^{-t-e^{tx}}\,dt. \tag{4}$$

If $x = \xi + i\eta$,

$$|e^{-e^{tx}}| = e^{-e^{\xi t}\cos \eta t}.$$

It is easy to see that, if $\xi > 0$, the integral (4) is convergent if and only if $\eta = 0$. On the other hand, if $\xi \leqq 0$, it is convergent for all values of η. Thus the integral is convergent

(i) along the positive real axis and (ii) in the half-plane to the left of the imaginary axis.

First suppose $x = \xi > 0$. Then, putting $e^t = u$, we obtain

$$f(x) = \int_0^\infty e^{-t-e^{\xi t}}\, dt = \int_1^\infty e^{-u^\xi} \frac{du}{u^2}$$

$$= \frac{1}{\xi} \int_1^\infty e^{-w} w^{-(1/\xi)-1}\, dw$$

$$= -y \int_1^\infty e^{-w} w^{y-1}\, dw,$$

where $y = -1/\xi = -1/x$. If, in the ordinary notation of the theory of the Gamma function, we write

$$\Gamma(s) = \int_0^\infty e^{-w} w^{s-1}\, dw = \int_0^1 e^{-w} w^{s-1}\, dw + \int_1^\infty e^{-w} w^{s-1}\, dw$$

$$= P(s) + Q(s),$$

when the real part of s is positive, then $Q(s)$ is an integral function of s; and, for x real and positive, we have

(5) $$f(x) = \frac{1}{x} Q\left(-\frac{1}{x}\right).$$

Secondly, suppose $\xi < 0$, and $x = \xi = -\lambda$. Then

$$f(x) = \int_0^\infty e^{-t-e^{-\lambda t}}\, dt = \int_1^\infty e^{-u^{-\lambda}} \frac{du}{u^2}$$

$$= \frac{1}{\lambda} \int_0^1 e^{-w} w^{(1/\lambda)-1}\, dw$$

$$= y \int_0^1 e^{-w} w^{y-1}\, dw,$$

where $y = 1/\lambda = -1/\xi = -1/x$. Thus, for real negative values of x,

(6) $$f(x) = -\frac{1}{x} P\left(-\frac{1}{x}\right).$$

The function $P(s)$ is regular for all values of s save negative integral values (including zero), where it has simple poles. Thus

$$-\frac{1}{x} P\left(-\frac{1}{x}\right)$$

is regular in the half-plane which we are considering, and it is clear that equation (6) is valid throughout this half-plane. The equations (5) and (6) show that Borel's integral converges, for different values of x, to two different analytic functions.

COMMENTS

The examples in this paper illustrate 1911, 8; see D.S., pp. 184–90, and the Comments on 1911, 8.

SOME THEOREMS CONCERNING DIRICHLET'S SERIES.

By *G. H. Hardy* and *J. E. Littlewood.*

I.

1. THE present paper is intended as a supplement to a series of papers published during the last few years in the *Proceedings of the London Mathematical Society*.

These papers have been concerned, in the main, with what we have called "Tauberian" theorems, theorems whose general character is the same as that of Tauber's well-known converse of Abel's theorem on the continuity of a power-series. The most typical Tauberian theorems have, as one of their hypotheses, a hypothesis of the type

$$(1.1) \qquad a_n = O(n^\alpha),$$

where a_n is the general term of the series considered. It is a natural conjecture that there must be analogues of these theorems in which this hypothesis is replaced by one as to the convergence of a series of the type

$$(1.2) \qquad \Sigma\, n^\beta |a_n|^\gamma;$$

and the fundamental importance of such hypotheses in the theory of Fourier's series suggests that theorems of this character might prove to be very interesting.

One such theorem has been proved already by Fejér.* Fejér shows that

if (i) *the series* Σa_n *is summable* $(C1)$, (ii) *the series* $\Sigma\, n\,|a_n|^2$ *is convergent, then the series* Σa_n *is convergent.*

This theorem is the analogue, in the direction indicated above, of the simplest case of what we have called the "general Cesàro-Tauber theorem," from which it differs in that the hypothesis that $a_n = O(1/n)$ is replaced by the hypothesis (ii).

2. We do not propose now to work out systematically a whole theory analogous to that contained in our former papers. We shall confine ourselves to proving the analogues of two of our simplest theorems, viz.: (i) *if* $a_n = O(1/n)$ *and* $f(x) = \Sigma a_n x^n$ *tends to a limit as* x *tends to* 1 *through real values less than* 1, *then* Σa_n *is convergent*; (ii) *if* $a_n = O(1/n)$,

* *Comptes Rendus*, 6th January, 1913.

1914, 11 (with J. E. Littlewood) *Messenger of Mathematics*, 43, 134–47.

$b_n = O(1/n)$, *and the series* Σa_n, Σb_n *are convergent, then the product series* Σc_n, *formed in accordance with Cauchy's rule for multiplication, is convergent:* or rather of the generalisations of these two theorems which hold for Dirichlet's series and Dirichlet's multiplication.

One preliminary remark is required. In our previous researches there was a sharp distinction between "general" theorems, theorems whose hypotheses involve an O, and "special" theorems, theorems whose hypotheses involve an o. This distinction now disappears: the theorems which we shall prove are of a "special" character, and their proofs involve none of the characteristic difficulties of those of the "general" theorems; nor do they appear to be capable of any generalisation analogous to the passage from the "special" to the "general."

3. In what follows we shall, as usual, denote by (λ_n) an arbitrary increasing sequence of positive numbers, tending to infinity with n, and we shall be concerned with series Σa_n, such that the series

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}, \tag{3.1}$$

where p is a positive number, is convergent: this series reduces to $\Sigma n^p |a_n|^{p+1}$ when $\lambda_n = n$. It will be convenient to write $\lambda_0 = 0$.

It should be observed first that *the convergence of the series* (3.1) *for any particular value of p neither implies, nor is implied by, its convergence for any other value of p.* We can see this by considering the special case in which $\lambda_n = n$. Suppose first that

$$a_n = \frac{1}{n(\log n)^\alpha},$$

where $0 < \alpha \leq 1$. Then the series (3.1) is convergent if

$$p > (1/\alpha) - 1,$$

so that its chance of convergence is increased by an increase in p. If on the other hand we suppose that $a_n = \nu^{-\alpha}$ when $n = \nu^\beta$, α and β being positive integers, of which the latter is the greater, and that $a_n = 0$ when n is not a perfect β^{th} power, the series (3.1) assumes the form

$$\Sigma \nu^{p\beta-(p+1)\alpha},$$

and is convergent if

$$p < \frac{\alpha - 1}{\beta - \alpha}.$$

Thus in this case the chance of convergence is diminished by an increase in p.

Secondly, we observe that *if the series* (3.1) *is convergent, the series* $\Sigma a_n e^{-\lambda_n s}$ *is absolutely convergent for all positive values of* s. The proof of this depends on an inequality on which much of our subsequent analysis will depend, viz., the inequality

$$(3.2) \qquad \Sigma\, ab \leq (\Sigma\, a^{p+1})^{1/(p+1)} (\Sigma\, b^{(p+1)/p})^{p/(p+1)},$$

known as the "generalised inequality of Schwarz." In this inequality the a's, the b's, and p are positive.*

We have

$$\sum_1^n |a_\nu| e^{-\lambda_\nu s} = \sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^{p/(p+1)} |a_\nu| \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right)^{p/(p+1)} e^{-\lambda_\nu s}$$

$$\leq \left\{\sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^p |a_\nu|^{p+1}\right\}^{1/(p+1)} \left\{\sum_1^n \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right) e^{-\{(p+1)/p\}\lambda_\nu s}\right\}^{p/(p+1)}.$$

Also

$$\sum_1^n \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right) e^{-\{(p+1)/p\}\lambda_\nu s} \leq \frac{1}{\lambda_1} \sum_1^n (\lambda_\nu - \lambda_{\nu-1})\, e^{-\{(p+1)/p\}\lambda_\nu s}$$

$$< \frac{1}{\lambda_1} \int_0^{\lambda_n} e^{-\{(p+1)/p\} ts}\, dt < \frac{p}{(p+1)\lambda_1 s}.$$

From these inequalities our assertion follows immediately.

4. THEOREM A. *Suppose that the series*

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}$$

is convergent, and that the series $f(s) = \Sigma a_n e^{-\lambda_n s}$, *then certainly absolutely convergent for* $s > 0$, *tends to a limit* A *as* $s \to 0$. *Then the series* Σa_n *is convergent to the sum* A.

Choose m so that

$$\sum_{m+1}^\infty \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^p |a_n|^{p+1} < \epsilon^{p+1},$$

and s so that $s = 1/\lambda_n$, where $n > m$. Then, if

$$a_1 + a_2 + \ldots + a_n = A_n,$$

we have

$$A_n - f\left(\frac{1}{\lambda_n}\right) = \sum_1^m a_\nu (1 - e^{-\lambda_\nu s}) + \sum_{m+1}^n a_\nu (1 - e^{-\lambda_\nu s}) - \sum_{n+1}^\infty a_\nu e^{-\lambda_\nu s}$$

$$= S_1 + S_2 + S_3,$$

say. Then

* For a proof of the inequality see, *e.g.*, F. Riesz, *Math. Annalen*, vol. 69, p. 455.

$$|S_2| < s \sum_{m+1}^{n} \lambda_\nu |a_\nu|$$

$$\leq s \left\{ \sum_{m+1}^{n} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}} \right)^p |a_\nu|^{p+1} \right\}^{1/(p+1)} \left\{ \sum_{m+1}^{n} \lambda_\nu^{1/p} (\lambda_\nu - \lambda_{\nu-1}) \right\}^{p/(p+1)}$$

$$< \epsilon s \left\{ \sum_{1}^{n} \lambda_\nu^{1/p} (\lambda_\nu - \lambda_{\nu-1}) \right\}^{p/(p+1)} < \epsilon s \lambda_n = \epsilon.$$

Also

$$|S_3| \leq \left\{ \sum_{n+1}^{\infty} \frac{|a_\nu|^{p+1}}{(\lambda_\nu - \lambda_{\nu-1})^p} \right\}^{1/(p+1)} \left\{ \sum_{n+1}^{\infty} (\lambda_\nu - \lambda_{\nu-1}) e^{-\{(p+1)/p\}\lambda_\nu s} \right\}^{p/(p+1)}$$

$$< \epsilon \lambda_n^{-p/(p+1)} \left(\int_0^\infty e^{-\{(p+1)/p\}ts}\, dt \right)^{p/(p+1)}$$

$$= \epsilon \left(\frac{p}{p+1} \right)^{p/(p+1)} < \epsilon.$$

Finally it is evident that, if n is large enough in comparison with m, we have $|S_1| < \epsilon$, and so

$$\left| A_n - f\left(\frac{1}{\lambda_n} \right) \right| < 3\epsilon;$$

and the theorem is therefore proved.

In particular the convergence of $\Sigma n^p |a_n|^{p+1}$, and the existence of Abel's limit $\lim \Sigma a_n x^n$ when $x \to 1$, involve the convergence of Σa_n. Finally, since the summability (Cr) of Σa_n involves the existence of Abel's limit, a series Σa_n, such that $\Sigma n^p |a_n|^{p+1}$ is convergent, cannot be summable (Cr) unless convergent. For $p = 1$, $r = 1$, this reduces to Fejér's result.

5. Theorem B. *If Σa_n, Σb_n converge to sums A, B, and*

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right)^p |a_n|^{p+1}, \quad \Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right)^q |b_n|^{q+1}$$

are convergent, then the Dirichlet's product of the two series, formed according to the rule associated with Dirichlet's series of type (λ_n), converges to the sum AB.

The proof of this theorem is a modification of that of the theorem of which it is the analogue, given in one of our former papers.*

* *Proc. London Math. Soc.*, vol. 10, p. 399.

We shall use the notation

$$U(x) = \sum_{n \leq x} u_n$$

to denote the sum of those terms of a series $u_1 + u_2 + \ldots$ whose rank is not greater than a positive number x, not necessarily an integer. We shall denote by $\lambda(x)$ a continuous and steadily increasing function of x, which assumes the value λ_n for $x = n$, and by (ν_r) the sequence $(\lambda_m + \lambda_n)$, arranged in ascending order of magnitude.

The product series is Σc_r, where

$$c_r = \sum_{\lambda_m + \lambda_n = \nu_r} a_m b_n.$$

Thus

$$C(r) = \Sigma a_m b_n,$$

where the summation is bounded by the inequalities

$$m \geq 1, \quad n \geq 1, \quad \lambda_m + \lambda_n \leq \nu_r.$$

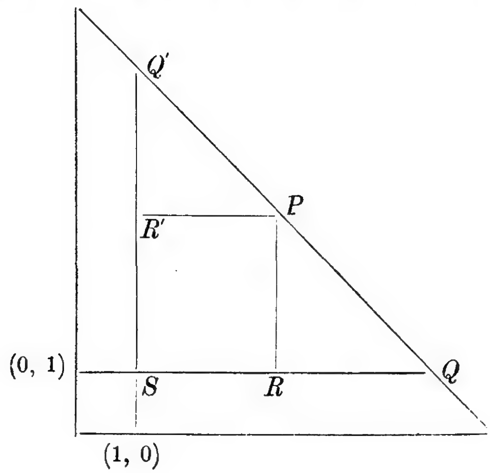
Let us draw the curve whose equation is

$$\lambda(x) + \lambda(y) = \nu_r,$$

and take on it the point P whose coordinates are

$$x_r = \bar{\lambda}(\tfrac{1}{2}\nu_r), \quad y_r = \bar{\lambda}(\tfrac{1}{2}\nu_r),$$

where $\bar{\lambda}$ is the function inverse to λ. Then $C(r)$ is the sum of all products $a_m b_n$ such that (m, n) lies in or on the boundary of the region SQQ', and $A(x_r) B(x_r)$ the sum of all such



that (m, n) lies in or on the boundary of $SRPR'$. Hence

$$C(r) - A(x_r)\, B(x_r) = \sum_{(D)} a_m b_n + \sum_{(D')} a_m b_n,$$

where D and D' denote the regions PQR, $P'Q'R'$, the boundaries of these regions being reckoned as part of them, except in so far as they are formed by the lines PR, PR'. It is plainly sufficient for our purpose to show that (*e.g.*)

$$\sum_{(D)} a_m b_n \to 0.$$

as $r \to \infty$.

Now $$\sum_{(D)} a_m b_n = \sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} a_m B\{\bar{\lambda}(\nu_r - \lambda_m)\},$$

the modulus of which is less than a constant multiple of

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} |a_m|.$$

We can choose r so that

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \left(\frac{\lambda_m}{\lambda_m - \lambda_{m-1}}\right)^p |a_m|^{p+1} < \epsilon^{p+1};$$

and then

$$\sum |a_m| \leq \left\{\sum \left(\frac{\lambda_m}{\lambda_m - \lambda_{m-1}}\right)^p |a_m|^{p+1}\right\}^{1/(p+1)} \left(\sum \frac{\lambda_m - \lambda_{m-1}}{\lambda_m}\right)^{p/(p+1)}$$

$$< \epsilon \left\{\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \frac{\lambda_m - \lambda_{m-1}}{\lambda_m}\right\}^{p/(p+1)}.$$

But

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \frac{\lambda_m - \lambda_{m-1}}{\lambda_m} < 1 + \int_{\frac{1}{2}\nu_r}^{\nu_r} \frac{dt}{t} = 1 + \log 2.$$

Hence $$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} |a_m| \to 0,$$

and so $$\sum_{(D)} a_m b_n \to 0,$$

as $r \to \infty$.

6. A comparison of the argument which precedes with that of our previous paper shows at once that *a series* Σa_n *for which*

$$\sum \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}$$

is convergent may be multiplied by a series Σb_n for which

$$b_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right).$$

Whether o may be replaced by O in this result we cannot say.

In another of our papers* we showed that our theorem concerning the multiplication, by Cauchy's rule, of series whose general terms are of order $1/n$ is a corollary of another theorem, viz., that *a series Σa_n, for which $a_n = O(1/n)$, if summable by any of Cesàro's means, is summable $(C, -1+\delta)$ for all positive values of δ.* It is naturally suggested that this theorem also has an analogue, and we have in fact proved the following result.

THEOREM C. *If Σa_n is summable (Ck) for any value of k, and*

$$\Sigma n^p |a_n|^{p+1}$$

is convergent, then Σa_n is summable $\left(C, -\dfrac{p}{p+1} + \delta\right)$ for all positive values of δ.

In order to prove this theorem, we observe† that the necessary and sufficient condition that a series Σa_n, known to be summable $(C, r+1)$, shall be summable (Cr), is that

$$t_n^r = o(n^{r+1}),$$

where

$$t_n^r = \binom{r+n-1}{r} a_1 + \binom{r+n-2}{r} 2a_2 + \ldots + \binom{r}{r} na_n.$$

Plainly

$$t_n^r = O\{n^r |a_1| + (n-1)^r 2|a_2| + \ldots + n|a_n|\}.$$

We divide the expression inside the brackets into the two parts

$$S_1 = \sum_{\nu=1}^{m} (n-\nu+1)^r \nu |a_\nu|, \quad S_2 = \sum_{m+1}^{n} (n-\nu+1)^r \nu |a_\nu|;$$

and we choose m so that

$$\sum_{m+1}^{\infty} \nu^p |a_\nu|^{p+1} < \epsilon^{p+1}.$$

* *Proc. London Math. Soc.*, vol. 11, p. 462.
† *Proc. London Math. Soc.*, vol. 8, p. 304.

Then

$$|S_2| \leq \left(\sum_{m+1}^{n} \nu^p |a_\nu|^{p+1}\right)^{1/(p+1)} \left\{\sum_{m+1}^{n} (n-\nu+1)^{\{(p+1)r\}/p}\, \nu^{1/p}\right\}^{p/(p+1)}$$

$$< \epsilon \left\{\sum_{1}^{n} (n-\nu+1)^{\{(p+1)r\}/p}\, \nu^{1/p}\right\}^{p/(p+1)} < \epsilon K n^{r+1},$$

where K is a constant. Also

$$|S_1| < n^r \sum_{1}^{m} \nu\, |a_\nu| < \epsilon n^{r+1},$$

if n is large enough in comparison with m. These inequalities obviously suffice to establish Theorem C.

II.

7. The theorem with which we shall conclude this paper is of a deeper character.

We have shown* that *if* $f(x) = \Sigma a_n x^n$ *is a power series, all of whose coefficients are positive, and which is convergent when* $0 < x < 1$, *and if*

$$f(x) \sim \frac{A}{(1-x)^\alpha} \quad (A > 0,\ \alpha > 0),$$

as $x \to 1$, *then*

$$A_n = a_1 + a_2 + \ldots + a_n \sim \frac{An^\alpha}{\Gamma(1+\alpha)}\ .\dagger$$

Further, we showed that *the hypothesis that* $a_n \geq 0$ *may be replaced by the more general hypothesis that* $a_n > -Kn^{\alpha-1}$.

8. We shall now prove

THEOREM D. *If* $f(s) = \Sigma a_n e^{-\lambda_n s}$ *is a Dirichlet's series convergent for* $s > 0$, *of type* (λ_n) *such that*

$$\frac{\lambda_{n+1}}{\lambda_n} \to 1$$

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* *Proc. London Math. Soc.*, vol. 13. This paper has not yet been published.

† In the paper referred to above we consider relations of the type

$$f(x) \sim \frac{A}{(1-x)^\alpha}\left\{\log\left(\frac{1}{1-x}\right)\right\}^{\alpha_1}\left\{\log\log\left(\frac{1}{1-x}\right)\right\}^{\alpha_2}\ldots .$$

The differences introduced into the proof by the adoption of the more general hypothesis are of the nature of trivial complications, and we shall confine ourselves now to the case in which $\alpha_1 = \alpha_2 = \ldots = 0$. The reader will easily satisfy himself of the truth of the more general results which are at once suggested.

as $n \to \infty$, *and with positive coefficients; if further*

$$f(s) \sim As^{-\alpha} \quad (A > 0,\ \alpha \geq 0)$$

as $s \to 0$*: then*

$$A_n = a_1 + a_2 + \ldots + a_n \sim \frac{A\lambda_n^{\alpha}}{\Gamma(1+\alpha)}$$

as $n \to \infty$.

We shall base our proof on the following lemma.

LEMMA D1. *If the series*

$$F(s) = \Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx$$

is convergent for $s > 0$*; if further* $a_n \geq 0$ *and*

$$F(s) \sim As^{-\alpha} \quad (A > 0,\ \alpha > 0)$$

as $s \to 0$, *then*
$$-F'(s) \sim A\alpha s^{-\alpha-1}.$$

Let
$$G(s) = \frac{F(s)}{s};$$

then

$$sG'(s) = F'(s) - \frac{F(s)}{s} = -\Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} xe^{-sx}\, dx - \frac{1}{s}\,\Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx$$

plainly decreases steadily as $s \to 0$. Hence, by a theorem of Landau,*

$$G'(s) \sim \frac{d}{ds}(As^{-\alpha-1}) = -A(\alpha+1)s^{-\alpha-2},$$

$$-F'(s) = -\frac{F(s)}{s} - G'(s) \sim A\alpha s^{-\alpha-1}.$$

There is also another lemma which we shall find useful, although it is of no particular intrinsic interest.

LEMMA D2. *If* ζ *and* ρ *are positive, and*

$$\zeta \to 0, \quad \rho \to \infty, \quad \zeta^2\rho \to \infty,$$

then

$$\frac{1}{\Gamma(\rho+1)} \int_0^{\rho(1-\zeta)} e^{-u} u^{\rho}\, du \to 0, \qquad \frac{1}{\Gamma(\rho+1)} \int_{\rho(1+\zeta)}^{\infty} e^{-u} u^{\rho}\, du \to 0.$$

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* *Rendiconti di Palermo*, vol. 26, p. 218; see also *Proc. London Math. Soc.*, vol. 13.

Consider the second integral, for example. It is

$$\frac{1}{\Gamma(\rho+1)}\int_{\rho(1+\zeta)}^{\infty} u^{\rho}e^{-u/(1+\zeta)}e^{-\zeta u/(1+\zeta)}\,du$$

$$< \frac{\{\rho(1+\zeta)\}^{\rho}e^{-\rho}}{\Gamma(\rho+1)}\int_{\rho(1+\zeta)}^{\infty} e^{-\zeta u/(1+\zeta)}\,du$$

$$< K\frac{(1+\zeta)^{\rho}}{\zeta\sqrt{\rho}}e^{-\zeta\rho}$$

$$= \frac{K}{\zeta\sqrt{\rho}}e^{-\rho\{\zeta-\log(1+\zeta)\}}$$

$$< \frac{K}{\zeta\sqrt{\rho}},$$

where K is a constant. That the other integral tends to zero may be proved in a similar manner.

9. Before proceeding to the proof of the main theorem we add the following preliminary remarks.

(i) Our argument will involve three variables, ζ, r, and s. Of these ζ and r are definite functions of one another, and $\zeta \to 0$, $r \to \infty$, $\zeta^3 r \to \infty$. We may, for example, suppose that $\zeta^3 r = 1$. The choice of a value of s will always be subsequent to that of ζ and r.

(ii) We shall make a number of assertions of the type

$$|f(\zeta, r, s)| < \epsilon,$$

or, more generally,

$$\phi(\zeta, r, s, \epsilon) < 0.$$

All such assertions are to be interpreted as follows: "given any positive number ϵ, we can choose r_0 so that, when any definite r greater than r_0 is taken, we can then choose s_0 so that $\phi < 0$ for $0 < s \leq s_0$, or for all such values of s as satisfy some further condition or conditions previously laid down."

It follows, of course, that when ϵ occurs in each of a succession of inequalities it must not be regarded as a definite number having the same value in each inequality.

(iii) We may plainly take $A = 1$.

10. We observe first that

(10.1) $$A_n = O(\lambda_n^{a});$$

since $$A_n < e\sum_{1}^{n} a_\nu e^{-(\lambda_\nu/\lambda_n)} < ef\left(\frac{1}{\lambda_n}\right).$$

Next, we have

$$f(s) = \Sigma a_n e^{-\lambda_n s} = \Sigma A_n (e^{-\lambda_n s} - e^{-\lambda_{n+1} s}) = s \Sigma A_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx,$$

and so
$$\Sigma A_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx \sim s^{-\alpha-1}.$$

Hence, by Lemma D1,

$$(10.2) \quad \Sigma A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx}\, dx \sim \frac{\Gamma(\alpha + r + 1)}{\Gamma(\alpha + 1)} s^{-\alpha-r-1}$$

for any value of r.

We shall suppose that r and s are such that

$$\frac{r+\alpha}{s} = \lambda_m,$$

and we shall denote by $\lambda_{m-\nu}$ and $\lambda_{m+\nu}$ the last and first respectively of the λ's such that

$$(10.3) \qquad \lambda_{m-\nu} < (1-\zeta)\lambda_m, \quad \lambda_{m+\nu} > (1+\zeta)\lambda_m.$$

It is important to observe that *it is possible to choose r and s so that either $m-\nu$ or $m+\nu$ shall be equal to any assigned large integer p.* For example, $m-\nu = p$ if

$$\lambda_p < \frac{(1-\zeta)(r+\alpha)}{s}, \qquad \lambda_{p+1} \geq \frac{(1-\zeta)(r+\alpha)}{s},$$

and we can certainly choose r and s so that these inequalities shall be satisfied. Thus $m-\nu$ and $m+\nu$ may be regarded as variables which assume all integral values, from a certain point onwards, as they tend to ∞.

Now

$$\sum_{m+\nu}^{\infty} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < K \sum_{m+\nu}^{\infty} \lambda_n^{\alpha} \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx$$

$$< K \int_{\lambda_{m+\nu}}^{\infty} x^{r+\alpha} e^{-sx}\, dx,$$

$$= K s^{-r-\alpha-1} \int_{s\lambda_{m+\nu}}^{\infty} u^{r+\alpha} e^{-u}\, du,$$

where K is a constant. The lower limit is greater than $(1+\zeta)(r+\alpha)$. Hence, by Lemma D2, we have

$$(10.4) \qquad \sum_{m+\nu}^{\infty} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < \epsilon \Gamma(r+\alpha+1)\, s^{-r-\alpha-1};$$

and a similar argument shows that

$$(10.5)\qquad \sum_{1}^{m-\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < \epsilon \Gamma(r+\alpha+1) s^{-r-\alpha-1},$$

11. From (10.2), (10.4), and (10.5) it follows that

$$(11.11)\qquad \sum_{m-\nu}^{m+\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx > (1-\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1},$$

$$(11.12)\qquad \sum_{m-\nu}^{m+\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1}.$$

But, since $a_n \geq 0$, A_n is a steadily increasing function of n. Hence

$$A_{m-\nu} \int_{\lambda_{m-\nu}}^{\lambda_{m+\nu}} x^r e^{-sx} dx < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1},$$

$$A_{m+\nu} \int_{\lambda_{m-\nu}}^{\lambda_{m+\nu}} x^r e^{-sx} dx > (1-\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1}.$$

In virtue of Lemma D2, we may replace the limits in these integrals by 0 and ∞. The first inequality then gives

$$A_{m-\nu} < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(r+1)\Gamma(\alpha+1)} s^{-\alpha},$$

$$A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \left(\frac{r}{s}\right)^{\alpha},$$

$$(11.2)\qquad A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \lambda_m^{\alpha}.$$

Now $\quad \lambda_{m-\nu} < (1-\zeta)\lambda_m, \quad \lambda_{m-\nu+1} \geq (1-\zeta)\lambda_m,$

and
$$\frac{\lambda_{n+1}}{\lambda_n} \to 1.*$$

Hence

$$(11.3)\qquad A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \lambda_{m-\nu}^{\alpha};$$

and similarly we can show that

$$A_{m+\nu} > \frac{1-\epsilon}{\Gamma(\alpha+1)} \lambda_{m+\nu}^{\alpha}.$$

* It is interesting to observe that this is the only point in the proof at which any use is made of this hypothesis.

It now follows from the remark made early in § 10 that, given any positive ϵ, we can choose p_0 so that

$$\frac{1-\epsilon}{\Gamma(\alpha+1)}\lambda_p^\alpha < A_p < \frac{1+\epsilon}{\Gamma(\alpha+1)}\lambda_p^\alpha$$

for $p>p_0$, and the proof of the theorem is accordingly completed.

12. It is easy to deduce from Theorem D a more general theorem.

THEOREM E. *The conclusion of Theorem D is still valid when $\alpha>0$ and the condition that a_n is positive is replaced by the more general condition*

$$a_n > -K\lambda_n^{\alpha-1}(\lambda_n-\lambda_{n-1}).$$

Let $$\phi(s) = \Sigma\lambda_n^{\alpha-1}(\lambda_n-\lambda_{n-1})e^{-\lambda_n s}.$$

Then it is easily proved that the series is convergent for $s>0$ and that

$$\phi(s) \sim \Gamma(\alpha)s^{-\alpha}$$

as $s \to 0$.*

The series $g(s) = f(s) + K\phi(s) = \Sigma b_n e^{-\lambda_n s}$,

where $$b_n = a_n + K\lambda_n^{\alpha-1}(\lambda_n-\lambda_{n-1}),$$

satisfies the condition

$$b_n \geq 0, \quad g(s) \sim \{A + K\Gamma(\alpha)\}s^{-\alpha}.$$

Hence $$\sum_1^n b_\nu \sim \left\{\frac{A}{\Gamma(\alpha+1)} + \frac{K}{\alpha}\right\}\lambda_n^\alpha;$$

and since $$\sum_1^n \lambda_\nu^{\alpha-1}(\lambda_\nu-\lambda_{\nu-1}) \sim \frac{\lambda_n^\alpha}{\alpha},$$

it follows that $$A_n \sim \frac{A\lambda_n^\alpha}{\Gamma(\alpha+1)}.$$

13. THEOREM F. *The conclusion of Theorem E is still valid when $\alpha = 0$.*

The proof given in the last section depends essentially on the hypothesis $\alpha>0$. The result is true when $\alpha=0$, but the proof is more subtle.†

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* Cf. Knopp, "Divergenzcharactere gewisser Dirichlet'scher Reihen," *Acta Mathematica*, vol. 34, pp. 165–204 (especially pp. 191–294).

† Cf. *Proc. London Math. Soc.*, vol. 13.

We have to prove that *if*

$$\text{(i)} \quad a_n > -K\frac{\lambda_n - \lambda_{n-1}}{\lambda_n},$$

$$\text{(ii)} \quad f(x) = \Sigma a_n e^{-\lambda_n s} \to A,$$

as $s \to 0$, *then* Σa_n *is convergent.*

We have $$f(s) = A + o(1),$$
and
$$f''(s) = \Sigma a_n \lambda_n^2 e^{-\lambda_n s} > -K\Sigma\lambda_n(\lambda_n - \lambda_{n-1})\, e^{-\lambda_n s} > -K/s^2.$$

Hence* $$f'(s) = o(1/s),$$

$$\Sigma a_n \lambda_n e^{-\lambda_n s} = o(1/s).$$

To this series we can apply Theorem E; and so we obtain

$$a_1\lambda_1 + a_2\lambda_2 + \ldots + a_n\lambda_n = o(\lambda_n).$$

But this equation, together with condition (ii), secures the convergence of the series Σa_n†; so that the theorem is proved. This theorem is of considerable interest as embodying the widest direct extension at present known of Tauber's original converse of Abel's theorem.‡

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* *Proc. London Math. Soc., l.c. supra.*

† Schnee, *Rendiconti di Palermo*, vol. 27, p. 87.

‡ In our earlier writings on this subject we have made considerable use of the following preliminary lemma: *If* $f(x)$ *has continuous derivatives of the first two orders, and* $f(x) = A + o(1)$, $f''(x) = O(1)$, *as* $x \to \infty$, *then* $f'(x) = o(1)$. Prof. J. Hadamard has very kindly pointed out to us that this result had already been proved independently, in the course of certain dynamical investigations, by himself ("Sur certaines propriétés des trajectories en Dynamique," *Journal de Mathématiques*, ser. 5, vol. 3, 1897, p. 334), and by Herr A. Kneser ("Studien über die Bewegungsvorgänge in der Umgebung instabiler Gleichgewichtslagen, *Journal für Mathematik*, vol. 118, 1897, p. 199). Hadamard and Kneser indeed prove the result, as Prof. Landau asks us to state, in the more general form in which it appears in his paper "Einige Ungleichungen für zweimal differentiierbare Funktionen" (*Proc. London Math. Soc.*, ser. 2, vol. 13, 1913, p. 43), where only the existence and not the continuity of $f''(x)$ is presupposed.

Both in our own writings and in Landau's paper the theorem in question appears only as a preliminary to a series of numbered theorems, the novelty of which is in no way affected by this anticipation.

We take this opportunity of referring also to a recent paper by Mr. A. Rosenblatt ("Über die Multiplikation der unendlichen Reihen," *Bulletin de l'Académie des Sciences de Cracovie*, 1913, p. 603), which contains a number of very interesting generalisations of some of our theorems on the multiplication of series.

CORRECTIONS

p. 139, *line* 11 *up* (3 times). For $\sum$ read $\sum\limits_{\frac{1}{2}\nu_r < \lambda_m \leqslant \nu_r}$.

p. 142, *line* 7 *up*. For $G'(s)$ read $sG'(s)$.

p. 146, 1*st footnote*. Read Vol. 34, pp. 165–204 (especially pp. 191–204).

p. 147, *line* 3. For $f(x)$ read $f(s)$.

COMMENTS

Part I

In § 1, Fejér's theorem is described as an analogue of the 'general Cesàro–Tauber theorem', but in § 2 the theorems of Part I are said to be of a 'special' character. In fact, Fejér's Tauberian condition implies the o-condition $\sum\limits_1^n \nu a_\nu = o(n)$, by the proof of Theorem C, with $r = 0$, $p = 1$.

In § 2, it is remarked that the theorems of Part I do not 'appear to be capable of any generalisation analogous to the passage from the "special" to the "general"'. However, Szasz and Iyer† extended the Tauberian condition in Theorem A, where $p > 0$, to

$$\sum_1^n (\lambda_r - \lambda_{r-1})^{-p} \lambda_r^{p+1} |a_r|^{p+1} = O(\lambda_n),$$

which is satisfied when $a_n = O((\lambda_n - \lambda_{n-1})/\lambda_n)$. Szasz (loc. cit., paper (2)) extended this to a one-sided form, in which a_r is replaced by $|a_r| - a_r$, with the extra condition $\underline{\lim}\, a_n \geqslant 0$. This is satisfied when $a_n > -K(\lambda_n - \lambda_{n-1})/\lambda_n$ and $\underline{\lim}\, a_n \geqslant 0$.

Theorem B is analogous to Theorem I of 1912, 2. In each case the Tauberian conditions are included in the pair of conditions

$$\sum_{\frac{1}{2}x \leqslant \lambda_n \leqslant x} |c_n| = o(1),$$

where $c_n = a_n$ or b_n.

Neder‡ replaced the o by O. This answers a question raised in § 6, and also includes Rosenblatt's extension of Theorem II of 1912, 2 (see the Comments on 1912, 2). For further results see 1927, 10 and 1944, 2.

In an addendum to 1915, 6 (in Vol. V), Hardy remarks that in the proof of Theorem C, § 6, 'the argument obviously depends on the assumption that $r > -p/(p+1)$, which is not explicitly stated'.

† Szasz (1) *Sitz. d. Bayerischen Akad. d. Wiss.* 59 (1929), 325–40, with the superfluous condition $\underline{\lim}\, a_n \geqslant 0$. Without the extra condition, Iyer, *Ann. of Math.* (2), 36 (1935), 100–16; Szasz (2), *Trans. American Math. Soc.* 39 (1936), 117–30.

‡ *Proc. London Math. Soc.* (2), 23 (1925), 172–84.

PART II

Theorem D, § 8, is an extension of the 'positive' theorem proved in 1914, 4. Szasz (loc cit., paper (1)) showed that the hypothesis $\lambda_{n+1}/\lambda_n \to 1$ may be omitted. On the other hand, both he and Ananda-Rau§ pointed out that, if $\alpha > 0$, $A > 0$, the condition is in fact a consequence of the other hypotheses. This is because they imply that

$$A(w) = \sum_{\lambda_n \leqslant w} a_n \sim Aw^{\alpha}/\Gamma(\alpha+1),$$

and hence that

$$A(\lambda_n)/\lambda_n^{\alpha} \sim A(\lambda_n)/\lambda_{n+1}^{\alpha} \sim A/\Gamma(\alpha+1).$$

Szasz (loc. cit., paper (1)) also remarked that Theorem D holds for $A = 0$, with the condition omitted; this is implicit in the proof of (10.1), § 10.

In Theorems E and F, §§ 12–13, the condition $\lambda_{n+1}/\lambda_n \to 1$ is still assumed. A method of Ingham‖ may be adapted to show that in Theorem E, where $\alpha > 0$, $A > 0$, the condition is again implied by the other hypotheses. First (cf. Ingham, loc. cit., p. 476) these imply that $A(w) > -Kw^{\alpha}$ $(w \geqslant \lambda_1)$ and

$$\int_0^{\infty} A(t)e^{-st}\,dt \sim \frac{A}{s^{\alpha+1}}.$$

Then, by the 'positive' theorem for integrals,††

$$A_1(w) = \int_0^{w} A(t)\,dt \sim \frac{A}{\Gamma(\alpha+2)} w^{\alpha+1},$$

and hence

$$(w-\lambda_n)A(\lambda_n) = A_1(w)-A_1(\lambda_n) = \frac{A}{\Gamma(\alpha+2)}(w^{\alpha+1}-\lambda_n^{\alpha+1})+o(w^{\alpha+1}),$$

where $\lambda_n \leqslant w < \lambda_{n+1}$. Putting $\Lambda_n = \frac{1}{2}(\lambda_n+\lambda_{n+1})$, $\delta_n = \frac{1}{4}(\lambda_{n+1}-\lambda_n)$, we obtain, since $\alpha > 0$,

$$A_1(\Lambda_n+\delta_n)-2A_1(\Lambda_n)+A_1(\Lambda_n-\delta_n) = 0 = A\delta_n^2\,\xi^{\alpha-1}/\Gamma(\alpha)+o(\lambda_{n+1}^{\alpha+1}),$$

where $\Lambda_n-\delta_n \leqslant \xi \leqslant \Lambda_n+\delta_n$. It follows that $\delta_n = o(\lambda_{n+1})$, i.e. $\lambda_{n+1}/\lambda_n \to 1$.

An example of Ananda-Rau‡‡ shows that Theorem F becomes false if the condition $\lambda_{n+1}/\lambda_n \to 1$ is omitted; see D.S., p. 161 and also the Comments on 1913, 3. The proof of Theorem F uses Theorem E for $A = 0$. The proof of Theorem E holds for $A = 0$, but becomes false if the condition $\lambda_{n+1}/\lambda_n \to 1$ is omitted. This may be shown by a modification of Ananda–Rau's example; take $\lambda_0 = 0$ and, for $m \geqslant 1$,

$$\lambda_{2m-1} = 2^m, \quad \lambda_{2m} = 2^m+1, \quad a_{2m-1} = -a_{2m} = -2^{m\alpha}.$$

§ Szasz (1); Ananda-Rau, *Rend. del. Circolo Mat. di Palermo* 54 (1930), 455–61.

‖ *Proc. London Math. Soc.* (2), 38 (1935), 458–80.

†† See 1930, 4, and the Comments on 1930, 4.

‡‡ *Proc. London Math. Soc.* (2), 30 (1930), 367–72.

Szasz (loc. cit., paper (1)) extended the Tauberian condition in Theorem F to: $A(w)$ *slowly decreasing*, and deduced that the condition $\lambda_{n+1}/\lambda_n \to 1$ in Theorem F may be replaced by $\underline{\lim}\, a_n \geqslant 0$. Similarly, in Theorem E for $A = 0$, the condition may be replaced by

$$\underline{\lim}\, a_n/\lambda_n^{\alpha} \geqslant 0;$$

compare the addendum to 1930, 1 and Ingham, loc. cit., p. 480.

In the final footnote, § 12, it is mentioned that Littlewood's Tauberian lemma for derivatives§§ was first obtained by Hadamard and Kneser in 1897. Kneser gave a detailed argument using Taylor's theorem, as in Landau's proof quoted in the footnote. Hadamard argued as follows:

> Nous avons à faire voir que la dérivée première est, pour t suffisamment grand, plus petite en valeur absolue qu'un nombre quelconque donné ϵ.
>
> Soit, à cet effet, l un nombre choisi arbitrairement. Dans la suite des valeurs de t, il ne peut exister une infinité d'intervalles ayant chacun une étendue supérieure à l et où $|df/dt|$ soit plus grand que $\epsilon/2$: car, dans un pareil intervalle, f varierait de plus de $l\epsilon/2$, ce qui ne peut se produire indéfiniment puisque f tend vers une limite. A partir du moment où ces intervalles cesseront de se rencontrer, le module de df/dt sera manifestement plus petit que ϵ si nous avons pris pour l un nombre qui, multiplié par la limite supérieure de $|d^2f/dt^2|$, donne un produit inférieur à $\epsilon/2$.

§§ *Proc. London Math. Soc.* (2), 9 (1911), 434–48, Theorem (A); see also Littlewood's *A mathematician's miscellany*, p. 36. Methuen, London, 1953.

Example to illustrate a Point in the Theory of Dirichlet's Series,

by

G. H. HARDY in Cambridge, England.

1. Suppose that

$$(1) \qquad \sum a_n e^{-\lambda_n s},$$

where $s=\sigma+ti$, is a Dirichlet's series whose region of convergence is the half-plane $\sigma>\sigma_0$; and that the function $f(s)$ defined by the series is regular for $\sigma>\eta$, where $\eta<\sigma_0$. In such cases it is often possible to obtain the analytic continuation of $f(s)$, throughout the whole or part of the region $\eta<\sigma\leqq\sigma_0$, by one or other of the methods of summation employed in the theory of divergent series, as for example by the use of Cesàro's means or the more general typical means of Marcel Riesz(1). It results, in fact, from the researches of Bohr and Riesz, that *the necessary and sufficient condition that the series* (1) *should be summable by typical means for $\sigma>\lambda$ is that $f(s)$ should be regular and of finite order for $\sigma>\lambda$*, that is to say that, given any positive number δ, we should have

$$f(s)=O(|t|^k),$$

where k is a number which depends only on δ, uniformly for

$$\sigma\geqq\lambda+\delta, \qquad |t|\geqq 1\ (^2).$$

The simplest illustration of the theorem just quoted is provided by the series

$$1^{-s}-2^{-s}+3^{-s}-4^{-s}+\cdots\cdots .$$

This series converges for $\sigma>0$, and represents the function

$$(1-2^{1-s})\,\zeta(s),$$

which is regular all over the plane and of finite order in any half-plane

(1) For a systematic account of the theory of typical means see G. H. Hardy and Marcel Riesz, 'The general theory of Dirichlet's series,' *Cambridge Tracts in Mathematics and Mathematical Physics*, 1915.

(2) Hardy and Riesz, loc. cit., Theorem 45. The result was first given for 'ordinary' Dirichlet's series $\Sigma a_n n^{-s}$, by Bohr (Göttinger Nachrichten, 1909).

$\sigma > \beta$. The theorem then asserts that the series is summable by Cesàro's means (1) for all values of s; and it is in fact summable by means of order k for $\sigma > -k$ (2).

It is of some theoretical interest to give simple examples in which the method of summation by typical means entirely fails to effect the analytic continuation of the function $f(s)$. In this note I give two such examples; in each $f(s)$ is regular all over the plane, but the half-plane of summability coincides with the half-plane of convergence.

2. The series

$$(2) \qquad \sum_{1}^{\infty} (-1)^{n-1} e^{-sn^a},$$

where $0 < a < 1$, is convergent if and only if $\sigma > 0$, and then absolutely convergent. Suppose now that $\sigma > 0$, $k > 0$ and that $(sn^a)^{-u}$ has its principal value. Then, in virtue of a well known formula of Mellin (1), we have

$$e^{-sn^a} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} (sn^a)^{-u} \Gamma(u)\, du .$$

If further $k > \frac{1}{a}$, we may multiply by $(-1)^{n-1}$ and sum under the sign of integration from 1 to ∞ (4), obtaining

$$(3) \qquad f(s) = \sum_{1}^{\infty} (-1)^{n-1} e^{-sn^a} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} s^{-u} \Gamma(u)\, \eta(au)\, du,$$

(1) There are two kinds of typical means, the 'first' and the 'second' kind, associated with a Dirichlet's series of given type. In the case of an ordinary Dirichlet's series, the means of the second kind are equivalent to Cesàro's. It is indifferent, in the enunciation of the theorem quoted above, whether the typical means referred to are of the first or the second kind. See Hardy and Riesz, loc. cit., and the memoirs of Bohr and of Riesz there quoted, in particular the note of Riesz entitled 'Une méthode de sommation équivalente à la méthode des moyennes arithmétiques' (*Comptes Rendus*, 12 June 1911).

(2) The substance of this result is due to Bohr, who however considered integral orders of summation only. The extension to non-integral orders is due to Chapman. For fuller information see Hardy and Riesz, loc. cit., pp. 20, 24.

(3) See e. g. Mellin, *Math. Annalen*, vol. 68.

(4) Since

$$\int_{k-i\infty}^{k+i\infty} |s^{-u}|\, |\Gamma(u)| \sum_{1}^{\infty} |n^{-au}|\, |du|$$

is convergent.

where

$$\eta(u) = \sum_{1}^{\infty} (-1)^{n-1} n^{-u} = (1 - 2^{1-u})\,\zeta(u).$$

It is moreover easy to prove that the integral (3) is equal to the sum of the residues of the subject of integration at the poles $u=0$, -1, $-2, \ldots$ (1). We thus obtain the equation

$$f(s) = \sum_{0}^{\infty} \frac{(-s)^n}{n!}\,\eta(-an), \tag{4}$$

which shows that $f(s)$ is an integral function of s (2).

But the series (2) is not summable by Cesàro's means for any value of s whose real part is negative. For, if a series Σc_n is summable by Cesàro's means of order k, then

$$c_n = o(n^k) \text{ (3)};$$

and it is obvious that this condition is not satisfied in this case. The typical means of the first kind, for a Dirichlet's series of type n^a, are equivalent to Cesàro's (4). The series (2) is therefore not summable by typical means for any value of s whose real part is negative. For the sake of completeness I may add that, when $\sigma=0$, the series is summable by Cesàro's means of any positive order.

3. The series (2) is not an ordinary Dirichlet's series. I shall now give an example of an ordinary Dirichlet's series which possesses similar properties.

The series

$$\sum_{2}^{\infty} \frac{e^{\mu i (\log n)^a}}{n^s}, \tag{5}$$

where $\mu>0$ and $a>1$, is absolutely convergent if $\sigma>1$ and convergent, though not absolutely convergent, if $\sigma=1$. It is not convergent for any value of s whose real part is less than 1, and it is easy to prove that

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(1) I suppress the details of the proof, which rests merely on an application of Cauchy's theorem entirely similar to many others made already by Mellin. See for example, Mellin, Acta Societatis Fennicae, vol. 31, no. 2.

(2) The series is convergent for all values of s. This would not be true if $a \geqq 1$.

(3) Chapman, *Proc. Lond. Math. Soc.*, ser. 2, vol. 9. See Hardy and Riesz, loc. cit., Theorem 21.

(4) Hardy and Riesz, loc. cit., Theorem 19.

it is not summable by any of Cesàro's means for any such value of s. For it is known ([1]) that if $\Sigma a_n n^{-s}$ is an ordinary Dirichlet's series whose line of absolute convergence is $\sigma = 1$, and $a_n = O(n^\delta)$ for all positive values of δ, then the distance between two successive lines of summability of integral order cannot exceed that between the lines of convergence and absolute convergence. In this case the lines of convergence and absolute convergence are coincident, and so the lines of summability must all also coincide with them.

It remains to prove that the series (5) represents an integral function of s.

4. I shall prove first that the integral

$$\phi(s) = \int_1^\infty \frac{e^{\mu i (\log x)^a}}{x^s} dx, \tag{6}$$

convergent if $\sigma \geqq 1$, represents an integral function of s.

Let us suppose for the moment that s is real and greater than 1. Putting $x = e^u$, we obtain

$$\phi(s) = \int_0^\infty e^{\mu i u^a - (s-1) u} du. \tag{7}$$

Now let a be a positive number such that $\alpha < \frac{\pi}{2}$, $a\alpha < \pi$. Then

$$\boldsymbol{R}\left\{ \mu i r^a e^{ai\theta} - (s-1) r e^{i\theta} \right\}$$

$$= -\mu r^a \sin a\theta - (s-1) r \cos\theta \leqq -(s-1) r \cos\alpha$$

for $0 \leqq \theta \leqq \alpha$. It follows that the integral (6) may be taken along the radius vector $\theta = \alpha$, in the plane of the complex variable $u = r e^{i\theta}$, instead of along the real axis. We thus obtain

$$\phi(s) = e^{i\theta} \int_0^\infty e^{\mu i r^a e^{ai\theta} - (s-1) r e^{i\theta}} dr. \tag{8}$$

But the last integral is uniformly convergent throughout any finite domain of values of s, since

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([1]) Hardy and Littlewood, Contributions to the arithmetic theory of series, *Proc. Lond. Math. Soc.*, ser. 2, vol. 11, Theorem 23.

$$| e^{\mu i r^a e^{ai\theta}} | = e^{-\mu r^a \sin a\theta}$$

and $\sin a\theta>0$ and $a>1$. Thus $\phi(s)$ is an integral function of s.

It is evident that the integral

$$\int_1^g \frac{e^{\mu i (\log x)^a}}{x^s}\, dx,$$

where $1<g<\infty$, is an integral function of s. We have therefore proved that

$$(9) \qquad \psi(s)=\int_g^\infty \frac{e^{\mu i (\log x)^a}}{x^s}\, dx,$$

where $g \geqq 1$, is an integral function of s.

5. We return to the series (5).

Suppose first that $s>1$, and that g and h are positive and $1<g<2$. Further, suppose that C is the contour formed by the three straight lines

$$(\infty+hi,\ g+hi), \qquad (g+hi,\ g-hi), \qquad (g-hi,\ \infty-hi)$$

in the plane of the complex variable x; and that C_1 and C_2 are the parts of C which lie respectively above and below the real axis. Then

$$f(s) = \sum_2^\infty \frac{e^{\mu i (\log n)^a}}{n^s} = \frac{1}{2\pi i}\int_C \pi \cot \pi x\, \frac{e^{\mu i (\log x)^a}}{x^s}\, dx.$$

We divide C into the two parts C_1 and C_2, and we write

$$\pi \cot \pi x = -\pi i + \frac{2\pi i}{1-e^{-2\pi i x}}$$

on C_1, and

$$\pi \cot \pi x = \pi i + \frac{2\pi i}{e^{2\pi i x}-1}$$

on C_2. We thus obtain

$$(10) \quad f(s) = -\frac{1}{2}\int_{C_1} \frac{e^{\mu i (\log x)^a}}{x^s}\, dx + \int_{C_1} \frac{e^{\mu i (\log x)^a}}{x^s} \cdot \frac{dx}{1-e^{-2\pi i x}}$$
$$+\frac{1}{2}\int_{C_2} \frac{e^{\mu i (\log x)^a}}{x^s}\, dx + \int_{C_2} \frac{e^{\mu i (\log x)^a}}{x^s} \cdot \frac{dx}{e^{2\pi i x}-1}$$

$$=\int_g^\infty \frac{e^{\mu i (\log x)^a}}{x^s}\,dx+f_1(s)+f_2(s),$$

where

(11) $$f_1(s)=\int_{C_1} \frac{e^{\mu i (\log x)^a}}{x^s}\cdot\frac{dx}{1-e^{-2\pi i x}},$$

(12) $$f_2(s)=\int_{C_2} \frac{e^{\mu i (\log x)^a}}{x^s}\cdot\frac{dx}{e^{2\pi i x}-1}.$$

We have proved already that the integral on the right hand side of (10) represents an integral function of s. It will therefore be sufficient for our purpose to prove that each of $f_1(s)$ and $f_2(s)$ is an integral function of s.

6. I shall prove first that in the equations (11) and (12), we may replace the contours C_1 and C_2 by the straight lines

$$(g+\infty i,\ g) \qquad \text{and} \qquad (g,\ g-\infty i)$$

respectively.

Let $$x=\xi+\eta i.$$

I shall prove that a constant K exists such that

(13) $$\left|\frac{e^{\mu i (\log x)^a}}{1-e^{-2\pi i x}}\right|<K$$

throughout the domain D_1, defined by $\xi \geqq g$, $\eta \geqq h$, and

(14) $$\left|\frac{e^{\mu i (\log x)^a}}{e^{2\pi i x}-1}\right|<K$$

throughout the domain D_2, defined by $\xi \geqq g$, $\eta \leqq -h$. The truth of the assertion made at the beginning of this section will then follow by a simple application of Cauchy's theorem, since $s>1$.

If $x=r\,e^{i\theta}$, where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, we have

$$\log x=\rho\, e^{i\phi},$$

where

$$\rho=\sqrt{\left\{(\log r)^2+\theta^2\right\}}, \qquad \phi=\arc\tan\left(\frac{\theta}{\log r}\right);$$

and

$$(15) \qquad \left| e^{\mu i (\log x)^a} \right| = e^{-\mu\rho^a \sin a\phi}.$$

The last function is bounded in D_1; and, since

$$\left| \frac{1}{1-e^{-2\pi i x}} \right|$$

is also bounded in D_1, we at once obtain the inequality (13). The inequality (14) is a little less obvious. The angle θ is negative in D_2, and ϕ and $\sin a\phi$ are negative and small when r is large. It follows from (15) that a constant L exists such that

$$\left| e^{\mu i (\log x)^a} \right| < e^{L (\log r)^{a-1} |\theta|},$$

if x lies in D_2 and r is large enough. On the other hand

$$\left| \frac{1}{e^{2\pi i x}-1} \right|$$

is, throughout D_2, less than a constant multiple of $e^{2\pi r \sin\theta}$; and a positive constant M exists such that

$$\left| \frac{1}{e^{2\pi i x}-1} \right| < e^{-Mr|\theta|}.$$

Also

$$Mr > 2L(\log r)^{a-1},$$

if r is large enough. It follows that

$$\left| \frac{e^{\mu i (\log x)^a}}{e^{2\pi i x}-1} \right| < e^{-\frac{1}{2}Mr|\theta|} < 1,$$

if x lies in D_2 and r is large enough; and from this follows immediately the truth of the inequality (14).

7. We have therefore

$$(16) \qquad f_1(s) = \int_{g+\infty i}^{g} \frac{e^{\mu i (\log x)^a}}{x^s} \cdot \frac{dx}{1-e^{-2\pi i x}},$$

$$(17) \qquad f_2(s) = \int_{g}^{g-\infty i} \frac{e^{\mu i (\log x)^a}}{x^s} \cdot \frac{dx}{e^{2\pi i x}-1},$$

if $s>1$. But each of these integrals is absolutely and uniformly con-

vergent throughout any finite domain of values of s (¹). Hence $f_1(s)$ and $f_2(s)$ are integral functions of s; and so $f(s)$ is an integral function of s.

It should be observed that the hypothesis that $a>1$ is essential for the truth of the conclusion. If $a=1$ we have

$$f(s)=\sum_2^\infty \frac{1}{n^{s-\mu i}}=\zeta(s-\mu i)-1,$$

and $f(s)$ is not an integral function.

(¹) Since the subject of integration in each integral contains a factor which tends to zero like $e^{-2\pi|\eta|}$.

COMMENTS

The properties of the series (5), established in §§ 3–6, were stated by Hardy and Littlewood in 1913, 2, p. 436, as an example. If $R(s) < 1$, the Cesàro means of order r ($r = 0, 1, \ldots$) oscillate like

$$n^{-s+1}(\log n)^{-(r+1)(a-1)}e^{\mu i(\log n)^a}.$$

The series is discussed further in 1916, 1; see also 1916, 7.

THE APPLICATION OF ABEL'S METHOD OF SUMMATION TO DIRICHLET'S SERIES.

By G. H. HARDY.

§ 1. A SERIES Σa_n may be said to be summable by Abel's method, or *summable* (A), to sum l, if

$$\Sigma a_n e^{-ny}$$

is convergent for all positive values of y and

$$\phi(y) = \Sigma a_n e^{-ny} \rightarrow l$$

as $y \rightarrow 0$. In this paper* I propose to discuss the application of this method of summation to the theory of ordinary Dirichlet's series

$$\Sigma \frac{a_n}{n^s}.$$

It is curious, considering the amount that has been written concerning the application of Cesàro's method, and the allied methods of Marcel Riesz, to such series, that the similar problems connected with Abel's method should not have been discussed before.

I suppose that

$$a_n = O(n^K)$$

for some value of K. The series has then a region of convergence, and represents an analytic function $f(s) = f(\sigma + it)$.

The region of summability (C) of the series $\Sigma a_n n^{-s}$ has been determined by Bohr. *The series is summable* (C) *for* $\sigma > \Lambda$ *if, and only if,* $f(s)$ *is regular and of finite order for* $\sigma > \Lambda$; that is to say, if to every positive number ϵ corresponds a number $k = k(\epsilon)$ such that

$$|f(s)| = O\{|t|^{k(\epsilon)}\}$$

for $$\sigma \geq \Lambda + \epsilon.$$

Any series summable (C) is summable (A). The region of summability (A) therefore includes the half-plane $\sigma > \Lambda$.

* A short account of some of the chief results of the paper appeared, under the title 'Sur la sommation des séries de Dirichlet', in the *Comptes Rendus*, 27 March 1916.

In § 2 I prove that the region of summability (A) is also a half-plane. In §§ 3–4 I show that this half-plane $\sigma > \mathfrak{A}$ is *the half-plane in which $f(s)$ is regular and of the form*

$$O(e^{H|t|}),$$

where $$H < \tfrac{1}{2}\pi.$$

In §§ 5–7 I use the idea of summability (A) to prove and extend an important theorem of Bohr. Bohr has shown that if

$$f(s) = O(1)$$

for $\sigma \geq \omega + \delta > \omega$, whatever the value of δ, then $\Sigma a_n n^{-s}$ is absolutely convergent for $\sigma > \omega + \frac{1}{2}$. Here I prove that the same conclusion follows if we are given only that

$$\frac{1}{2T}\int_{-T}^{T} |f(s)|^2\, dt = O(1),$$

as $T \to \infty$, in every half-plane $\sigma \geq \omega + \delta$.

In §§ 8–10 I state some further results concerning mean values of the type just mentioned, and in §§ 11–12 I illustrate some of the theorems which precede by means of some special functions, and in particular the function

$$f(s) = \Sigma \frac{e^{Ai(\log n)^2}}{n^s},$$

where $A > 0$.

I conclude the paper by indicating, in § 13, some generalisations of my results for other methods of summation and for the theory of Dirichlet's series of the most general type.

The region of summability (A).

§ 2. THEOREM 1. *If Σa_n is summable (A), then $\Sigma a_n n^{-s}$ is summable (A) for all values of s whose real part is positive.*

Let $$\phi(y) = \Sigma a_n e^{-ny} \quad (y > 0).$$

Then $\phi(y)$ is continuous for $y > 0$, and

$$\phi(y) \to \phi(0),$$

say, as $y \to 0$. Also

$$\phi(y, s) = \Sigma a_n n^{-s} e^{-ny} = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \phi(u+y)\, du$$

for $\sigma > 0$.

The integral $$\int_0^\infty u^{s-1}\phi(u+y)\,du$$
is uniformly convergent for $0 \leq y \leq y_0$. For
$$\phi(u+y) = O(e^{-u})$$
uniformly in y, so that
$$\int_1^\infty u^{s-1}\phi(u+y)\,du$$
is uniformly convergent. And
$$|\phi(u+y)| < K$$
for $$0 \leq u \leq 1,\quad 0 \leq y \leq y_0,$$
so that $$\int_0^1 u^{s-1}\phi(u+y)\,du$$
is uniformly convergent. Hence
$$\lim_{y\to 0}\phi(y, s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\phi(u)\,du,$$
and the theorem is proved.

From Theorem 1 we can at once deduce

THEOREM 2. *The region of summability* (A) *is a half-plane.*

§ 3. Before proceeding further we require some preliminary theorems. Suppose that $f(s)$ is an analytic function regular at all points of the line $\sigma = \beta$, and that $H = H(\beta)$ is the lower bound of the numbers ξ such that
$$f(s) = O(e^{\xi|t|}).$$
We call *H the index of $f(s)$ for $\sigma = \beta$.* We may have $H = \infty$ or $H = -\infty$.

THEOREM 3. *Suppose that $f(s)$ is regular for $\beta_1 \leq \sigma \leq \beta_2$, and that*
$$f(s) = O(e^{K|t|}),$$
for some value of K and uniformly for $\beta_1 \leq \sigma \leq \beta_2$, so that $H(\sigma)$ is bounded above in this interval. Then
$$H(\sigma) \leq \frac{\beta_2 - \sigma}{\beta_2 - \beta_1} H(\beta_1) + \frac{\sigma - \beta_1}{\beta_2 - \beta_1} H(\beta_2),$$
for $\beta_1 \leq \sigma \leq \beta_2$.

We may evidently suppose, without loss of generality, that $\beta_1 = 0$, $\beta_2 = 1$. Let

$$H(0) = H_1, \quad H(1) = H_2,$$

and
$$h = h(\sigma) = (1-\sigma) H_1 + \sigma H_2;$$

and let us suppose first that t is positive.

Let
$$g(s) = e^{\frac{1}{2}ais^2 + bis},$$

where
$$a = H_2 - H_1, \quad b = H_1 + \delta,$$

and δ is any positive number. Then

$$|g(s)| = e^{-a\sigma t - bt},$$

$$|g(it)| = e^{-(H_1+\delta)t}, \quad |g(1+it)| = e^{-(H_2+\delta)t}.$$

Hence, if
$$f(s) g(s) = F(s),$$

we have
$$F(it) = O(1), \quad F(1+it) = O(1),$$

and
$$F(\sigma + it) = O(e^{Kt}),$$

for some value of K and uniformly for $0 \leq \sigma \leq 1$. It follows* that

$$F(\sigma + it) = O(1),$$

uniformly for $0 \leq \sigma \leq 1$. Hence

$$f(\sigma + it) = O\{e^{(a\sigma + b)t}\}$$
$$= O\{e^{(h+\delta)t}\},$$

uniformly for $0 \leq \sigma \leq 1$.

A similar argument may be applied for negative values of t, taking

$$g(s) = e^{-\frac{1}{2}ais^2 - bis}.$$

Thus
$$f(\sigma + it) = O\{e^{(h+\delta)|t|}\}$$

for all positive values of δ; *i.e.*

$$H(\sigma) \leq h(\sigma) = (1-\sigma) H(0) + \sigma H(1),$$

which proves the theorem.

---

* By Lindelöf's Theorem (Theorem 14 of the tract "The general theory of Dirichlet's series", by M. Riesz and myself). As is indicated in the tract, it is enough that we should have

$$f(s) = O(e^{\epsilon t^2}),$$

for $0 \leq \sigma \leq 1$ and for every positive value of ϵ.

THEOREM 4. *Suppose that $\Sigma a_n n^{-s}$ is absolutely convergent for $\sigma > \bar{\sigma}$, and that $f(s)$ is regular and has a finite index for $\sigma > \gamma$, where $\gamma < \bar{\sigma}$. Then either $H(\sigma)$ is zero for $\sigma > \gamma$; or it is zero for $\sigma \geq \gamma_0$, where $\gamma < \gamma_0 \leq \bar{\sigma}$, while for $\gamma < \sigma < \gamma_0$ it is a positive, strictly decreasing, convex, and continuous function of σ.*

The proof of this theorem requires merely a repetition of the arguments of pp. 16 *et seq.* of the tract just referred to.

§ 4. THEOREM 5. *Suppose that $\Sigma a_n n^{-s}$ is summable (A) for $\sigma > \mathfrak{A}$. Then $H(\sigma) < \frac{1}{2}\pi$ for $\sigma > \mathfrak{A}$.*

Since $H(\sigma)$ is a strictly decreasing function of σ, it is enough to prove that $H(\sigma) \leq \frac{1}{2}\pi$, *i.e.* that

$$|f(s)| = O\{e^{(\frac{1}{2}\pi+\epsilon)|t|}\}$$

for every σ greater than $\mathfrak{A}$ and every positive ϵ. And it is evidently enough, in order to prove this, to prove that *if Σa_n is summable (A), then*

$$f(\sigma + ti) = O\{e^{(\frac{1}{2}\pi+\epsilon)|t|}\}$$

for every positive pair of values of σ and ϵ.

Now
$$f(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \phi(x)\, dx,$$
where
$$\phi(x) = \Sigma a_n e^{-nx},$$
and
$$\int_0^\infty x^{\sigma-1} |\phi(x)|\, dx$$
is convergent. Hence

$$f(s) = O\left\{\frac{1}{|\Gamma(s)|}\right\} = O(e^{\frac{1}{2}\pi|t|} |t|^{\frac{1}{2}-\sigma}) = O\{e^{(\frac{1}{2}\pi+\epsilon)|t|}\}.$$

THEOREM 6. *If $f(s)$ is regular, and $H(\sigma) < \frac{1}{2}\pi$, for $\sigma > \eta$, then $f(s)$ is summable (A) for $\sigma > \eta$, so that $\mathfrak{A} \leq \eta$.*

We have
$$e^{-y} = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} y^{-u} \Gamma(u)\, du$$

if $y > 0$, $\kappa > 0$, and y^{-u} has its principal value. If now we suppose that $\kappa > \bar{\sigma}$, write ny for y, multiply by $a_n n^{-s}$, and sum, we obtain

$$\phi(y, s) = \Sigma a_n n^{-s} e^{-ny} = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} y^{-u} \Gamma(u)\, \Sigma a_n n^{-s-u} du$$
$$= \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} y^{-u} \Gamma(u) f(s+u)\, du,$$

the inversion of the order of integration and summation being justified by the fact that

$$\int_{\kappa-i\infty}^{\kappa+i\infty} |y^{-u}||\Gamma(u)|\Sigma|a_n||n^{-s-u}||du|$$

is convergent.

The formula

$$(1) \qquad \phi(y, s) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} y^{-u}\Gamma(u)f(s+u)\,du$$

has been proved for every positive κ and all values of s whose real part is sufficiently large. But it is easy to prove that it holds provided only $\kappa > 0$ and $\sigma > \eta$. For let D be any finite region of values of s throughout which

$$\sigma \geq \eta + \delta > \eta,$$

and let $u = \kappa + i\tau$. Then $\sigma + \kappa > \eta + \delta$, and so

$$f(s+u) = O\{e^{(\frac{1}{2}\pi-\zeta)|\tau|}\},$$

where $\zeta > 0$, uniformly throughout D. On the other hand

$$y^{-u} = O(1), \quad \Gamma(u) = O\{e^{-\frac{1}{2}(\pi-\zeta)|\tau|}\};$$

so that

$$y^{-u}\Gamma(u)f(s+u) = O(e^{-\frac{1}{2}\zeta|\tau|})$$

uniformly throughout D. The integral on the right-hand side of (1) is therefore uniformly convergent throughout D; and so (1) holds for $\sigma > \eta$.

Suppose now that $\sigma \geq \eta + \delta$, and let λ be a positive number less than $\frac{1}{2}\delta$. Consider the integral

$$\frac{1}{2\pi i}\int y^{-u}\Gamma(u)f(s+u)\,du,$$

taken round the rectangle whose angles are at the points

$$\kappa - iT, \quad \kappa + iT, \quad -\lambda + iT, \quad -\lambda + iT.$$

Since $$\sigma - \lambda > \eta + \tfrac{1}{2}\delta,$$

there is a positive number ζ such that

$$f(s+u) = O\{e^{(\frac{1}{2}\pi-\zeta)|\tau|}\}$$

uniformly for $-\lambda \leq \mathfrak{R}(u) \leq \kappa$; and an argument similar to that used immediately above shows that the contributions to the integral of the horizontal sides of the rectangle tend to zero. We have therefore

$$(2) \qquad \phi(y, s) = f(s) + \frac{1}{2\pi i}\int_{-\lambda-i\infty}^{-\lambda+i\infty} y^{-u}\Gamma(u)f(s+u)\,du.$$

N 2

Suppose now that $y \to 0$. Then

$$\frac{1}{2\pi i}\int_{-\lambda-i\infty}^{-\lambda+i\infty} y^{-u}\Gamma(u)f(s+u)\,du = O\left(y^{\lambda}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\zeta|\tau|}\,d\tau\right) = o(1).$$

Hence
$$\phi(y, s) \to f(s),$$
and the proof of the theorem is completed.

Combining Theorems 5 and 6, we obtain

THEOREM 7. *The half-plane of summability (A) of the series $\Sigma a_n n^{-s}$ is identical with the half-plane throughout which $f(s)$ is regular and of the form*
$$O(e^{H|t|}),$$
where
$$H < \tfrac{1}{2}\pi.$$

Bohr's Theorem and its extension.

§ 5. THEOREM 8. *Suppose that $f(s)$ is regular and bounded in every half-plane $\sigma \geq \eta + \delta > \eta$. Then $\Sigma a_n n^{-s}$ is absolutely convergent for $\sigma > \eta + \frac{1}{2}$.*

This important theorem, due to Bohr,* is very easily proved by means of the ideas of the preceding sections. We require the following preliminary theorem.

THEOREM 9. *If $f(s)$ is bounded in every half-plane $\sigma \geq \eta + \delta$, then $\Sigma a_n n^{-s}$ is uniformly summable (A) in every such half-plane.*

This theorem is a corollary of another well-known theorem of Bohr,† which asserts that the series is uniformly convergent in every such half-plane. It is more consonant with the plan of this paper to prove the theorem directly as follows.

We have, by equation (2) of § 4,

$$\phi(y, s) = f(s) + \frac{1}{2\pi i}\int_{-\lambda-i\infty}^{-\lambda+i\infty} y^{-u}\Gamma(u)f(s+u)\,du;$$

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* "Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\Sigma a_n n^{-s}$", *Göttinger Nachrichten*, 1913, pp. 441–488 (p. 480).

† "Sur la convergence des séries de Dirichlet", *Comptes Rendus*, 1 Aug. 1910; "Über die gleichmässige Konvergenz Dirichletscher Reihen", *Journal für Mathematik*, vol. cxliii., 1913, pp. 203–211.

and hence, when $y \to 0$, we have

$$\phi(y, s) - f(s) = O\left\{ y^{\lambda} \int_{-\infty}^{\infty} |\Gamma(-\lambda + i\tau)| \, d\tau \right\}$$
$$= o(1),$$

uniformly for $\sigma \geq \eta + \delta$.*

Incidentally we have proved that *to every δ corresponds a K such that*

$$|\phi(y, s)| < K$$

for $\sigma \geq \eta + \delta, \quad y > 0.$

§ 6. We can now prove Bohr's Theorem. We have,† for any positive value of y,

$$\Sigma \frac{|a_n|^2}{n^{2\sigma}} e^{-2ny} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi(y, s)|^2 \, dt,$$

whatever the value of σ. We have therefore

$$\Sigma \frac{|a_n|^2}{n^{2\sigma}} e^{-2ny} \leq K^2$$

for $\sigma \geq \eta + \delta$, $y > 0$. Making $y \to 0$, we see that

$$\Sigma \frac{|a_n|^2}{n^{2\sigma}}$$

is convergent for $\sigma \geq \eta + \delta$. But

$$\left(\Sigma \frac{|a_n|}{n^{\eta + \frac{1}{2} + \delta + \delta_1}} \right)^2 < \Sigma \frac{|a_n|^2}{n^{2\eta + 2\delta}} \Sigma \frac{1}{n^{1 + 2\delta_1}},$$

and $\Sigma a_n n^{-s}$ is therefore absolutely convergent for

$$\sigma = \eta + \tfrac{1}{2} + \delta + \delta_1,$$

whatever be the positive numbers δ and δ_1, and therefore for $\sigma > \eta + \frac{1}{2}$.

Bohr's Theorem may be generalised as follows.

---

* If $f(s) = O(1)$ then $f(s+u) = O(1)$; this is the point of the proof. If we were only given an equation of the type $f(s) = O(\chi\,|t|)$—for example, as in § 4, $f(s) = O\,[\exp\{(\frac{1}{2}\pi - \zeta)|t|\}]$—then we could only assert that

$$f(s+u) = O\{\chi(|t| + |\tau|)\},$$

and the proof would fail on the point of uniformity.

† Landau, *Handbuch*, p. 776.

§ 7. Theorem 10. *If*

$$\frac{1}{2T}\int_{-T}^{T}|f(s)|^2\,dt$$

is bounded in every half-plane $\sigma \geq \eta + \delta$, *then* $\Sigma a_n n^{-s}$ *is absolutely convergent for* $\sigma > \eta + \frac{1}{2}$.

Suppose that

$$\frac{1}{2T}\int_{-T}^{T}|f(s)|^2\,dt < K$$

for $\sigma \geq \eta + \frac{1}{2}\delta$. Suppose further that $\sigma \geq \eta + \delta$, $T > 1$, and $0 < \lambda < \frac{1}{2}\delta$. Then, as in § 5, we have

$$\phi(y, s) = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} y^{-u}\,\Gamma(u) f(s+u)\,du$$
$$= f(s) + \frac{1}{2\pi i}\int_{-\lambda - i\infty}^{-\lambda + i\infty} y^{-u}\,\Gamma(u) f(s+u)\,du.$$

Hence

$$\frac{1}{2T}\int_{-T}^{T}|\phi(y, s)|^2\,dt$$
$$= \frac{1}{2T}\int_{-T}^{T}\left|f(s) + \frac{1}{2\pi i}\int_{-\lambda - i\infty}^{-\lambda + i\infty} y^{-u}\,\Gamma(u) f(s+u)\,du\right|^2 dt$$
$$\leq \frac{1}{T}\int_{-T}^{T}|f(s)|^2\,dt + \frac{1}{4\pi^2 T}\int_{-T}^{T}\left|\int_{-\lambda - i\infty}^{-\lambda + i\infty} y^{-u}\,\Gamma(u) f(s+u)\,du\right|^2 dt$$
$$< 2K + J,$$

say, where

$$J = \frac{1}{4\pi^2 T}\int_{-T}^{T}\left|\int_{-\infty}^{\infty} y^{\lambda - i\tau}\,\Gamma(-\lambda + i\tau) f\{\sigma - \lambda + i(t+\tau)\}\,d\tau\right|^2 dt$$
$$\leq \frac{y^{2\lambda}}{4\pi^2 T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty}|\Gamma(-\lambda + i\tau)|\,|f\{\sigma - \lambda + i(t+\tau)\}|\,d\tau\right]^2 dt$$
$$< \frac{L}{T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty}|\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\,|f\{\sigma - \lambda + i(t+\tau)\}|\,d\tau\right]^2 dt,$$

L being another constant.

Applying 'Schwarz's inequality' to the integral with respect to τ, we obtain

$$J < \frac{L}{T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty}|\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\,d\tau \times \int_{-\infty}^{\infty}|\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\,|f\{\sigma - \lambda + i(t+\tau)\}|^2\,d\tau\right] dt$$
$$< \frac{M}{T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty}|\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\,|f\{\sigma - \lambda + i(t+\tau)\}|^2\,d\tau\right] dt.$$

where M is a constant. Thus

$$J < \frac{M}{T}\int_{-\infty}^{\infty} |\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\left[\int_{-T}^{T} |f\{\sigma-\lambda+i(t+\tau)\}|^2 dt\right] d\tau$$

$$= \frac{M}{T}\int_{-\infty}^{\infty} |\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|}\left[\int_{-T+\tau}^{T+\tau} |f(\sigma-\lambda+i\omega)|^2 d\omega\right] d\tau$$

$$< \frac{KM}{T}\int_{-\infty}^{\infty} |\tau|^{-\frac{1}{2}-\lambda}(T+|\tau|)e^{-\frac{1}{2}\pi|\tau|} d\tau$$

$$= KM\int_{-\infty}^{\infty} |\tau|^{-\frac{1}{2}-\lambda} e^{-\frac{1}{2}\pi|\tau|} d\tau + \frac{KM}{T}\int_{-\infty}^{\infty} |\tau|^{\frac{1}{2}-\lambda} e^{\frac{1}{2}\pi|\tau|} d\tau < N,$$

where N also is a constant.

Thus a constant P exists such that

$$\frac{1}{2T}\int_{-T}^{T} |\phi(y, s)|^2 dt < P$$

for $$\sigma \geq \eta + \delta, \quad T \geq 1, \quad y > 0.$$

It follows that

$$\Sigma \frac{|a_n|^2}{n^{2\sigma}} e^{-2ny} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\phi(y, s)|^2 dt \leq P.$$

The proof of the theorem may now be completed as in § 6.

§ 8. In Theorem 7 we assumed that $f(s)$ is bounded *in every half-plane* $\sigma \geq \eta + \delta$. It is an easy deduction from Lindelöf's Theorem that this condition is fulfilled whenever $f(s)$ is bounded *on every line* $\sigma = \eta + \delta$. This condition may therefore replace the former one.

A similar modification may be made in Theorem 10. For it may be shown that *if*

$$\frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt$$

is bounded on the line $\sigma = \sigma_0$, *then it is bounded in the half-plane* $\sigma \geq \sigma_0$. I do not propose to give a proof of this result in the present paper. It will find a more natural place among a number of general theorems that I have proved, constructed on the model of Lindelöf's Theorem, but dealing with mean values. The typical theorem of this type is the following.

THEOREM 11. *If $f(s)$ is regular for* $\beta_1 \leq \lambda \leq \beta_2$, *and*

$$f(s) = O(e^{\epsilon t^2}),$$

for every positive ϵ and uniformly for $\beta_1 \leq \sigma \leq \beta_2$; and if

$$\frac{1}{2T}\int_{-T}^{T}|f(\beta_1+it)|^2 dt = O(T^{\alpha_1}),$$

$$\frac{1}{2T}\int_{-T}^{T}|f(\beta_2+it)|^2 dt = O(T^{\alpha_2}),$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$; then

$$\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2 dt = O(T^{\alpha}),$$

where

$$\alpha = \alpha(\sigma) = \frac{\beta_2-\sigma}{\beta_2-\beta_1}\alpha_1 + \frac{\sigma-\beta_1}{\beta_2-\beta_1}\alpha_2.$$

§ 9. If we bear these remarks in mind we obtain the following interesting consequence.

THEOREM 12. *Suppose that σ_0 and $\bar{\sigma}$ are the abscissae of convergence and absolute convergence of the series $\Sigma a_n n^{-s}$, and that*

$$\bar{\sigma} = \sigma_0 + 1.$$

Suppose further that ω is the lower bound of the values of σ for which

$$\frac{1}{2T}\int_{-T}^{T}|f(s)|^2 dt = O(1).$$

Then $$\omega = \sigma_0 + \tfrac{1}{2}.$$

In the first place, it follows from Theorem 10 (when modified in the manner indicated in the last section) that

$$\bar{\sigma} \leq \omega + \tfrac{1}{2}, \quad \omega \geq \sigma_0 + \tfrac{1}{2}.$$

Secondly we have

$$\frac{1}{2T}\int_{-T}^{T}|f(s)|^2 dt \sim \Sigma\frac{|a_n|^2}{n^{2\sigma}}$$

for $$\sigma > \sigma_0 + \tfrac{1}{2}.*$$

Hence $$\omega \leq \sigma_0 + \tfrac{1}{2}.$$

It should be observed that the result that ω falls half-way between σ_0 and $\bar{\sigma}$ depends on the hypothesis that the difference

* Landau, *Handbuch*, pp. 798–799 (Satz 41).

between σ and σ_0 is 1, the maximum possible. Thus, for the function

$$\eta(s) = 1^{-s} - 2^{-s} + 3^{-s} - \ldots = (1 - 2^{1-s}) \zeta(s),$$

we have $\sigma_0 = 0, \quad \bar{\sigma} = 1, \quad \omega = \tfrac{1}{2};$

but for $\eta(s) + \zeta(s + \tfrac{1}{2})$

we have $\sigma_0 = \tfrac{1}{2}, \quad \bar{\sigma} = 1, \quad \omega = \tfrac{1}{2}.$

On the other hand ω *may* fall half-way between σ_0 and $\bar{\sigma}$ even when $\bar{\sigma} < \sigma_0 + 1$. I have proved for example that, for the function

$$\Sigma \frac{e^{Ain^a}}{n^s},$$

where $A > 0$, $0 < a < \tfrac{1}{2}$, we have

$$\sigma_0 = 1 - a, \quad \bar{\sigma} = 1, \quad \omega = 1 - \tfrac{1}{2}a.$$

§ 10. I take this opportunity of stating some further results concerning 'mean values' of functions represented by Dirichlet's series, to which I have been led in the course of these researches. These results involve a parameter p, and some are valid for all positive values of p, others for $p \geq 1$, and others again only when p is an even integer. For the sake of simplicity of statement I shall suppose this last condition satisfied. I also suppose that $f(s)$ is a function represented by an ordinary Dirichlet's series which possesses a region of convergence, and that $f(s)$ is regular for all values of s under consideration. If

(1) $$\frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^p \, dt = O(T^{p\xi}),$$

I write

(2) $$f(\sigma + it) = O_p(|t|^\xi),$$

and I denote by $\mu_p(\sigma)$ the lower bound of the values of ξ for which (2) is true. If $\mu(\sigma)$ is the ordinary μ-function* of $f(s)$, we have

$$\mu_2 \leq \mu_4 \leq \mu_6 \leq \cdots \leq \mu.$$

Further, we have

$$\mu_2 + \tfrac{1}{2} \geq \mu_4 + \tfrac{1}{4} \geq \mu_6 + \tfrac{1}{6} \geq \cdots \geq \mu,$$

and $$\lim_{p \to \infty} \mu_p = \mu.$$

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* See Hardy and Riesz, *l. c. supra*, p. 14.

Each of the functions $\mu_p(\sigma)$ is ultimately zero, and, when not zero or infinite, is a positive, strictly decreasing, convex and continuous function of σ.

Many important general theorems in the theory of Dirichlet's series, usually stated with a hypothesis of the type

$$f(s)=O(|t|^k),$$

remain valid when this hypothesis is generalised to

$$f(s)=O_p(|t|)^k.$$

In particular this is so with the 'Schnee-Landau' theorem,* and with Theorem 41 of Riesz's and my tract.†

Examples.

§11. If $f(s)$ is regular and of finite order all over the plane, the series $\Sigma a_n n^{-s}$ is summable (A) all over the plane. Examples are furnished by the series

$$\Sigma \frac{(-1)^{n-1}}{n^s},$$

and $$\Sigma \frac{e^{Ain^a}}{n^s} \quad (A>0,\ \ 0<a<1).$$

These functions are, however, of finite order, and the series summable (C), all over the plane, so that it is not necessary to use Abel's method.

A particularly interesting example of the scope of the method is afforded by the function

$$f(s)=\Sigma \frac{e^{Ai(\log n)^2}}{n^s} \quad (A>0).$$

I have shown elsewhere‡ that $f(s)$ is regular all over the plane, and that its lines of absolute convergence, convergence, and summability (C), all coincide in the line $\sigma=1$.

In the paper referred to I proved that

$$f(s)=\sum_2^\infty \frac{e^{Ai(\log n)^2}}{n^s}=\phi(s)-f_1(s)+f_2(s),$$

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* Landau, *Handbuch*, p. 853.

† Hardy and Riesz, *l. c.*, p. 53.

‡ Hardy, "Examples to illustrate a point in the theory of Dirichlet's series", *The Tôhoku Mathematical Journal*, vol. viii., pp. 59–66.

where $\phi(s)$, $f_1(s)$, and $f_2(s)$ are integral functions of s, $\phi(s)$ is defined for $\sigma > 1$ by the formula

$$\phi(s) = \int_g^\infty \frac{e^{Ai(\log x)^2}}{x^s}\,dx,$$

$f_1(s)$ and $f_2(s)$ are defined for all values of s by the formulæ

$$f_1(s) = \int_g^{g+\infty e^{i\alpha}} \frac{e^{Ai(\log x)^2}}{x^s}\,\frac{dx}{1-e^{-2\pi ix}},$$

$$f_2(s) = \int_g^{g+\infty e^{-i\alpha}} \frac{e^{Ai(\log x)^2}}{x^s}\,\frac{dx}{e^{2\pi ix}-1},$$

and g and α are numbers subject to the inequalities

$$1 < g < 2, \quad 0 < \alpha < \tfrac{1}{2}\pi.$$

We may suppose the paths of integration to be parallel to the imaginary axis until they reach the lines $\operatorname{am} x = \alpha$ or $\operatorname{am} x = -\alpha$, and then to go off to infinity along these lines.

Let us first consider $f_1(s)$. On the vertical part of the contour we have $0 \le \operatorname{am} x \le \alpha$, and so

$$|x^s| = |x|^\sigma e^{-t \operatorname{am} x} \ge |x|^\sigma e^{-\alpha t},$$

and the contribution of this part is therefore, whatever be σ, of the form $O(e^{\alpha|t|})$. On the remaining part of the contour we have

$$x = re^{i\alpha},$$

$$|x^s| = r^\sigma e^{-\alpha t},$$

$$\left|\frac{1}{1-e^{-2\pi ix}}\right| < Ke^{-2\pi r \sin\alpha},$$

$$|e^{Ai(\log x)^2}| = |e^{Ai(\log r + i\alpha)^2}|$$

$$= e^{-2A\alpha \log r} = r^{-2A\alpha}.$$

Hence

$$f_1(s) = O(e^{\alpha|t|}) + O(e^{\alpha|t|}) \int_{g\sec\alpha}^\infty r^{-\sigma-2A\alpha} e^{-2\pi r \sin\alpha}\,dr,$$

$$= O(e^{\alpha|t|}).$$

In a similar manner it may be proved that

$$f_2(s) = O(e^{\alpha|t|}),$$

whatever the value of σ. Since α may be as small as we please, we have

$$f(s) = \phi(s) + O(e^{\epsilon|t|}),$$

for all values of σ and every positive ϵ.

Now

$$\phi(s) = \int_g^\infty \frac{e^{Ai(\log x)^2}}{x^s}\,dx = \int_{\log g}^\infty e^{Aiu^2-(s-1)u}\,du.$$

This integral is convergent only for $\sigma \geq 1$, but $\phi(s)$ is, as I showed in the paper referred to above, an integral function of s. Similarly the integral

$$\psi(s) = \int_0^g \frac{e^{Ai(\log x)^2}}{x^s}\,dx$$

$$= \int_{-\infty}^{\log g} e^{Aiu^2-(s-1)u}\,du,$$

which is convergent only for $\sigma \leq 1$, represents an integral function of s. We can calculate

$$\phi(s) + \psi(s)$$

by supposing that

$$s = 1 + it,$$

when both integrals are convergent. We have then

$$\phi(s) + \psi(s) = \int_{-\infty}^\infty e^{Aiu^2-itu}\,du$$

$$= \sqrt{\left(\frac{\pi}{A}\right)} \exp\left(\frac{i\pi}{4} - \frac{it^2}{4A}\right)$$

$$= \sqrt{\left(\frac{\pi}{A}\right)} \exp\left\{\frac{i\pi}{4} + \frac{i(s-1)^2}{4A}\right\},$$

and this equation must hold for all values of s.

Suppose now that $\sigma < 1$. Then

$$\psi(s) = O(1),$$

when $|t| \to \infty$. Hence

$$\phi(s) = \sqrt{\left(\frac{\pi}{A}\right)} \exp\left\{\frac{i\pi}{4} + \frac{i(s-1)^2}{4A}\right\} + O(1)$$

$$= \sqrt{\left(\frac{\pi}{A}\right)} \exp\left[\frac{(1-\sigma)t}{2A} + \frac{i}{4}\left\{\pi + \frac{(\sigma-1)^2 - t^2}{A}\right\}\right] + O(1).$$

Let

$$T = \frac{\pi}{4} + \frac{(\sigma-1)^2 - t^2}{A}.$$

Then, combining our previous results, we see that

$$f(s) = \sqrt{\left(\frac{\pi}{A}\right)} \exp\left\{\frac{(1-\sigma)t}{2A} + iT\right\} + O(e^{\epsilon|t|}).$$

Thus $$f(s) = O(e^{\epsilon|t|}),$$
when $t \to -\infty$, while

$$|f(s)| \sim \sqrt{\left(\frac{\pi}{A}\right)} e^{(1-\sigma)t/2A},$$

when $t \to \infty$; and *the index of* $f(s)$ *is*

$$\frac{1-\sigma}{2A}$$

when $\sigma < 1$.

§ 12. It follows that *the series*

$$\Sigma \frac{e^{Ai(\log n)^2}}{n^s}$$

is summable (A) *if* $\sigma > 1 - A\pi$, *but not if* $\sigma < 1 - A\pi$. *The breadth of the strip in which the series is summable* (A), *but not convergent or summable* (C), *is* $A\pi$.

A similar investigation shows that *the series*

$$\Sigma \frac{e^{Ai(\log n)^a}}{n^s}$$

is summable (A) *all over the plane if* $a > 2$, *but never summable* (A) *if* $1 < a < 2$ *and* $\sigma < 1$. In none of these cases is the series summable (C) anywhere except in its region of convergence.

Generalisations.

§ 12. We say that Σa_n is *summable* (A, ν) if

$$\Sigma a_n e^{-yn^\nu}$$

is convergent for all positive values of y and

$$\phi(y) = \Sigma a_n e^{-yn^\nu}$$

tends to a limit as $y \to 0$. We have then

$$\phi(y, s) = \Sigma a_n n^{-s} e^{-yn^\nu} = \frac{1}{\Gamma(s/\nu)} \int_0^\infty u^{(s/\nu)-1} \phi(u+y)\, du,$$

if $\sigma > 0$. Arguing as in § 2, we see that *the region of summability* (A, ν) *is a half-plane.*

The argument used in proving Theorem 5 shows that *if* $\Sigma a_n n^{-s}$ *is summable* (A, ν) *for* $\sigma > \mathfrak{A}_\nu$ *then* $H(\sigma) < \pi/2\nu$ *for* $\sigma > \mathfrak{A}_\nu$.

A similar modification can be made in Theorem 6. *If $H(\sigma) < \pi/2\nu$ for $\sigma > \eta$, then the series is summable (A, ν) for $\sigma > \eta$.* For

$$e^{-yn^{\nu}} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} (yn^{\nu})^{-u}\,\Gamma(u)\,du,$$

$$\phi(y, s) = \Sigma a_n n^{-s} e^{-yn^{\nu}} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} y^{-u}\,\Gamma(u) f(s+\nu u)\,du,$$

for sufficiently large values of κ. The remainder of the argument follows the course of the proof of Theorem 6.

Combining these results, we see that *the region of summability (A, ν) is the half-plane in which*

$$H(\sigma) < \frac{\pi}{2\nu}.$$

Thus, for example, *the series*

$$\Sigma \frac{e^{Ai(\log n)^2}}{n^s} \quad (A > 0)$$

is summable (A, ν) if $\sigma > 1 - \dfrac{A\pi}{\nu}$, but not if $\sigma < 1 - \dfrac{A\pi}{\nu}$.

§ 13. Similar methods may be applied to the general series

$$\Sigma a_n e^{-\lambda_n s} = \Sigma a_n l_n^{-s}.$$

We may say that the series Σa_n is *summable (A, l)* if $\Sigma a_n e^{-yl_n}$ is convergent for all positive values of y and tends to a limit when $y \to 0$. Thus summability (A) is the same as summability (A, n), and summability (A, ν) as summability (A, n^{ν}).

If we suppose that $\Sigma a_n e^{-\lambda_n s}$ has a region of absolute convergence, we can prove a series of results analogous to those proved in the preceding sections for ordinary Dirichlet's series. *The region of summability (A, l) is the half-plane in which*

$$H(\sigma) < \tfrac{1}{2}\pi.$$

If

$$\Sigma a_n e^{-yl_n^{\nu}}$$

tends to a limit, we may say that Σa_n is *summable (A, l, ν)*. *The region of summability (A, l, ν) of the series $\Sigma a_n e^{-\lambda_n s}$ is the half-plane in which*

$$H(\sigma) < \frac{\pi}{2\nu}.$$

CORRECTIONS

p. 180, *line* 4 *up*. For $\kappa > \bar{\sigma}$ read $\sigma > \bar{\sigma}$.

p. 181, *line* 9 *up*. For the 2nd $-\lambda+iT$ read $-\lambda-iT$.

p. 185, *line* 5. For $e^{\frac{1}{2}\pi|\tau|}$ read $e^{-\frac{1}{2}\pi|\tau|}$.

—— *line* 13. For Theorem 7 read Theorem 8.

—— *line* 2 *up*. For λ read σ.

pp. 191–2. The last two paragraphs should be numbered 13 and 14.

COMMENTS

A summary of the main results of this paper is given in 1916, 7.

The condition $a_n = O(n^K)$ (§ 1) is necessary and sufficient for the ordinary Dirichlet series $\sum a_n n^{-s}$ to have half-planes of absolute convergence ($\sigma > \bar{\sigma}$), uniform convergence ($\sigma > \sigma_u$) and convergence ($\sigma > \sigma_c$). This also ensures the existence of $f(s)$, and of half-planes of regularity of $f(s)$ ($\sigma > \rho$), 'index' $< \frac{1}{2}\pi$ ($\sigma > \gamma$), 'finite order' ($\sigma > \lambda$), and boundedness of $f(s)$ ($\sigma > \beta$), each half-plane being the union of the open half-planes with the appropriate property. Then

$$-\infty \leqslant \rho \leqslant \gamma \leqslant \lambda \leqslant \sigma_c \leqslant \sigma_u \leqslant \beta \leqslant \bar{\sigma} < \infty,$$

where the relation $\sigma_u \leqslant \beta$ is Bohr's theorem mentioned after the statement of Theorem 9, § 5. The existence of $\bar{\sigma}$ is assumed throughout §§ 1–10, except in Theorems 3 and 11, which are independent of Dirichlet series. Bohr proved in *Bidrag* . . . that the half-plane of Cesàro summability ($\sigma > \Lambda$) is identical with $\sigma > \lambda$. Here Hardy proves that the half-plane of Abel summability ($\sigma > \mathfrak{A}$) is identical with $\sigma > \gamma$.

In the proof of Theorem 1, the remark that the integral over (0, 1) is 'uniformly convergent' seems to refer to its convergence as an improper integral at $u = 0$. The analysis in the proof of Theorem 1 leads to the further conclusion that, when $\mathfrak{A} = 0$, the formula

$$f(s) = \frac{1}{\Gamma(s)} \int\limits_0^\infty u^{s-1}\phi(u)\, du$$

gives the analytic continuation of $\sum a_n n^{-s}$ onto the strip $0 < \sigma \leqslant \sigma_0$, and hence shows that $\rho \leqslant \mathfrak{A}$. Theorem 5 shows that $\rho \leqslant \gamma \leqslant \mathfrak{A}$, and Theorem 6 that $\rho \leqslant \mathfrak{A} \leqslant \gamma$.

The mean value formula quoted from Landau,† in the proof of Theorem 8, § 6, may be written

$$\sum |b_n|^2 n^{-2\sigma} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\sum b_n n^{-s}|^2 \, dt,$$

where $b_n = a_n e^{-ny}$ $(y > 0)$. The series $\sum b_n n^{-s}$ is absolutely convergent for every s, since $a_n = O(n^K)$. A simple proof of this formula is given in Titchmarsh's *Theory of functions*, § 9.5;‡ Landau's proof is for general Dirichlet series.

The modification of Theorem 8, stated in § 8 (see Corrections), follows from H.R., Theorem 14, since $\bar{\sigma}$ exists. The corresponding modification of Theorem 10 follows in the same way from Theorem 11. Theorem 11, which is not proved here, is included in Theorem 6 of 1927, 5, by Hardy, Ingham, and Pólya (in Vol. IV). They say, § 1.3, that 'the question of the existence of theorems, for mean values, of the Phragmén–Lindelöf type was apparently first raised by Hardy' in the present paper. They also say that Hardy states theorems, which are 'included, in forms both more general and more precise, in Carlson's independent researches'.§

Some results from Carlson (papers (1) and (2)) are given by Titchmarsh (loc. cit.), §§ 9.51–5. In particular, he gives a theorem which includes Theorem 10, § 7, and the modification of it in § 8. Carlson's paper (3) contains extensions of the Schnee–Landau theorem, as foreseen in § 10. In paper (4), Carlson states some results for pth power means, which overlap the results in 1927, 5.

The first string of inequalities in § 10 may be obtained in the form

$$\mu_p \leqslant \mu_q \leqslant \mu \quad (0 < p < q),$$

from the inequalities

$$M_p \leqslant M_q \leqslant M \quad (0 < p < q),$$

where $M_p = \left(\frac{1}{b-a} \int_a^b |g|^p \, dt\right)^{1/p}$ and $M = \operatorname*{ess.sup.}_{a\leqslant t\leqslant b} |g(t)|$;

† *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 2, pp. 776–9. Teubner, Leipzig, 1909.

‡ Oxford University Press, 1932.

§ Carlson (1) *Comptes rendus* 172 (1921), 838–40; (2) *Arkiv för Mat.* 16 (1922), No. 18, 1–19; (3) ibid. 19 (1926), No. 25, 1–17; (4) *Comptes rendus* 181 (1925), 397–9.

see *Inequalities,*‖ Theorems 192–3. We have

$$\mu_p = \mu_p(\sigma) = \overline{\lim_{T\to\infty}} \log\left(\frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^p\,dt\right)^{1/p} \Big/ \log T,$$

$$\mu = \mu(\sigma) = \overline{\lim_{t\to\infty}} \log|f(\sigma+it)|/\log t.$$

The second string may be obtained in the form

$$\mu_p+1/p \geqslant \mu_q+1/q \geqslant \mu \quad (0 < p < q)$$

in two steps. The first step is a proof that $\mu_p+1/p \geqslant \mu$ $(p > 0)$. This was stated by Carlson (paper (4)), and is a corollary of Theorem 2 in 1927, 5. Since $\mu_p(\sigma)$ is a decreasing function of σ for $\sigma > \lambda$, whenever $\bar{\sigma}$ exists, Hardy, Ingham, and Pólya's theorem shows that

$$\mu(\sigma) \leqslant \mu_p(\sigma-\epsilon)+1/p$$

for $\epsilon > 0$. Since $\mu_p(\sigma)$ is continuous, it follows that

$$\mu(\sigma) \leqslant \mu_p(\sigma)+1/p.$$

The second step is to prove that

$$\mu_p+1/p \geqslant \mu_q+1/q \quad (0 < p < q).$$

For this we may use the *Lemma*: *if* $b-a > 1$, *and* $S_q \geqslant M$ *for some* $q > 0$, *where* $S_q = (b-a)^{1/q}M_q$, *then* $S_p \geqslant S_q$ *for* $0 < p < q$. The proof is analogous to that of Jensen's inequality; see *Inequalities*, Theorem 19. Excluding the trivial case $M = 0$, we have, for $0 < p < q$, since $S_q/M \geqslant 1$ and $|g|/M \leqslant 1$ p.p.,

$$M \leqslant S_q = M\left(\int_a^b |g/M|^q\,dt\right)^{1/q} \leqslant M\left(\int_a^b |g/M|^q\,dt\right)^{1/p}$$

$$\leqslant M\left(\int_a^b |g/M|^p\,dt\right)^{1/p} = S_p.$$

If $b-a = 1$, then $S_p \leqslant S_q$ $(0 < p < q)$; see Hardy 1929, 2 (in Vol. III), p. 74. In this case $S_p = M_p$, and the condition $S_p \geqslant M$ cannot be satisfied unless $g \equiv$ constant.

Some remarks in Carlson (4) indicate another way of completing step two. He says that (i) $\nu_p = p\mu_p+1$ *is convex*; (ii) ν_p/p *is a convex function of* $1/p$; (iii) $\lim \nu_p/p = \mu$. Now (i) follows from the logarithmic convexity of S_p^p for $p > 0$; see *Inequalities*, Theorem 196. Also (i) is equivalent to (ii)$'$: *the convexity of* $p\nu_{1/p}$ for $p > 0$, by Theorem

‖ Hardy, Littlewood, and Pólya's *Inequalities.* Cambridge University Press, 1934.

119 of *Inequalities*, while (ii)$'$ is equivalent to (ii). Further, (iii) follows from step one and the first string of inequalities. Finally, since ν_p/p is never less than its limit as $p \to \infty$, it follows from (ii) that ν_p/p must decrease as p increases.

In § 11, results are quoted from 1915, 11; see also 1913, 2, p. 436. The results in the second § 12 have been extended by Cartwright.†† Hardy gives further extensions in D.S., Appendix V.

†† Cartwright (1), *J. London Math. Soc.* 3 (1928), 262–7; (2), *Proc. London Math. Soc.* (2), 31 (1930), 81–96.

THE SECOND THEOREM OF CONSISTENCY FOR SUMMABLE SERIES

By G. H. Hardy.

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1. My object in writing this paper is to give a full proof of a theorem enunciated without proof in the tract "The general theory of Dirichlet's series", recently published by Dr. Marcel Riesz and myself*. The theorem is as follows:

If (i) *the series* Σc_n *is summable* (λ, κ), *to sum* C;

(ii) μ *is a logarithmico-exponential function of* λ *such that*

$$\mu = O(\lambda^{\Delta}),$$

where Δ *is a constant*; *then the series* Σc_n *is summable* (μ, κ) *to sum* C.

2. I begin by recalling Riesz's definition† of summability (λ, κ), *i.e.* summability by means of type λ and order κ. Suppose that (λ_n) is an ascending sequence of positive numbers whose limit is infinity; and let

$$C_\lambda(\tau) = c_1 + c_2 + \ldots + c_n$$

if

$$\lambda_n < \tau \leqslant \lambda_{n+1}.$$

Further let

(2.1)
$$C_\lambda^\kappa(\omega) = C_\lambda(\omega)$$

if $\kappa = 0$, and

(2.2)
$$C_\lambda^\kappa(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\kappa c_n = \kappa \int_0^\omega C_\lambda(\tau)(\omega - \tau)^{\kappa - 1}\, d\tau$$

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* *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 18, p. 33 (Theorem 19). I refer to this tract as "H. and R."

† H. and R., p. 21.

if $\kappa > 0$. Then
$$\omega^{-\kappa} C_\lambda^\kappa(\omega)$$
is called the *typical mean of type* λ *and order* κ formed from the series Σc_n, and the series is said to be *summable* (λ, κ), to sum C, if this "typical mean" tends to the limit C when $\omega \to \infty$.

When $\lambda_n = n$, the means are said to be *arithmetic.* Arithmetic means are equivalent to Cesàro's means, or to the generalisations of Cesàro's means considered by Knopp and Chapman: a series is summable (n, κ) if and only if it is summable (C, κ).*

The *first theorem of consistency*† asserts that, if a series is summable (λ, κ), then it is summable (λ, κ'), to the same sum, for any value of κ' greater than κ. In particular a convergent series is summable by typical means of any positive order, since summability $(\lambda, 0)$ is equivalent to convergence. The general idea expressed by the first theorem of consistency is that, so long as the type remains the same, *the efficacy of a method of summation increases with the order.*

The *second theorem of consistency* lies somewhat deeper. The general idea which it expresses is that, when the order of a method of summation remains the same, *its efficacy increases as the type decreases,* that is to say as the rate of increase of the function λ_n which defines the type decreases. If
$$\lambda_n - \lambda_{n-1} > K\lambda_n,$$
that is to say if the rate of increase of λ_n is as great as that of an exponential e^{An}, then the efficacy of the method is *nil*: it will sum convergent series and no others‡. If λ_n runs through the functions of the logarithmico-exponential scale, such as
$$e^n,\ n,\ \log n,\ \log\log n,\ \dots,$$
then we obtain a succession of systems of methods of gradually increasing efficacy.

The theorem suggested by this general idea is that *if a series is summable* (λ, κ) *then it is summable* (μ, κ), μ *being any function of* n *whose rate of increase is less than that of* λ. The actual theorem stated in § 1 is in one way less general and in another more general than this. In the first place, in order to ensure the truth of the theorem, we must suppose that the relation between the rates of increase of μ and λ is characterised by a certain regularity; and the most convenient way of

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* Riesz, "Sur une méthode de sommation équivalente à la méthode des moyennes arithmétiques", *Comptes Rendus*, 12 June 1911.

† H. and R., p. 29 (Theorem 16).

‡ H. and R., p. 46 (Theorem 36).

ensuring this is to suppose that μ is a logarithmico-exponential function of λ, a phrase which we will define more precisely in a moment. But, when this limitation is made, we are able to assert rather more than our general principle suggests. The efficacy of the method increases, or at any rate does not decrease, as the rate of increase of the type decreases; a series summable (λ, κ) is certainly summable (μ, κ) if μ increases more slowly than λ. But the converse implication will also be true, and the two methods completely equivalent, if the difference between the rates of increase of λ and μ is not too pronounced, if in fact either function increases with a rapidity comparable to that of a power of the other. If, for example, both λ and μ are powers of n, then any series summable (λ, κ) will be summable (μ, κ), and conversely.

3. Proofs of certain special cases of this theorem have already been published. The most important case is that in which

$$\mu = \log \lambda. \tag{3.1}$$

This case of the theorem was enunciated in 1909 by Riesz*; and his proof was published for the first time in our tract†. Another case is that in which

$$\mu = P(\lambda), \tag{3.2}$$

where P is a polynomial. This case has been treated by Berwald‡, when $\lambda = n$ and κ is an integer. A third case§ is that in which $\kappa = 1$: the theorem then amounts to little more than a restatement in different language of a theorem of Cesàro.

I had conjectured the truth of the general theorem some years ago, when engaged, in collaboration with Mr. Chapman, on a paper dealing with the general theory of summability‖. At that time I had a proof not of the theorem itself, but of its analogue for integrals, and only in the two cases in which (i) κ is an integer or (ii) $0 < \kappa < 1$. As soon as I became familiar with Riesz's methods it became clear to me that my proof applied to series as well; but I was still unable to overcome the algebraical difficulties presented by the proof of the theorem in its most general form.

* "Sur la sommation des séries de Dirichlet", *Comptes Rendus*, 5 July 1909.

† H. and R., p. 30 (Theorem 17).

‡ "Solution nouvelle d'un problème de Fourier", *Arkiv för Matematik*, Vol. 9, 1913, No. 14.

§ Hardy, "On certain oscillating series", *Quarterly Journal*, Vol. 38, 1907, pp. 269–288.

‖ Hardy and Chapman, "A general view of the theory of summable series", *Quarterly Journal*, Vol. 42, 1911, pp. 181-216.

It was only when Riesz, in the course of the preparation of our tract, discovered an important simplification of his method of treatment of the case in which $\mu = \log \lambda$, that I was able to find a completely general proof.

*Definitions and lemmas from the Infinitärcalcul.**

4. A *logarithmico-exponential function*, or, shortly, *an L-function*, is a real one-valued function which can be defined by an explicit formula involving, each only a finite number of times, the ordinary algebraical symbols

$$+, \quad -, \quad \times, \quad \div, \quad \surd,$$

and the symbols $\qquad \log(\ldots), \quad e^{\cdots}$

of the logarithmic and exponential functions.

The properties of L-functions which are required for the argument of this paper are as follows.

4.1. *Any L-function $\mu(\lambda)$ is continuous, of constant sign, and monotonic, from a certain value of λ onwards; and the same is true of any of its derivatives.*†

We may suppose, without real loss of generality, that $\mu(\lambda)$, and such of its derivatives as occur in the argument, satisfy these conditions for all values of λ in question.

4.2. *If $\mu \to \infty$, and a number Δ exists such that*

(4.21) $$\mu = O(\lambda^{\Delta}),$$

then $$\mu^{(r)} = O\left(\frac{\mu}{\lambda^r}\right),$$

$\mu^{(r)}$ denoting the r-th derivative of $\mu(\lambda)$.‡

4.3. *If μ satisfies the conditions of* 4.2, *and ν lies between two posi-*

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* See Hardy, "Orders of infinity", *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 12, 1910, or "Properties of logarithmico-exponential functions", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1912, pp. 54–90. I refer to the first of these publications as "O. I.".

† O. I., p. 18. See also "Properties &c.", p. 40

‡ O. I., pp. 38 *et seq.*

tive numbers g and G, then positive numbers h and H exist, such that

$$h \leqslant \frac{\mu(\lambda\nu)}{\mu(\lambda)} \leqslant H.\text{*} \tag{4.31}$$

For
$$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\log\mu(\lambda\nu)-\log\mu(\lambda)} = e^{\left\{\lambda(\nu-1)\frac{\mu'(\theta)}{\mu(\theta)}\right\}},$$

where θ lies between λ and $\lambda\nu$. Hence

$$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\lambda O(1/\lambda)} = e^{O(1)}.$$

It is evident that the same result holds for *decreasing* functions which decrease less rapidly than $\lambda^{-\Delta}$ for some value of Δ.

Proof of the theorem when κ is an integer.

5.1. In proving the theorem we may suppose c_n to be *real*: if c_n is complex we can consider the real and imaginary parts of the series separately. We may also suppose, without real loss of generality, that $C = 0$. If C is not zero we begin by proving the theorem for the series

$$(c_1 - C) + c_2 + c_3 + \ldots,$$

and afterwards add to this series the convergent series

$$C + 0 + 0 + \ldots .$$

We are given that

$$\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\,d\sigma = o(\eta^\kappa),\dagger \tag{5.11}$$

and we wish to prove that

$$\int_{\mu_1}^{\zeta} C_\mu(\tau)(\zeta-\tau)^{\kappa-1}\,d\tau = o(\zeta^\kappa). \tag{5.12}$$

In (5.12) we put
$$\tau = \mu(\sigma),$$

and observe that
$$C_\mu\{\mu(\sigma)\} = C_\lambda(\sigma).$$

* More precise results of this character will be found in my paper "Oscillating Dirichlet's integrals", *Quarterly Journal*, Vol. 44, 1913, pp. 1–40 (see p. 23 *et seq.*).

† Since $C_\lambda(\sigma) = 0$ for $0 \leqslant \sigma \leqslant \lambda_1$, it is a matter of indifference whether the lower limit is 0 or λ_1.

We thus obtain

$$(5.13)\qquad \int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu' d\sigma = o(\zeta^\kappa),$$

where $\mu = \mu(\sigma), \qquad \zeta = \mu(\eta).$

From this point onwards our argument depends on the nature of κ. I shall suppose first that κ is an integer.

5.2. If κ is an integer, we have

$$C_\lambda(\sigma) = \frac{1}{\kappa!}\left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma).*$$

The integral (5.13) is therefore a constant multiple of

$$(5.21)\qquad J = \int_{\lambda_1}^{\eta} \frac{d}{d\sigma}(\zeta-\mu)^\kappa \left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma)\, d\sigma.$$

We transform this integral by κ integrations by parts. Observing that $C_\lambda(\sigma)$ and its first $\kappa-1$ derivatives vanish for $\sigma = \lambda_1$, and that $\zeta-\mu$ vanishes for $\sigma = \eta$, we obtain

$$(5.22)\qquad \begin{aligned} J &= (-1)^{\kappa-1} C_\lambda^\kappa(\eta)\left[\left(\frac{d}{d\sigma}\right)^\kappa (\zeta-\mu)^\kappa\right]_{\sigma=\eta} \\ &\quad + (-1)^\kappa \int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\zeta-\mu)^\kappa\, d\sigma \\ &= J_1 + J_2, \end{aligned}$$

say.

5.3. In the first place

$$(5.31)\qquad J_1 = -\kappa!\, C_\lambda^\kappa(\eta)(\zeta')^\kappa,$$

where ζ' is the value of μ' when $\sigma = \eta$, $\mu = \zeta$. Hence

$$J_1 = -\kappa!\,\frac{C_\lambda^\kappa(\eta)}{\eta^\kappa}\left(\frac{\eta\zeta'}{\zeta}\right)^\kappa \zeta^\kappa = o(1)\,O(1)\,\zeta^\kappa = o(\zeta^\kappa),$$

by 4.2.

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* H. and R., p. 28.

On the other hand, it is easily verified that

(5.32) $$\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\zeta-\mu)^{\kappa} = \Sigma A\zeta^{\kappa-r}\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\ldots,$$

where the A's are constants, and

(5.331) $$0 < s+s_1+s_2+\ldots = r \leqslant \kappa,$$

(5.332) $$s_1+2s_2+3s_3+\ldots = \kappa+1.$$

Hence our integral reduces to a sum of constant multiples of integrals of the types

(5.34) $$\zeta^{\kappa-r}\int_{\lambda_1}^{\eta} C_{\lambda}^{\kappa}(\sigma)\,\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\ldots\,d\sigma.$$

Observing that $$C_{\lambda}(\sigma) = o(\sigma^{\kappa}),$$

and $$\mu^{(r)} = O\left(\frac{\mu}{\lambda^{r}}\right),$$

by 4.2, we see that (5.34) is of the form

(5.35) $$o\left(\zeta^{\kappa-r}\int_{\lambda_1}^{\eta}\sigma^{\kappa-s_1-2s_2-3s_3-\ldots+1}\,\mu^{s+s_1+s_2+\ldots-1}\mu'\,d\sigma\right)$$

$$= o\left(\zeta^{\kappa-r}\int_{\lambda_1}^{\eta}\mu^{r-1}\mu'\,d\sigma\right) = o(\zeta^{\kappa}).$$

This completes the proof of the theorem when κ is an integer.

Proof when $0 < \kappa < 1$.

6.1. We consider next the case in which $0<\kappa<1$. I shall suppose first that the increase of μ is greater than that of $\log\lambda$, so that $\lambda\mu'$ tends steadily to infinity with λ.

We observe first that, in virtue of the first theorem of consistency, Σc_n is summable $(\lambda, 1)$, to sum zero, so that

(6.11) $$C_{\lambda}^{1}(\sigma) = o(\sigma).$$

Now let A be any positive constant. Then we can, when η is large enough, determine a unique number η_1 such that

(6.12) $$\lambda_1 < \eta_1 < \eta, \qquad \zeta-\zeta_1 = A\eta_1\zeta_1',$$

where

$$\zeta_1 = \mu(\eta_1), \qquad \zeta_1' = \mu'(\eta_1). \tag{6.121}$$

For, as σ increases from λ_1 to η, $\zeta-\mu$ decreases steadily from $\zeta-\mu_1$ (which is large when η is large) to zero, whereas $\sigma\mu'$ increases steadily with σ.

Let us suppose that $0 < A < 1$, and that η_1 has been chosen so as to satisfy (6.12). Then we can determine a positive constant h such that

$$h\eta < \eta_1 < \eta. \tag{6.13}$$

For
$$\zeta - \zeta_1 = (\eta-\eta_1)\,\zeta_2',$$

where ζ_2' is the value of μ' when σ has a certain value η_2 between η_1 and η. Thus

$$A\eta_1\zeta_1' = \zeta-\zeta_1 = (\eta-\eta_1)\zeta_2' = \frac{\eta-\eta_1}{\eta_2}\,\eta_2\zeta_2' > \frac{\eta-\eta_1}{\eta}\,\eta_1\zeta_1',$$

since $\sigma\mu'$ increases with λ. Hence

$$A > \frac{\eta-\eta_1}{\eta}, \qquad \eta_1 > (1-A)\,\eta,$$

and we may take
$$h = 1-A.$$

6.2. We now write

$$J = \int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma = \int_{\lambda_1}^{\eta_1} + \int_{\eta_1}^{\eta} = J_1 + J_2, \tag{6.21}$$

say. We begin by considering J_1. Integrating by parts, we obtain

$$\begin{aligned} J_1 &= C_\lambda^1(\eta_1)(\zeta-\zeta_1)^{\kappa-1}\zeta_1' - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)\frac{d}{d\sigma}\{(\zeta-\mu)^{\kappa-1}\mu'\}\,d\sigma \\ &= J_{1,1} + J_{1,2}, \end{aligned} \tag{6.22}$$

say. In the first place

$$J_{1,1} = \frac{C_\lambda^1(\eta_1)}{\eta_1}(A\eta_1\zeta_1')^{\kappa-1}\eta_1\zeta_1' = o(\eta_1\zeta_1')^{\kappa} = o(\zeta_1^{\kappa}) = o(\zeta^{\kappa}). \tag{6.23}$$

Secondly,

$$\begin{aligned} J_{1,2} &= (\kappa-1)\int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\zeta-\mu)^{\kappa-2}\mu'^2\,d\sigma - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\zeta-\mu)^{\kappa-1}\mu''\,d\sigma \\ &= J_{1,2,1} + J_{1,2,2}, \end{aligned} \tag{6.24}$$

say. Now

(6.25)
$$\frac{\sigma\mu'}{\zeta-\mu} < \frac{\sigma\mu'}{\zeta-\zeta_1} < \frac{\sigma\mu'}{A\eta_1\zeta_1'} < \frac{1}{A},$$

since $\lambda_1 < \sigma < \eta_1$ and $\sigma\mu'$ increases with σ. Hence

(6.26)
$$J_{1,2,1} = (\kappa-1)\int_{\lambda_1}^{\eta_1} o(\sigma)(\zeta-\mu)^{\kappa-1}\mu'\frac{\sigma\mu'}{\zeta-\mu}\,d\sigma$$
$$= o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= o(\zeta^{\kappa}).$$

Also

(6.27)
$$J_{1,2,2} = -\int_{\lambda_1}^{\eta_1} o(\sigma)(\zeta-\mu)^{\kappa-1} O\left(\frac{\mu'}{\sigma}\right)d\sigma$$
$$= o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= o(\zeta^{\kappa}).$$

From (6.22)–(6.27) it follows that

(6.28)
$$J_1 = o(\zeta^{\kappa}).$$

6.3. It remains to consider J_2. We have

(6.311)
$$J_2 = \int_{\eta}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma$$
$$= \zeta'\int_{\eta_2}^{\eta} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\,d\sigma$$

if μ' increases; and

(6.312)
$$J_2 = \zeta_1'\int_{\eta_1}^{\eta_2} C_\lambda(\sigma)(\zeta-\mu)^{\kappa-1}\,d\sigma$$

if μ' decreases, η_2 denoting in either case a number between η_1 and η. Now

$$\frac{d}{d\sigma}\frac{\zeta-\mu}{\eta-\sigma} = \frac{\zeta-\mu-(\eta-\sigma)\mu'}{(\eta-\sigma)^2} = \frac{\bar{\zeta}'-\mu'}{\eta-\sigma},$$

$\bar{\zeta}'$ being the value of μ' for a value $\bar{\eta}$ of its argument between σ and η.

This is positive if μ' increases and negative if μ' decreases. Hence

$$\frac{\zeta-\mu}{\eta-\sigma}$$

is monotonic, and varies in the same sense as μ'; so that

$$\left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1}$$

is monotonic in the sense opposite to that of the variation of μ'. We have therefore

$$(6.321) \qquad J_2 = \zeta' \int_{\eta_2}^{\eta} C_\lambda(\sigma) \left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma$$

$$= \zeta' \left(\frac{\zeta-\zeta_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_2}^{\eta_3} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' increases; and

$$(6.322) \qquad J_2 = \zeta_1' \int_{\eta_1}^{\eta_2} C_\lambda(\sigma) \left(\frac{\zeta-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma$$

$$= \zeta_1' \left(\frac{\zeta-\zeta_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' decreases, η_3 being in either case another number between η_1 and η. And in either case

$$\frac{\zeta-\zeta_2}{\eta-\eta_2}$$

lies between ζ_1' and ζ'.

The order of μ lies between $\log\lambda$ and λ^Δ, and that of μ' between $1/\lambda$ and $\lambda^{\Delta-1}$, and *a fortiori* between $\lambda^{-\Delta}$ and λ^{Δ}.* Also

$$h\eta < \eta_1 < \eta.$$

It follows from 4.3 that the ratio ζ_1'/ζ' lies between fixed positive limits We have therefore in any case

$$(6.33) \qquad J_2 = O(\zeta')^\kappa \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1} d\sigma.$$

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* Evidently we may suppose $\Delta > 1$.

But*

$$\left|\int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}d\sigma\right| < 2 \underset{\lambda_1 \leqslant \tau \leqslant \eta}{\text{Max}} C_\lambda^\kappa(\tau). \tag{6.34}$$

Hence

$$J_2 = O(\zeta')^\kappa o(\eta^\kappa) = o(\eta\zeta')^\kappa = o(\zeta^\kappa). \tag{6.35}$$

From (6.21), (6.28), and (6.35) it follows that

$$J = o(\zeta^\kappa). \tag{6.36}$$

6.4. We have thus proved the theorem when $0 < \kappa < 1$ and the order of μ lies between $\log \lambda$ and λ^Δ. Suppose next that the order of μ lies between $\log \log \lambda$ and $(\log \lambda)^\Delta$; and let

$$\nu = (\log \lambda)^a,$$

where $a > 1$. The series is summable (λ, κ), and therefore, by what precedes, summable (ν, κ). But

$$\mu(\lambda) = \mu(e^{\nu^{1/a}})$$

is an L-function of ν whose order lies between $(1/a)\log \nu$ and $\nu^{\Delta/a}$. Hence the series is summable (μ, κ). The theorem is thus proved when $0 < \kappa < 1$ and the order of μ is greater than $\log \log \lambda$. Repeating the argument, we prove it whenever the order of μ is greater than any one of

$$\log \log \log \lambda, \quad \log \log \log \log \lambda, \quad \dots .$$

Since any L-function which tends to infinity must increase more rapidly than some one of the repeated logarithmic functions†, the theorem is true without restriction on μ. The proof when $0 < \kappa < 1$ is thus complete.

Proof when κ is greater than 1 and not integral.

7.1. The proof of the theorem when κ is greater than 1, but not an integer, presents no fresh difficulty of principle. All that is necessary is to combine in an appropriate manner the arguments used in 5 and 6.

We suppose that

$$k < \kappa < k+1, \tag{7.11}$$

* H. and R., p. 28 (Lemma 7).

† O. I., p. 20; "Properties &c.", pp. 63 *et seq.*

k being an integer, and (for the moment) that μ is of higher order than $\log\lambda$. Integrating the integral (5.13) k times by parts, we obtain a constant multiple of

$$(7.12) \qquad J=\int_{\lambda_1}^{\eta} C_\lambda^k(\sigma)\left(\frac{d}{d\sigma}\right)^{k+1}(\zeta-\mu)^\kappa\,d\sigma.$$

We write, as in 6.2,

$$(7.13) \qquad J=\int_{\lambda_1}^{\eta_1}+\int_{\eta_1}^{\eta}=J_1+J_2.$$

7.2. In order to obtain an upper limit for J_1 we integrate once more by parts. We thus obtain

$$(7.21)\quad J_1=\frac{C_\lambda^{k+1}(\eta_1)}{k+1}\left(\frac{d}{d\eta_1}\right)^{k+1}(\zeta-\zeta_1)^\kappa-\frac{1}{k+1}\int_{\lambda_1}^{\eta_1}C_\lambda^{k+1}(\sigma)\left(\frac{d}{d\sigma}\right)^{k+2}(\zeta-\mu)^\kappa\,d\sigma$$
$$=J_{1,1}+J_{1,2},$$

say. Now

$$(7.22) \qquad \left(\frac{d}{d\sigma}\right)^{k+1}(\zeta-\mu)^\kappa=\Sigma A\,(\zeta-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\ldots,$$

where

$$(7.231) \qquad s+s_1+s_2+\ldots=\kappa,$$

$$(7.232) \qquad s_1+2s_2+3s_3+\ldots=k+1.$$

Hence

$$(7.24) \qquad J_{1,1}=\frac{C^{k+1}(\eta_1)}{k+1}\Sigma A\,(\zeta-\zeta_1)^s(\zeta_1')^{s_1}(\zeta_1'')^{s_2}\ldots.$$

But
$$C^{k+1}(\eta_1)=o(\eta_1^{k+1}),$$

since the series is summable (λ, κ), and *a fortiori* summable $(\lambda, k+1)$, to sum 0. Also

$$\zeta-\zeta_1=O(\eta_1\zeta_1'),$$

by (6.12), and

$$\zeta_1^{(r)}=O\left(\frac{\zeta_1'}{\eta_1^{r-1}}\right),$$

by 4.2. Hence

$$(7.25) \qquad J_{1,1}=\Sigma o\left\{\eta_1^{k+1}(\eta_1\zeta_1')^s\,\eta_1^{-s_2-2s_3-\ldots}(\zeta_1')^{s_1+s_2+\ldots}\right\}$$
$$=\Sigma o\left\{\eta_1^{s+s_1+2s_2+\ldots-s_2-2s_3-\ldots}(\zeta_1')^{s+s_1+s_2+\ldots}\right\}$$
$$=\Sigma o(\eta_1\zeta_1')^{s+s_1+s_2+\ldots}=o(\eta_1\zeta_1')^\kappa$$
$$=o(\zeta_1^\kappa)=o(\zeta^\kappa).$$

7.3. The integral $J_{1,2}$ is a sum of constant multiples of integrals of the type

$$\int_{\lambda_1}^{\eta_1} C_\lambda^{k+1}(\sigma)(\zeta-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\ldots d\sigma, \tag{7.31}$$

where now

$$s+s_1+s_2+\ldots=\kappa, \tag{7.321}$$

$$s_1+2s_2+3s_3+\ldots=k+2. \tag{7.322}$$

The integral (7.31) is of the form

$$\int_{\lambda_1}^{\eta_1} o(\sigma^{k+1})(\zeta-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\ldots(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma. \tag{7.33}$$

Now
$$\zeta-\mu \geqslant \zeta-\zeta_1 = A\eta_1\zeta_1' \geqslant A\sigma\mu',$$

for $\lambda_1 \leqslant \sigma \leqslant \eta_1$; and

$$s-\kappa+1 = 1-s_1-s_2-\ldots \leqslant 0.$$

Hence

$$(\zeta-\mu)^{s-\kappa+1} = O(\sigma\mu')^{s-\kappa+1}; \tag{7.34}$$

and

$$\begin{aligned}\sigma^{k+1}(\zeta-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\ldots \\ &= O\left\{\sigma^{k+1}(\sigma\mu')^{s-\kappa+1}\sigma^{-s_2-2s_3-\ldots}(\mu')^{s_1+s_2+\ldots-1}\right\} \\ &= O(1),\end{aligned} \tag{7.35}$$

since
$$k+1+s-\kappa+1-s_2-2s_3-\ldots=0$$

and
$$s-\kappa+1+s_1+s_2+\ldots-1=0,$$

in virtue of (7.321) and (7.322). Thus the integral (7.33) is of the form

$$o\int_{\lambda_1}^{\eta_1}(\zeta-\mu)^{\kappa-1}\mu'\,d\sigma = o(\zeta_1^\kappa) = o(\zeta^\kappa);$$

so that

$$J_{1,2} = o(\zeta^\kappa). \tag{7.36}$$

From (7.25) and (7.36) it follows that

$$J_1 = o(\zeta^\kappa). \tag{7.37}$$

7.4. It remains to consider

(7.41) $$J_2 = \int_{\eta_1}^{\eta} C^k(\sigma) \left(\frac{d}{d\sigma}\right)^{k+1} (\zeta-\mu)^\kappa \, d\sigma,$$

which is a sum of constant multiples of integrals of the type

(7.42) $$\int_{\eta_1}^{\eta} C^k(\sigma)(\zeta-\mu)^s (\mu')^{s_1} (\mu'')^{s_2} \dots \, d\sigma,$$

where

(7.431) $$s+s_1+s_2+\dots = \kappa,$$

(7.432) $$s_1+2s_2+3s_3+\dots = k+1.$$

The integral (7.42) may be written in one or other of the forms

(7.441) $$(\zeta')^{s_1} (\zeta'')^{s_2} \dots \int_{\eta_2}^{\eta} C^k(\sigma)(\zeta-\mu)^s \, d\sigma,$$

(7.442) $$(\zeta_1')^{s_1} (\zeta_1'')^{s_2} \dots \int_{\eta_1}^{\eta_2} C^k(\sigma)(\zeta-\mu)^s \, d\sigma.$$

Arguing as in 6.3, we replace each of these integrals by one of the form

(7.45) $$j = (\zeta_3')^{s_1} (\zeta_3'')^{s_2} \dots \left(\frac{\zeta-\zeta_4}{\eta-\eta_4}\right)^s \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma,$$

where η_3, η_4, ... are numbers between η_1 and η, and ζ_3, ζ_4, ..., ζ_3', ζ_4', ... are the corresponding values of μ and μ'. We write (7.45) in the form

(7.46) $$j = j_1 j_2,$$

where j_1 and j_2 denote the external factor and the integral in (7.45) respectively.

7.5. It follows from arguments similar to those employed in 6.3 that

(7.51) $$j_1 = O\{(\zeta')^{s+s_1}(\zeta'')^{s_2}\dots\} = O\{\eta^{-s_2-2s_3-\dots}(\zeta')^{s+s_1+s_2+\dots}\}.$$

In order to obtain an upper limit for j_2, we observe that

(7.52) $$[s] = k-s_1-s_2-\dots = k',$$

say, and integrate $k'+1$ times by parts. We thus obtain

(7.53) $$j_2 = \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma = F(\eta_6)-F(\eta_5)+j_4 = j_3+j_4$$

say, where

(7.54) $$F(\sigma) = \frac{1}{k+1} C^{k+1}(\sigma)(\eta-\sigma)^s + \frac{s}{(k+1)(k+2)} C^{k+2}(\sigma)(\eta-\sigma)^{s-1} + \dots$$
$$+ \frac{s(s-1)\dots(s-k'+1)}{(k+1)(k+2)\dots(k+k'+1)} C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'},$$

(7.55) $$j_4 = \frac{s(s-1)\dots(s-k')}{(k+1)(k+2)\dots(k+k'+1)} \int_{\eta_5}^{\eta_6} C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'-1}\,d\sigma.$$

Any term of $F(\sigma)$ is of the form

$$o(\eta^{k+r+s-r+1}) = o(\eta^{k+s+1});$$

and so

(7.56) $$j_3 = o(\eta^{k+s+1}).$$

On the other hand $s-k'$ lies between 0 and 1, and so*

(7.57) $$\left| \int_{\eta_5}^{\eta_6} C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'-1}\,d\sigma \right|$$
$$\leqslant 2\frac{\Gamma(k+k'+2)\,\Gamma(s-k')}{\Gamma(s+k+2)} \operatorname*{Max}_{\lambda_1 \leqslant \tau \leqslant \eta} |C^{s+k+1}(\tau)|$$
$$= o(\eta^{k+s+1}).$$

Thus both j_3 and j_4 are of this form, and so, therefore, is j_2; and therefore

(7.58) $$j = j_1 j_2 = o\left\{\eta^{k+s+1-s_2-2s_3-\dots}(\zeta')^{s+s_1+s_2+\dots}\right\}$$
$$= o(\eta\zeta')^{s+s_1+s_2+\dots}$$
$$= o(\eta\zeta')^{\kappa} = o(\zeta^{\kappa}).$$

Hence

(7.59) $$J_2 = o(\zeta^{\kappa}).$$

7.6. From (7.13), (7.37), and (7.59) it follows that

(7.61) $$J = o(\zeta^{\kappa}).$$

The proof of the theorem is thus completed, provided that the order of μ is greater than that of $\log\lambda$. In order to extend the result to cover all possible cases we have only to repeat the argument of 6.4.

* H. and R., p. 29 (Lemma 8).

Conclusion.

8. It remains only to show that the theorem is the best possible theorem of its kind. In order to prove this it is necessary to show that *if μ is any L-function of λ which tends to infinity more rapidly than any power of λ, then we can determine a number κ and a series Σc_n which is summable* (λ, κ) *and not summable* (μ, κ).

We may take $\lambda = n$, and we may suppose that μ is of lower order than e^n, since methods of type as high as e^n will sum convergent series only*. Consider the series

$$(8.1) \qquad \Sigma(-1)^n \left(\frac{\mu_n}{\mu'_n}\right)^\kappa,$$

which may be written in the form

$$\Sigma(-1)^n n^\kappa \left(\frac{\mu_n}{n\mu'_n}\right)^\kappa.$$

Since μ_n is of order higher than any power of n, we have

$$\frac{\mu_n}{n\mu'_n} = o(1).\dagger$$

Hence

$$\left(\frac{\mu_n}{n\mu_n}\right)^\kappa$$

is an L-function which tends to zero, and so also are all its derivatives. All of these derivatives, moreover, are ultimately of constant sign; and the same is true of all the successive differences of the function. Also the series

$$\Sigma(-1)^n n^\kappa$$

is *finite* (C, κ). It follows from known theorems‡ that the series (8.1) is summable (C, κ), *i.e.* (n, κ).

But the series (8.1) is *not* summable (μ, κ). For if it were, we should have

$$\left(\frac{\mu_n}{\mu'_n}\right)^\kappa = o\left(\frac{\mu_{n+1}}{\mu_{n+1}-\mu_n}\right)^\kappa \S,$$

* H. and R., p. 46 (Theorem 36).

† O.I., p. 38.

‡ The theorem required is a special case of Theorem 1a (p. 61) of Bohr's dissertation "Bidrag til de Dirichlet'ske Rækkers Theori" (Copenhagen, 1910).

§ H. and R., p. 36 (Theorem 21).

and this is untrue, since $$\mu_{n+1} \sim \mu_n$$

and $$\mu_{n+1} - \mu_n \sim \mu'_n,$$

when μ_n is an L-function of n of order less than e^n*.

For example, the series

$$1-1+1-1+\ldots$$

is summable $(1, \kappa)$ for any positive value of κ, but is not summable (e^n, κ) for any value of κ. The series

$$\sqrt{1}-\sqrt{2}+\sqrt{3}-\ldots$$

is summable $(n, 1)$ but not summable $(e^{\sqrt{n}}, 1)$; and so on.

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* O.I., pp. 41 *et seq.*

CORRECTIONS

p. 78, *line* 9. For $C_\lambda(\sigma)$ read $C_\lambda^\kappa(\sigma)$.
p. 80, *line* 4. For $o(\sigma)$ read $o(1)$.
p. 82, *line* 1. Insert $| \; |$.
p. 87, *line* 15. For μ_n in the denominator read μ_n'.

COMMENTS

Special cases and partial conjectures of the 'second consistency theorem' may be found in 1907, 5, §§ 5–6; 1910, 1, § 8; and 1910, 3, § 8.

The theorem of Cesàro, referred to in § 3, is the theorem rediscovered by Hardy in 1907, 5, § 4, concerning the relative strength of weighted means. In 1910, 3, § 8, where Riesz means of order 1 are defined as weighted means, with weights $\mu_{m+1} = \lambda_{m+1} - \lambda_m$, Hardy uses case (*a*) of the theorem; case (*b*) may also be used, or the general case of Garabedian and Randels. For references, see the Comments on 1907, 5.

The conditions in the 'second consistency theorem' were extended by Hirst† who gave sufficient conditions for a function ϕ, with a certain number of derivatives, to be such that (ϕ, k) implies $(\phi(\lambda), k)$ for all choices of λ_n. Later, Kuttner‡ obtained necessary and sufficient conditions, which, in the case k an integer, are essentially the same as Hirst's conditions.

The formulae 5.32–5.332 in § 5 and 7.22–7.232 in § 7 are cases of the formula of Faa di Bruno§ for the nth derivative of a function of a function.

The theorem quoted in § 8 is Bohr's version of the Bohr–Hardy theorem|| that: *if* $\sum a_n$ *is bounded* (C, k), *and* (i) $d_n = o(1)$, (ii) $\sum n^s |\Delta^{s+1} d_n| < \infty$ $(s = 0, 1, \ldots, k)$, *then* $\sum a_n d_n$ *is summable* (C, k). Here $d_n = d(n) = (\mu_n / n\mu_n')^k$, and Hardy's conditions imply that $(-1)^r d^{(r)}(x)$ is ultimately positive, and hence that $\Delta^s d_n$ is ultimately positive, for $s = 0, 1, \ldots, k$. Since $d_n \to 0$, this implies that Bohr's hypotheses are satisfied; see 1907, 6, § 4, Lemma A.

† *Proc. London Math. Soc.* (2), 33 (1932), 353–66.
‡ *J. London Math. Soc.* 26 (1951), 104–11 and ibid. 27 (1952), 207–17.
§ See De la Vallée Poussin, *Cours d'analyse infinitésimale*, Vol. I (edn. 7, 1930), pp. 89–90.
|| For references see the Comments on 1908, 1.

ANALYSE MATHÉMATIQUE. — *Sur la sommation des séries de Dirichlet.* Note de M. G.-H. Hardy.

1. On peut dire qu'une série Σa_n est sommable par la méthode d'Abel, ou *sommable* (A), si la série $\Sigma a_n e^{-ny}$ est convergente pour $y > 0$ et tend vers une limite finie quand y tend vers zéro. L'application de cette méthode de sommation aux séries de Dirichlet $\Sigma a_n n^{-s}$ donne des résultats assez intéressants, qu'on ne semble pas avoir remarqués jusqu'ici.

Je suppose qu'il existe un nombre M tel que $a_n = O(n^M)$; il y a donc un demi-plan de convergence absolue, et la série représente une fonction analytique

$$f(s) = f(\sigma + ti).$$

On a étudié surtout les fonctions qui sont d'*ordre fini* sur une parallèle à l'axe des imaginaires. Supposons que $f(s)$ est régulière pour $\sigma \geqq \sigma_0$ et désignons par $K = K(\sigma_0)$ la borne inférieure des nombres ξ tels que

$$f(\sigma + ti) = O(|t|^\xi),$$

uniformément pour $\sigma \geqq \sigma_0$, et supposons, de plus, que $K < \infty$. Cela étant, on dit que $f(s)$ est d'ordre fini K pour $\sigma = \sigma_0$. Les séries de telles fonctions sont sommables (C), c'est-à-dire par les moyennes de Cesàro; on sait, en effet, d'après M. Harald Bohr, que *la condition nécessaire et suffisante pour que $\Sigma a_n n^{-s}$ soit sommable* (C), *pour $\sigma > \sigma_0$, est que $f(s)$ soit régulière et d'ordre fini pour $\sigma > \sigma_0$.*

2. Toute série sommable (C) est sommable (A) : la méthode d'Abel est donc applicable à toutes les séries correspondant aux fonctions d'ordre fini, mais son application en ce cas n'est jamais nécessaire. Au contraire, pour les fonctions d'ordre infini, les méthodes de Cesàro, et les méthodes plus générales de M. Marcel Riesz, sont en défaut. On a donc besoin de méthodes plus puissantes, telles que la méthode d'Abel.

Soit $f(s)$ régulière pour $\sigma \geqq \sigma_0$, et soit $H = H(\sigma_0)$ la borne inférieure des nombres ξ tels que

$$f(\sigma + ti) = o(\varepsilon^{\xi|t|}),$$

1916, 7 Comptes rendus, 162, 463–5.

uniformément pour $\sigma \geqq \sigma_0$. J'appelle ce nombre H l'*indice* de $f(s)$ pour $\sigma = \sigma_0$. On démontre sans peine que la fonction $H(\sigma)$ est nulle à partir d'une valeur σ_1 de σ et qu'elle est, de plus, positive, décroissante, convexe et continue (tant qu'elle reste finie) pour $\sigma < \sigma_1$.

En faisant usage de la formule

$$\Sigma a_n n^{-s} e^{-ny} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (\Sigma a_n e^{-n(x+y)})\, dx,$$

valable sous des conditions qu'il n'y a pas besoin de récapituler, on trouve aisément que, *si la série* Σa_n *est sommable* (A), *la série* $\Sigma a_n n^{-s}$ *est sommable* (A) *pour* $\sigma > \sigma_0$.

Il s'ensuit que *le domaine de sommabilité* (A) *est un demi-plan*. On obtient aussi le théorème :

I. *Si la série est sommable* (A) *pour* $\sigma > \sigma_0$, $f(s)$ *est régulière, et* $H(\sigma) < \frac{\pi}{2}$, *pour* $\sigma > \sigma_0$.

D'autre part, en partant de la formule

$$\Sigma a_n n^{-s} e^{-ny} = \frac{1}{2\pi i} \int_{K-i\infty}^{K+i\infty} y^{-u}\, \Gamma(u) f(s+u)\, du,$$

où K désigne un nombre positif assez grand, et en suivant une marche qui ne diffère pas, au fond, de celle que M. Littlewood et moi avons suivie dans quelques recherches récentes sur les nombres premiers ([1]), on aboutit au théorème réciproque :

II. *Si* $f(s)$ *est régulière, et* $H(\sigma) < \frac{\pi}{2}$, *pour* $\sigma > \sigma_0$, *la série est sommable* (A) *pour* $\sigma > \sigma_0$.

De ces deux théorèmes il résulte que *le domaine de sommabilité* (A) *est le demi-plan dans lequel l'indice de la fonction est plus petit que* $\frac{\pi}{2}$.

3. On peut généraliser la méthode d'Abel en supposant que $\Sigma a_n e^{-n^\lambda y}$, où λ est un nombre positif quelconque, converge pour $y > 0$ et tend vers une limite finie quand y tend vers zéro. Je conviendrai alors de dire que la série Σa_n est *sommable* (A, λ). On augmente la puissance de la méthode en diminuant λ.

([1]) Voir une courte Note dans *The Quarterly Journal of Mathematics*, t. 45, et un Mémoire étendu qui va paraître dans les *Acta mathematica*

Les résultats de l'application de la méthode généralisée sont tout à fait analogues aux théorèmes que je viens d'énoncer : il faut seulement remplacer $\frac{\pi}{2}$ par $\frac{\pi}{2\lambda}$.

Tous ces théorèmes admettent aussi des généralisations pour des séries $\Sigma a_n e^{-\lambda_n s}$ d'un type quelconque.

4. Considérons, par exemple, la série

$$f(s) = \Sigma\, e^{\mu i (\log n)^a} n^{-s} \qquad (\mu > 0,\ a > 1).$$

La série est absolument convergente pour $\sigma > 1$ et convergente pour $\sigma \geqq 1$; les droites de convergence et de sommabilité (C) coïncident dans la droite $\sigma = 1$. La fonction $f(s)$ est entière et d'ordre infini si $\sigma < 1$ (¹).

On peut démontrer que $H(\sigma) = 0$, pour toute valeur de σ, si $a > 2$; la série est alors sommable (A) dans tout le plan. Si $1 < a < 2$, on a $H(\sigma) = 0$ pour $\sigma > 1$, et $H(\sigma) = \infty$ pour $\sigma < 1$; la série n'est plus sommable (A) que dans le demi-plan de convergence. Enfin, si $a = 2$, on a $H(\sigma) = 0$ pour $\sigma \geqq 1$ et $H(\sigma) = \frac{1-\sigma}{2\mu}$ pour $\sigma < 1$. La série en ce cas est donc sommable (A) pour $\sigma > 1 - \mu\pi$, ce que l'on peut vérifier par un calcul direct. Elle est sommable (A, λ) pour $\sigma > 1 - \frac{\mu\pi}{\lambda}$.

(¹) J'ai donné les démonstrations de ces propositions dans *The Tohoku Mathematical Journal*, t. 8.

(*Comptes rendus*, t. 162, p. 463, séance du 27 mars 1916.)

CORRECTION

p. 464, *line* 9. For $\sigma > \sigma_0$, read $\sigma > 0$.

COMMENTS

This is a summary of results published in 1916, 1. The reference in § 4 is to 1915, 11.

THEOREMS CONCERNING THE SUMMABILITY OF SERIES BY BOREL'S EXPONENTIAL METHOD.

By **G. H. Hardy** and **J. E. Littlewood** (Cambridge).

§ 1.

Introduction.

1.1. In a paper [1]) published in the *Proceedings of the London Mathematical Society* in 1912 we proved that *a series* $\sum a_n$ *in which*

(1.1) $$a_n = o\left(\frac{1}{\sqrt{n}}\right)$$

cannot be summable by BOREL'*s method unless it is convergent;* and we raised the question whether the theorem remains true if the o in the condition (1.1) is replaced by an O. We stated that we had no doubt as to the truth of the theorem thus suggested, but that we were unable to find a proof.

We are now able to supply the proof that was then lacking, and to do this is the principal object of the present paper. The proof is given in section 2. In sections 3 and 4 we consider some theorems of a different character but also relating to BOREL'S method.

§ 2.

Proof of the general Borel-Tauber [2]) Theorem.

2.1. The series $\sum a_n$ is said to be *summable* (B), *to sum s*, if

$$e^{-x} \sum s_n \frac{x^n}{n!},$$

[1]) G. H. HARDY and J. E. LITTLEWOOD, *The Relations between* BOREL'*s and* CESÀRO'*s Methods of Summation* [Proceedings of the London Mathematical Society, series II, vol. XI (1912-1913), pp. 1-16].

[2]) For an explanation of our reasons for giving this name to the theorem, see G. H. HARDY and J. E. LITTLEWOOD, *Contributions to the arithmetic Theory of Series* [Proceedings of the London Mathematical Society, series II, vol. XI (1912-1913), pp. 411-478 (p. 413)].

where

$$s_n = a_0 + a_1 + \cdots + a_n,$$

tends to the limit s when $x \to \infty$.

THEOREM 2.1. — *If $\sum a_n$ is summable (B), and*

(2.11) $$a_n = O\left(\frac{1}{\sqrt{n}}\right),$$

then $\sum a_n$ is convergent.

The proof of this theorem, as of any TAUBERian theorem of the O type, is decidedly difficult. We shall base it on a number of preliminary lemmas.

LEMMA 2.11. — *If $\sum a_n$ is summable (B), and*

(2.111) $$a_n = o(1),$$
then
(2.112) $$s_n = a_0 + a_1 + \cdots + a_n = o(\sqrt{n}).$$

This is Theorem 3 if our previous paper [3]), with $k = 0$. The conclusion is true *a fortiori* if a_n satisfies (2.11).

LEMMA 2.12. — *If $\sum a_n$ is summable (B) to sum s, and s_n satisfies* (2.112), *then*

$$\frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} e^{-n^2/2\mu} s_{n+\mu} \to s,$$

when $\mu \to \infty$ by integral values.

It is to be understood that $s_{n+\mu} = 0$ when the suffix is negative.

We have

(2.121) $$S = e^{-\mu} \sum_{0}^{\infty} s_n \frac{\mu^n}{n!} \to s.$$

We write

(2.122) $$S = e^{-\mu}\left(\sum_{0}^{(1-H)\mu} + \sum_{(1-H)\mu}^{(1+H)\mu} + \sum_{(1+H)\mu}^{\infty}\right) = S_1 + S_2 + S_3,$$

say, H being a constant, positive, irrational, and less than unity. Then we can find a positive constant δ such that

(2.123) $$S_1 = O(e^{-\delta\mu}), \qquad S_3 = O(e^{-\delta\mu}) \text{ [4]}).$$

Thus

(2.124) $$S_2 = e^{-\mu} \sum_{-H\mu}^{H\mu} s_{h+\mu} \frac{\mu^{h+\mu}}{(h+\mu)!} \to s.$$

Now it is easy to deduce from STIRLING's Theorem that

(2.125) $$e^{-\mu} \frac{\mu^{h+\mu}}{(h+\mu)!} = \frac{1}{\sqrt{2\pi\mu}} e^{-h^2/2\mu} \left\{1 + O\left(\frac{|h|}{\mu}\right) + O\left(\frac{|h|^3}{\mu^2}\right)\right\};$$

uniformly for $-H\mu < h < H\mu$.

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[3]) loc. cit. [1]), p. 8.
[4]) loc. cit. [1]), p. 6.

Substituting in (2.124), we see that

(2.126) $$S_2 = \frac{1}{\sqrt{2\pi\mu}} \sum_{-H\mu}^{H\mu} e^{-h^2/2\mu} s_{h+\mu} + S_2' + S_2'',$$

where

(2.1261) $$\begin{cases} S_2' = O\left(\mu^{-\frac{3}{2}} \sum_{-H\mu}^{H\mu} |h|\, e^{-h^2/2\mu} |s_{h+\mu}|\right) \\ \quad = o\left(\frac{1}{\mu} \sum_{-\infty}^{\infty} |h|\, e^{-h^2/2\mu}\right) \\ \quad = o\left(\frac{1}{\mu} \int_0^{\infty} x\, e^{-x^2/2\mu}\, dx\right) \\ \quad = o\left(\int_0^{\infty} y\, e^{-y^2}\, dy\right) \\ \quad = o(1), \end{cases}$$

and

(2.1262) $$\begin{cases} S_2'' = O\left(\mu^{-\frac{5}{2}} \sum_{-H\mu}^{H\mu} |h|^3 e^{-h^2/2\mu} |s_{h+\mu}|\right) \\ \quad = o\left(\frac{1}{\mu^2} \sum_{-\infty}^{\infty} |h|^3 e^{-h^2/2\mu}\right) \\ \quad = o\left(\frac{1}{\mu^2} \int_0^{\infty} x^3 e^{-x^2/2\mu}\, dx\right) \\ \quad = o\left(\int_0^{\infty} y^3 e^{-y^2}\, dy\right) \\ \quad = o(1). \end{cases}$$

From (2.124), (2.126), (2.1261), and (2.1262), it follows that

(2.127) $$\bar{S}_2 = \frac{1}{\sqrt{2\pi\mu}} \sum_{-H\mu}^{H\mu} e^{-h^2/2\mu} s_{h+\mu} \rightarrow s.$$

Now

(2.128) $$\begin{cases} \bar{S} = \frac{1}{\sqrt{2\pi\mu}} \sum_{-\infty}^{\infty} e^{-h^2/2\mu} s_{h+\mu} \\ \quad = \frac{1}{\sqrt{2\pi\mu}} \left(\sum_{-\infty}^{-H\mu} + \sum_{-H\mu}^{H\mu} + \sum_{H\mu}^{\infty} \right) \\ \quad = \bar{S}_1 + \bar{S}_2 + \bar{S}_3, \end{cases}$$

say. Also

(2.1281) $$\begin{cases} \bar{S}_1 = O\left(\frac{1}{\sqrt{\mu}} \sum_{-\infty}^{-H\mu} \sqrt{\mu}\, e^{-h^2/2\mu}\right) \\ \quad = O\left(\int_{H\mu}^{\infty} e^{-x^2/2\mu}\, dx\right) \\ \quad = O\left(\sqrt{\mu} \int_{H\sqrt{\mu/2}}^{\infty} e^{-y^2}\, dy\right) \\ \quad = O(e^{-\delta\mu}), \end{cases}$$

where δ is a positive constant; and

$$(2.1282)\quad \left\{\begin{aligned} \bar{S}_3 &= O\left(\frac{1}{\sqrt{\mu}}\sum_{H\mu}^{\infty}\sqrt{h}\,e^{-h^2/2\mu}\right)\\ &= O\left(\frac{1}{\sqrt{\mu}}\int_{H\mu}^{\infty}\sqrt{x}\,e^{-x^2/2\mu}\,dx\right)\\ &= O\left(\mu^{\frac{1}{4}}\int_{H\sqrt{\mu/2}}^{\infty}\sqrt{y}\,e^{-y^2}\,dy\right)\\ &= O(e^{-\delta\mu}).\end{aligned}\right.$$

From (2.127), (2.128), (2.1281) and (2.1282) it follows that

$$(2.129)\qquad \bar{S} \twoheadrightarrow s.$$

This completes the proof of Lemma 2.12.

LEMMA 2.13. — *Suppose that $f(x)$ is the continuous function of x defined by the* equations

$$(2.1311)\qquad f(x) = s_n + (x-n)(s_{n+1} - s_n) \qquad (n \leqq x \leqq n+1),$$

$$(2.1312)\qquad f(x) = 0 \qquad (x < 0).$$

Suppose further that $\sum a_n$ is summable (B) to sum s, and that

$$(2.111)\qquad a_n = o(1).$$

Then

$$(2.132)\qquad \frac{1}{\sqrt{2\pi x}}\int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x)\,dt \twoheadrightarrow s$$

as $x \twoheadrightarrow \infty$.

We have already proved that

$$(2.133)\qquad \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty} e^{-h^2/2\mu} s_{h+\mu} \twoheadrightarrow s$$

when $\mu \twoheadrightarrow \infty$ by integral values. We shall prove first that (2.133) may be replaced by

$$(2.134)\qquad \frac{1}{\sqrt{2\pi\mu}}\int_{-\infty}^{\infty} e^{-t^2/2\mu} s(t+\mu)\,dt \twoheadrightarrow s,$$

where $s(x)$ is the discontinuous function which is equal to s_n when $n \leqq x < n+1$. To prove this we observe that the difference between the left hand sides of (2.133) and (2.134) may be written in the form

$$(2.1341)\qquad \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty}\int_{h}^{h+1}(e^{-h^2/2\mu} - e^{-t^2/2\mu})\,s(t+\mu)\,dt;$$

and that $s(t+\mu)$ is of the form $o(\sqrt{\mu})$ or $o(\sqrt{t})$, according as μ or t is numerically the greater, and so in any case of the form

$$o(\sqrt{\mu}) + o(\sqrt{t}).$$

Also

$$(2.1342)\qquad \left\{\begin{aligned} &\frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty}\int_h^{h+1}(e^{-h^2/2\mu}-e^{-t^2/2\mu})\,o(\sqrt{\mu})\,dt\\ &\quad = o\left(\frac{1}{\mu}\int_{-\infty}^{\infty}|t|\,e^{-t^2/2\mu}\,dt\right) = o(1),\end{aligned}\right.$$

and

$$(2.1343)\qquad \left\{\begin{aligned} &\frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^{\infty}\int_h^{h+1}(e^{-h^2/2\mu}-e^{-t^2/2\mu})\,o(\sqrt{t})\,dt\\ &\quad = o\left(\mu^{-\frac{3}{2}}\int_{-\infty}^{\infty}|t|^{\frac{3}{2}}\,e^{-t^2/2\mu}\,dt\right)\\ &\quad = o(\mu^{-\frac{1}{4}}) = o(1).\end{aligned}\right.$$

Hence (2.1341) tends to zero, and (2.133) may be replaced by (2.134). Secondly, (2.134) may be replaced by

$$(2.135)\qquad \frac{1}{\sqrt{2\pi\mu}}\int_{-\infty}^{\infty}e^{-t^2/2\mu}f(t+\mu)\,dt \twoheadrightarrow s.$$

For the difference of the left hand sides of (2.134) and (2.135) is, since $a_n = o(1)$, of the form

$$(2.1351)\qquad o\left(\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}e^{-t^2/2\mu}\,dt\right) = o(1).$$

Finally we may replace the integer μ in (2.135) by a continuous variable x. For suppose that

$$(2.136)\qquad x = \mu + \theta \qquad (0 < \theta < 1).$$

Then

$$(2.1361)\qquad \left\{\begin{aligned} &\int_{-\infty}^{\infty}e^{-t^2/2\mu}f(t+\mu)\,dt - \int_{-\infty}^{\infty}e^{-t^2/2x}f(t+x)\,dt\\ &= \int_{-\infty}^{\infty}e^{-t^2/2\mu}\{f(t+\mu)-f(t+x)\}\,dt\\ &+ \int_{-\infty}^{\infty}(e^{-t^2/2\mu}-e^{-t^2/2x})f(t+x)\,dt\\ &= o\left(\int_{-\infty}^{\infty}e^{-t^2/2\mu}\,dt\right) + o\left(\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}|t|\,e^{-t^2/2\mu}\,dt\right) + o\left(\frac{1}{\mu}\int_{-\infty}^{\infty}|t|^{\frac{3}{2}}\,e^{-t^2/2\mu}\,dt\right)\\ &= o(\sqrt{\mu}) + o(\sqrt{\mu}) + o(\mu^{\frac{1}{4}})\\ &= o(\sqrt{\mu}).\end{aligned}\right.$$

Hence (2.132) follows from (2.135).

2.21. So far we have never used the full condition (2.11): the hypothesis that $a_n = o(1)$ has been sufficient for our needs. It may be inferred that we have still to face the main difficulty of the problem. This difficulty lies in the proof of the lemma which follows.

LEMMA 2.21. — *Suppose that $f(x)$ is a real continuous function of x, with a derivative $f'(x)$ continuous except at isolated points. Suppose further that*

(2.211) $$f'(x) = O\left(\frac{1}{\sqrt{x}}\right).$$

Finally suppose that the relation

(2.212) $$\sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt \to s$$

holds for some positive value of a. Then (2.212) holds for all greater values of a.

We shall prove first that

(2.213) $$f(x) = O(1) \text{ [5]}.$$

We have

(2.214) $$\begin{cases} s + o(1) - f(x) = \sqrt{\dfrac{a}{\pi x}}\displaystyle\int_{-\infty}^{\infty} e^{-at^2/x}\{f(t+x) - f(x)\}\,dt \\ \qquad = \sqrt{\dfrac{a}{\pi x}}\left(\displaystyle\int_{-\infty}^{-Hx} + \int_{-Hx}^{Hx} + \int_{Hx}^{\infty}\right) \\ \qquad = J_1 + J_2 + J_3, \end{cases}$$

say. It is easy to prove, as in the proof of Lemma 2.12 [6], that

(2.215) $$J_1 = O(e^{-\delta x}), \qquad J_3 = O(e^{-\delta x}).$$

But, if $-Hx < t < Hx$, we have

(2.216) $$f(t+x) - f(x) = \int_x^{t+x} f'(u)\,du = O\left(\frac{|t|}{\sqrt{x}}\right),$$

and so

(2.217) $$J_2 = O\left(\frac{1}{x}\int_{-\infty}^{\infty} |t| e^{-at^2/x}\,dt\right) = O(1).$$

Hence

(2.218) $$s + o(1) - f(x) = O(1),$$

(2.219) $$f(x) = O(1).$$

2.22. Let

(2.221) $$F(x) = f(x) - s,$$

so that

(2.222) $$\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-at^2/x} F(t+x)\,dt = o(1).$$

Further, let

(2.223) $$I_n(x) = \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-at^2/x}\left(\frac{t^2}{x}\right)^n F(t+x)\,dt,$$

---

[5]) It would be sufficient for this purpose to suppose the left hand side of (2.212) *bounded.*

[6]) See 2.1281 and 2.1282.

where n is zero or half a positive integer. Since $F(x) = O(1)$, we have

$$I_n(x) = O\left(x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n}\, dt\right) = O(1). \tag{2.224}$$

We shall now prove that (2.224) may be replaced by

$$I_n(x) = o(1). \tag{2.225}$$

It will be well to point out explicitly that the argument by which this transition is justified contains the kernel of the whole proof of Lemma 2.2 and of our main theorem.

2.23. Suppose first that $n > 0$. We have

$$I_n(x) = x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F(t+x)\, dt \tag{2.231}$$

$$\left\{\begin{aligned} I_n'(x) = \frac{dI_n}{dx} = &-\left(n+\frac{1}{2}\right) x^{-n-\frac{3}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F(t+x)\, dt \\ &+ a x^{-n-\frac{5}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n+2} F(t+x)\, dt \\ &+ x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F'(t+x)\, dt \end{aligned}\right. \tag{2.232}$$ [7].

Also

$$\left\{\begin{aligned} &x^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n} F'(t+x)\, dt \\ &= -2nx^{-n-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n-1} F(t+x)\, dt \\ &+ 2ax^{-n-\frac{3}{2}} \int_{-\infty}^{\infty} e^{-at^2/x} t^{2n+1} F(t+x)\, dt, \end{aligned}\right. \tag{2.233}$$

by integration by parts. Substituting in (2.232), we obtain

$$\left\{\begin{aligned} I_n' &= -\frac{2n}{\sqrt{x}} I_{n-\frac{1}{2}} - \frac{n+\frac{1}{2}}{x} I_n + \frac{2a}{\sqrt{x}} I_{n+\frac{1}{2}} + \frac{a}{x} I_{n+1} \\ &= O\left(\frac{1}{\sqrt{x}}\right) + O\left(\frac{1}{x}\right) + O\left(\frac{1}{\sqrt{x}}\right) + O\left(\frac{1}{x}\right) \\ &= O\left(\frac{1}{\sqrt{x}}\right); \end{aligned}\right. \tag{2.234}$$

and it is plain that a repetition of this argument will lead to

$$I_n'' = O\left(\frac{1}{x}\right). \tag{2.235}$$

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[7]) There is no difficulty in the differentiation under the integral sign: all the integrals concerned are absolutely and uniformly convergent.

We have therefore

(2.236) $$I_n = O(1), \quad I'_n = O\left(\frac{1}{\sqrt{x}}\right), \quad I''_n = O\left(\frac{1}{x}\right)$$

for all positive values of n.

Secondly, suppose that $n = 0$. Then it is easy to see that (2.234) must be replaced by

(2.237) $$I'_0 = -\frac{1}{2x} I_0 + \frac{2a}{\sqrt{x}} I_{\frac{1}{2}} + \frac{a}{x} I_1,$$

and so that the relations (2.236) are true even when $n = 0$.

2.24. Suppose now first that $n = 0$.

Then

(2.2411) $$I_0 = o(1),$$

by (2.222), and

(2.2412) $$x^2 I''_0 = O(x)$$

by (2.236). Hence [8]

(2.2413) $$x I'_0 = o(\sqrt{x}), \quad I'_0 = o\left(\frac{1}{\sqrt{x}}\right).$$

But we have, from (2.237),

(2.242) $$\begin{cases} \dfrac{2a}{\sqrt{x}} I_{\frac{1}{2}} = I'_0 + \dfrac{1}{2x} I_0 - \dfrac{a}{x} I_1 \\ = o\left(\dfrac{1}{\sqrt{x}}\right) + O\left(\dfrac{1}{x}\right) + O\left(\dfrac{1}{x}\right) = o\left(\dfrac{1}{\sqrt{x}}\right), \end{cases}$$

and so

(2.243) $$I_{\frac{1}{2}} = o(1).$$

Thus (2.225) is established when $n = \frac{1}{2}$.

But it is fairly obvious that this argument may be repeated. We have now

(2.2441) $$I_{\frac{1}{2}} = o(1), \quad x^2 I''_{\frac{1}{2}} = O(x),$$

and therefore

(2.2442) $$x I'_{\frac{1}{2}} = o(\sqrt{x}), \quad I'_{\frac{1}{2}} = o\left(\frac{1}{\sqrt{x}}\right).$$

Hence, using (2.234), we have

(2.245) $$\begin{cases} \dfrac{2a}{\sqrt{x}} I_1 = I'_{\frac{1}{2}} + \dfrac{1}{\sqrt{x}} I_0 + \dfrac{1}{x} I_{\frac{1}{2}} - \dfrac{a}{x} I_{\frac{3}{2}} \\ = o\left(\dfrac{1}{\sqrt{x}}\right) + o\left(\dfrac{1}{\sqrt{x}}\right) + o\left(\dfrac{1}{x}\right) + O\left(\dfrac{1}{x}\right) \\ = o\left(\dfrac{1}{\sqrt{x}}\right), \end{cases}$$

(2.246) $$I_1 = o(1).$$

---

[8]) loc. cit. [2]). The Theorem used here is Theorem 6, with $f = I_0$, $\varphi = 1$, $\psi = x$, $r = 2$, $s = 1$.

And we need only repeat this argument indefinitely in order to complete the proof of (2.225).

2.25. We are now in a position to complete the proof of Lemma 2.21.

Suppose that δ is a positive number not greater than a. Then

$$e^{-\delta t^2/x} = \sum_{\nu=1}^{n} \frac{(-1)^\nu}{\nu!}\left(\frac{\delta t^2}{x}\right)^\nu + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{\delta t^2}{x}\right)^{n+1} e^{-\delta' t^2/x}, \tag{2.251}$$

where $0 < \delta' < \delta$. We have therefore

$$\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-(a+\delta)t^2/x} F(t+x)\,dt = \sum_{0}^{n} \frac{(-1)^\nu \delta^\nu}{\nu!} I_\nu(x) + \rho, \tag{2.252}$$

where

$$\left\{\begin{aligned} |\rho| &< \frac{K}{(n+1)!}\frac{\delta^{n+1}}{x^{n+\frac{3}{2}}}\int_{-\infty}^{\infty} e^{-at^2/x}\, t^{2n+2}\,dt \\ &= \frac{K}{(n+1)!}\left(\frac{\delta}{a}\right)^{n+1}\int_{-\infty}^{\infty} e^{-w^2}\, w^{2n+2}\,dw \\ &\leqq K\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+2)} < \frac{K}{\sqrt{n}}, \end{aligned}\right. \tag{2.2521}$$

and so ρ tends to zero as $n \to \infty$, *uniformly in x.*

Thus

$$\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-(a+\delta)t^2/x} F(t+x)\,dt = \sum_{0}^{\infty} \frac{(-1)^\nu \delta^\nu}{\nu!} I_\nu(x). \tag{2.253}$$

The series on the right hand side is uniformly convergent in x, say for $x \geqq x_0$, and each of its terms tends to zero as $x \to \infty$. It follows that

$$\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-(a+\delta)t^2/x} F(t+x)\,dt = o(1). \tag{2.254}$$

Thus (2.222) remains true when we substitute for a any number between a and $2a$ inclusive. As this argument may be repeated, it holds for all values of a greater than the original value. Hence (2.212) also holds, and the lemma is proved.

2.3. Theorem 2.1 is an easy deduction from Lemma 2.21. We have

$$\left\{\begin{aligned} &\sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt - f(x) \\ &= \sqrt{\frac{a}{\pi x}}\int_{-\infty}^{\infty} e^{-at^2/x}\{f(t+x) - f(x)\}\,dt. \end{aligned}\right. \tag{2.31}$$

Now

$$|f(t+x) - f(x)| = \left|\int_{x}^{t+x} f'(u)\,du\right| < \frac{K|t|}{\sqrt{x}}. \tag{2.32}$$

Hence

$$(2.33)\quad \begin{cases} \left|\sqrt{\dfrac{a}{\pi x}}\displaystyle\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt - f(x)\right| \\ \quad < \dfrac{K}{x}\sqrt{\dfrac{a}{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-at^2/x}\,|t|\,dt \\ \quad = \dfrac{K}{\sqrt{a}} < \varepsilon, \end{cases}$$

if a is large enough. If now we make $x \to \infty$, we obtain

$$(2.34)\quad \overline{\lim}\,|s - f(x)| \leqq \varepsilon,$$

for all positive values of ε, so that

$$(2.35)\quad f(x) \to s.$$

It follows that

$$s_n \to s;$$

and the proof of Theorem 2.1 is completed.

§ 3.

A new method of summation and its relations to Borel's method.

1.3. THEOREM 2.1, is a generalisation of Theorem 1 of our former paper on BOREL's method. It is naturally suggested that Theorems 2-5 of that paper are susceptible of similar generalisation. We shall content ourselves with stating that the generalisations thus suggested are in fact true, and with mentioning explicitly two of the most interesting of them, viz.,

THEOREM 3.11. — *If* $\sum a_n$ *is summable* (B), *and* $a_n = O(1)$, *then* $\sum a_n$ *is summable* $(C, 1)$.

THEOREM 3.12. — *If* $\sum a_n$ *is summable* (B), *and* $a_n = O(1)$, *then* $s_n = o(\sqrt{n})$.

We may remark that Theorem 3.12 may be deduced from Theorem 3.11 by using Theorem 11 of our paper quoted in note [2]).

3.2. We take this opportunity of correcting an error in the footnote to p. 10 of our former paper. It is stated there that the equation

$$(3.21)\quad \frac{k!\,s_n^k}{n^k} = A + o\left(\frac{1}{\sqrt{n}}\right) \quad ^{9)}$$

gives a sufficient condition for the summability of $\sum a_n$ by BOREL's method. It is easy to show that this statement is false.

Suppose that $\sum a_n n^{-s}$ is an ordinary DIRICHLET's series which represents a function $f(s) = f(\sigma + it)$ regular and of finite order for all values of σ. The series is

[9]) We write A instead of s, as we are about to use s in a different sense.

then, in virtue of a theorem of BOHR [10]), summable by CESÀRO's means all over the plane. We have therefore

$$\frac{k!\,s_n^k}{n^k} = f(s) + o(1), \tag{3.211}$$

where k is a number which depends upon s.

Let us denote by t_n^k CESÀRO's mean of order k formed from the series

$$\sum b_n n^{-s} = \sum \frac{a_n}{\sqrt{n}} n^{-s}. \tag{3.212}$$

It is easy to deduce from (3.211) that

$$\frac{k!\,t_n^k}{n^k} = f\left(s + \frac{1}{2}\right) + o\left(\frac{1}{\sqrt{n}}\right) \quad {}^{11}). \tag{3.213}$$

It follows that, if the theorem suggested is true, then the series $\sum b_n n^{-s}$, and therefore the series $\sum a_n n^{-s}$, must be summable (B) all over the plane.

But there are DIRICHLET's series which represent functions regular and of finite order all over the plane and which are *not* summable (B) all over the plane. For example the series

$$1^{-s} + 0 + 0 + \cdots - 8^{-s} + 0 + \cdots + 27^{-s} + 0 + \cdots$$

represents the function

$$(1 - 2^{1-3s})\zeta(3s),$$

and is summable when, and only when, it is convergent, *i. e.* when $\sigma > 0$ [12]).

The suggested theorem is therefore certainly false.

The theorem is true when $k = 1$ [13]), and the correct generalisation is as follows.

THEOREM 3.2. — *If*

$$\frac{(k+1)!\,s_n^{k+1}}{n^{k+1}} = A + o\left(\frac{1}{\sqrt{n}}\right), \tag{3.21}$$

then BOREL'*s integral*

$$\int_0^\infty e^{-x} \sum \frac{a_n x^n}{n!} dx$$

is summable (C, k), *i. e.*

$$\frac{1}{k!\,x^k}\left(\int_0^x (x-t)^k e^{-t} \sum \frac{a_n t^n}{n!} dt\right) \to A \tag{3.22}$$

as $x \to \infty$.

---

[10]) See G. H. HARDY and M. RIESZ, *The general Theory of* DIRICHLET'*s Series* [Cambridge Mathematical Tracts, no. 18, 1915], p. 56.

[11]) We cannot quote any general theorem of which this equation is a direct corollary: but the materials necessary for the proof will be found in our paper « *Contributions,* etc. », loc. cit. [2]), pp. 432 *et seq.*.

[12]) G. H. HARDY, *The Application to* DIRICHLET'*s Series of* BOREL'*s exponential Method of Summation* [Proceedings of the London Mathematical Society, series II, vol. VIII (1910), pp. 277-294].

[13]) G. H. HARDY, *Researches in the Theory of divergent Series and divergent Integrals* [Quarterly Journal, vol. XXXV (1904), pp. 22-66], p. 40; T. J. I'A. BROMWICH, *Infinite Series*, pp. 319-322.

We shall be content to sketch the proof of this theorem. We may suppose without loss of generality that $A = 0$. Then

$$s_0^k + s_1^k + \cdots + s_n^k = o(n^{k+\frac{1}{2}});$$

and it is easy to deduce successively

$$\frac{1}{n}\left\{s_0^k + \frac{s_1^k}{2^k} + \cdots + \frac{s_n^k}{(n+1)^k}\right\} = o\left(\frac{1}{\sqrt{n}}\right),$$

$$e^{-x}\sum \frac{s_n^k}{(n+1)^k}\frac{x^n}{n!} \twoheadrightarrow 0,$$

$$e^{-x}\sum s_n^k \frac{x^{n+k}}{(n+k)!} \twoheadrightarrow 0.$$

The last formula is equivalent to (3.22).

3.3. The work of section 2 suggests some new definitions which seem to us likely to be of considerable use in the theory of divergent series and integrals. We shall say that *the series* $\sum a_n$ *is summable* (E, a) *to sum* s, or that

$$s_n \twoheadrightarrow s \qquad (E, a),$$

if

$$\lim_{\mu \twoheadrightarrow \infty} \sqrt{\frac{a}{\pi\mu}} \sum_{-\infty}^{\infty} e^{-ah^2/\mu} s_{h+\mu} = s. \tag{3.31}$$

Similarly we shall say that

$$f(x) \twoheadrightarrow l \qquad (E, a)$$

if

$$\lim_{\xi \twoheadrightarrow \infty} \sqrt{\frac{a}{\pi\xi}} \int_{-\infty}^{\infty} e^{-at^2/\xi} f(t+\xi)\, dt = l \text{ [14]} \tag{3.32}$$

The properties of these definitions are, in so far as series or integrals near to the boundary of convergence are concerned, very similar to those of Borel's method. In particular the « Tauberian » properties of the definition (3.31) are the same as those of Borel's method. For example we have.

Theorem 3.3. — *If* (3.31) *is true, i. e., if* $\sum a_n$ *is summable* (E, a), *and*

$$a_n = O\left(\frac{1}{\sqrt{n}}\right),$$

then $\sum a_n$ *is convergent.*

3.4. Borel's method has however a peculiarly intimate connection with the method of type $(E, \frac{1}{2})$. This connection is expressed by the following theorem, the proof of which is contained implicitly in the analysis of section 2.

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[14]) It is to be understood that $s_{h+\mu} = 0$ when $h + \mu < 0$, and $f(t+\xi) = 0$ when $t + \xi$ is less than some fixed number.

THEOREM 3.4. — *Suppose that* $s_n = o(\sqrt{n})$. *Then the existence of any one of the limits*

$$\lim_{x\to\infty} e^{-x}\sum_0^\infty s_n\frac{x^n}{n!}, \tag{3.411}$$

$$\lim_{\mu\to\infty} \frac{1}{\sqrt{2\pi\mu}}\sum_{-\infty}^\infty e^{-h^2/2\mu}s_{h+\mu}, \tag{3.412}$$

$$\lim_{\xi\to\infty} \frac{1}{\sqrt{2\pi\xi}}\int_{-\infty}^\infty e^{-t^2/2\xi}s(t+\xi)\,dt, \tag{3.413}$$

where $s(x)$ *is the discontinuous function defined in* 2.1, *involves the existence of the remainder and the equality of all. In this proposition it is indifferent whether* x *or* ξ *tends to its limit continuously or by integral values.*

The condition $s_n = o(\sqrt{n})$ *is certainly satisfied if any one of the limits exist and* $a_n = o(1)$. *And then the existence of any one of the limits* (3.411)-(3.413) *implies, and is implied by, the convergence of* BOREL'S *integral*

$$\int_0^\infty e^{-x}\left(\sum a_n\frac{x^n}{n!}\right)dx, \tag{3.414}$$

or the existence of the limit

$$\lim_{\xi\to\infty} \frac{1}{\sqrt{2\pi\xi}}\int_{-\infty}^\infty e^{-t^2/2\xi}f(t+\xi)\,dt, \tag{3.415}$$

where $f(x)$ *is the continuous function defined in* 2.1.

3.5. We add a few remarks as to the relations between definitions of the type (E, a) corresponding to different values of a. We proved in 2.2. that, when

$$a_n = O\left(\frac{1}{\sqrt{n}}\right), \tag{2.11}$$

summability (E, a) for a particular value of a implies summability for any larger value of a; and in 2.3 that this implies convergence. Now it is easy to prove that these definitions obey the « condition of consistency », *i. e.* that *any convergent series is summable* (E, a) *for any positive value of* a. We see thus that, when (2.11) is satisfied, all methods of the type (E, a) are equivalent and will sum convergent series only. But this is not a sufficient account of the matter.

Suppose that $s = 0$, so that

$$\sqrt{\frac{1}{x}}\int_{-\infty}^\infty e^{-at^2/x}s(t+x)\,dt = o(1). \tag{3.51}$$

Putting

$$t = u\sqrt{\frac{1}{2a}} = \alpha u, \tag{3.52}$$

we obtain

$$\sqrt{\frac{1}{x}}\int_{-\infty}^\infty e^{-u^2/2x}s(u+\alpha x)\,du = o(1). \tag{3.53}$$

Let us suppose, for simplicity, that α is an integer k. When $k = 1$ our method

is, at any rate when $a_n = o(1)$, equivalent to BOREL's. When $k > 1$ it is equivalent to the method which consists in applying BOREL's method to the subsequence s_{kn}. Now $s_n \twoheadrightarrow 0$ implies $s_{kn} \twoheadrightarrow 0$, while of course the converse implication does not hold; and it would be natural to expect the same to be true when the limits are taken in BOREL's sense. It would therefore also be natural to expect that the truth of (3.53) for a given α should involve its truth for a larger (but not for a smaller) α either without any restriction on a_n or at any rate under some condition less narrow than the « TAUBERian » condition (2.11); in a word to expect the inference from a smaller to a larger α to be of an « ABELian » and not of a « TAUBERian » characte. And so we should expect the inference from summability (E, a) for a given a to summability for a *smaller* to be « ABELian », to be valid at any rate under a wider condition than the condition (2.11) required for the inference to a *larger* a, and to be of a less subtle character.

We shall see shortly that this conjecture is justified, and that *the inference from a larger to a smaller a holds at all events whenever* $a_n = o(1)$. We have however no direct proof of this assertion; our proof is indirect and depends on the methods of summation considered in the next section.

§ 4.

The "circle" method.

4.1. We shall conclude this paper by giving a short account of the results of some researches which started from a suggestion made to us in 1912 by Dr. MARCEL RIESZ.

Suppose that the series

$$f(x) = \sum a_n x^n \tag{4.11}$$

has unit radius of convergence, and that

$$f(x) = f(\tfrac{1}{2} + y) = \sum a_n(\tfrac{1}{2} + y)^n = \sum b_n y^n, \tag{4.12}$$

so that

$$\sum \frac{b_n}{2^n} \tag{4.13}$$

is the TAYLOR's series for $f(\frac{1}{2} + \frac{1}{2})$, *i. e.* the series obtained by putting $y = \frac{1}{2}$ in the expansion of $f(\frac{1}{2} + y)$. Then RIESZ's suggestion was that *the hypotheses (i) that* $\sum a_n$ *is summable (B), and (ii) that the series* (4.13) *is convergent, are equivalent,* at any rate under fairly general conditions as to the order of a_n.

We have succeeded in proving, by the use of some ideas already used in sections 2 and 3, that this very beautiful theorem is true whenever $s_n = o(\sqrt{n})$, and in particular whenever $a_n = o(1)$. We propose now, however, to give a proof not exactly

of RIESZ's theorem, but of another theorem in which the central idea is exactly the same and which differs from RIESZ's only in certain formal respects.

4.2. Suppose that

$$F(y) = \sum a_n e^{-ny} \tag{4.21}$$

is convergent for all values of y whose real part is positive; and consider the series

$$\sum b_n = \sum \frac{(-k)^n}{n!} F^{(n)}(k) \qquad (k > 0), \tag{4.22}$$

which we may denote symbolically by

$$F(k - k). \tag{4.221}$$

Then, if (4.22) is convergent, we shall say that $\sum a_n$ is *summable with radius k*.

Suppose that this is so, and that the sum is s. Then

$$\lim_{M \to \infty} \sum_0^M \frac{k^m}{m!} \sum_0^\infty a_n n^m e^{-kn} = s, \tag{4.231}$$

or

$$\lim_{M \to \infty} \sum_0^\infty a_n e^{-kn} \sum_0^M \frac{(kn)^m}{m!} = s, \tag{4.232}$$

or

$$\lim_{M \to \infty} \sum_0^\infty s_n \Delta \left\{ e^{-kn} \sum_0^M \frac{(kn)^m}{m!} \right\} = s \text{ [15]}, \tag{4.233}$$

or

$$\lim_{M \to \infty} \sum_0^\infty s_n \int_{kn}^{(k+1)n} e^{-t} \frac{t^M}{M!} dt = s. \tag{4.234}$$

This is the form of definition which we shall find it most convenient to adopt. It enables us to verify at once that our definition satisfies the condition of consistency, *i. e.* that *any convergent series is summable with any radius k*. More generally we have

THEOREM 4.2. — *A series which is summable with radius k is summable, to the same sum, with any smaller radius.*

This theorem is plainly equivalent to that which follows [16]).

THEOREM 4.21. — *Suppose that the series*

$$f(x) = \sum a_n x^n$$

is convergent at a point x_0 on its circle of convergence, and that $0 < \alpha < 1$. Then the TAYLOR'*s series for*

$$f\{\alpha x_0 + (1 - \alpha) x_0\},$$

viz.,

$$\sum \frac{f^{(n)}(\alpha x_0)}{n!} \{(1 - \alpha) x_0\}^n = \sum b_n \{(1 - \alpha) x_0\}^n$$

is also convergent.

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[15]) We write $u_n - u_{n+1} = \Delta u_n$.

[16]) We do not claim this theorem as new: it is certainly contained implicitly in earlier writings; but we do not remember having seen it stated explicitly.

We may suppose without loss of generality that $x_0 = 1$.

Then

$$t_n = b_0 + (1-\alpha)b_1 + \cdots + (1-\alpha)^n b_n \tag{4.24}$$

is equal to the coefficient of y^n in

$$\left\{\begin{aligned} \frac{(1-\alpha)^{n+1} - y^{n+1}}{1-\alpha-y} \sum b_n y^n &= \frac{(1+\alpha)^{n+1} - y^{n+1}}{1-\alpha-y} \sum a_n(\alpha+y)^n \\ &= \{(1-\alpha)^{n+1} - y^{n+1}\} \sum s_n (\alpha+y)^n, \end{aligned}\right. \tag{4.25}$$

and so

$$t_n = (1-\alpha)^{n+1}\left\{s_n + (n+1)\alpha s_{n+1} + \frac{(n+1)(n+2)}{1.2}\alpha^2 s_{n+2} + \cdots\right\}. \tag{4.26}$$

The theorem is a straightforward deduction from this identity.

4.3. The « circle » method of summation is related in a very simple manner to the method defined in section 3.

THEOREM 4.3. — *Suppose that $s_n = o(\sqrt{n})$. Then summability with radius k implies summability $(E, \frac{1}{2}k)$, and conversely.*

The proof of this theorem is so much like that of Lemma 2.12 that it will not be necessary to do more than summarize its general lines.

Suppose that $\sum a_n$ is summable with radius k, and that its sum is zero. Then

$$\sum s_n \int_{kn}^{k(n+1)} e^{-t}\frac{t^M}{M!}\,dt = o(1), \tag{4.31}$$

when $M \to \infty$ by integral values. Suppose now that

$$\frac{M}{k} = \mu = [\mu] + \nu = m + \nu,$$

so that $0 \leqq \nu < 1$, and $n = m + h$, where h, as in 2.1, ranges between $-H\mu$ and $H\mu$.

Then

$$\int_{k(m+h)}^{k(m+h+1)} e^{-t}\frac{t^{k\mu}}{(k\mu)!}\,dt = k\int_0^1 e^{-k(m+h+\xi)}\frac{k^{k\mu}(m+h+\xi)^{k\mu}}{(k\mu)!}\,d\xi. \tag{4.33}$$

But it is easy to deduce from STIRLING's Theorem that

$$e^{-k(m+h+\xi)}\frac{k^{k\mu}(m+h+\xi)^{k\mu}}{(k\mu)!} = \frac{e^{-kh^2/2\mu}}{\sqrt{2\pi k\mu}}\left\{1 + O\left(\frac{|h|^3}{\mu^2}\right)\right\}, \tag{4.34}$$

uniformly for $0 \leqq \nu < 1$ and $0 \leqq \xi \leqq 1$. It follows, by arguments similar to those used in the proof of Lemma 2.12, that

$$\sqrt{\frac{k}{2\pi\mu}}\sum_{-\infty}^{\infty} e^{-kh^2/2\mu} s_{h+m} = o(1), \tag{4.35}$$

and hence that $\sum a_n$ is summable $(E, \frac{1}{2}k)$ to sum zero.

A similar argument suffices to prove the converse proposition.

If in particular we suppose that $k = 1$, we obtain

THEOREM 4.31. — *If* $a_n = o(1)$, *then the existence of any one of the limits specified in Theorem 3.4 implies, and is implied by, the summability of the series* $\sum a_n$ *with radius* 1.

4.4. The summability of a series by BOREL's method is therefore equivalent to its summability with radius unity in all cases in which $a_n = o(1)$, and implies (though it is not necessarily implied by) its summability with any lesser radius.

We have also:

THEOREM 4.4. — *The system of* TAUBERian *theorems which holds for the « circle » method of summation is the same as that which holds for* BOREL'*s method or for a method of type* (E, a). *In particular, if a series* $\sum a_n$ *is summable by the circle method, then*

(i) *the condition*

$$a_n = O\left(\frac{1}{\sqrt{n}}\right)$$

implies convergence, and

(ii) *the condition*

$$a_n = O(1)$$

implies summability $(C, 1)$, *and*

(iii) *the condition*

$$a_n = O(1)$$

implies

$$s_n = o(\sqrt{n}).$$

The proof of these theorems involves no difficulty beyond that implied in an adaptation and rearrangement of arguments used already; and the same applies to

THEOREM 4.41. (RIESZ'*s Theorem*). — *If* $a_n = o(1)$ *then the summability* (B) *of* $\sum a_n$ *implies, and is implied by, the convergence of the series for* $f(\frac{1}{2} + y)$ *when* $y = \frac{1}{2}$.

4.5. We conclude with a few miscellaneous remarks.

(i) We asserted at the end of section 3 that summability (E, a) for a particular value of a involved summability for any smaller a whenever $a_n = o(1)$. The truth of this assertion follows now from Theorems 4.2 and 4.3.

(ii) The analysis employed in the proof of Theorem 4.2 suggests yet another definition of the sum of a divergent series viz., as the limit of

$$(1 - \alpha)^{n+1}\left\{s_n + (n + 1)\alpha s_{n+1} + \frac{(n + 1)(n + 2)}{1.2}\alpha^2 s_{n+2} + \cdots\right\},$$

where α is any number between 0 and 1. The properties of this definition would naturally resemble those of the other definitions which we have been discussing.

(iii). The « circle » method may be generalised by supposing that

$$F(y) = \sum a_n e^{-\lambda_n y}$$

where (λ_n) is any ascending sequence which has the limit ∞ and is such that the series is convergent for all values of y whose real part is positive. The « sum » of

$\sum a_n$ is then again defined as being the sum of (4.22), or as

$$\lim_{M \to \infty} \sum s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-t} \frac{t^M}{M!} dt.$$

The definition reduces to that already considered when λ_n is a constant multiple of n. These methods are connected with those used by RIESZ [17]) for effecting the analytic continuation of a function represented partially by a DIRICHLET's series.

The « TAUBERian » condition which corresponds to $a_n = o\left(\frac{1}{\sqrt{n}}\right)$ is now

$$a_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\sqrt{\lambda_n}}\right).$$

(iv) It is of some interest to find a theorem which shall enable us to infer the summability (B) of $\sum a_n$ from the properties of the analytic function $f(x)$. The following theorem, which we state without proof [18]), was found independently by RIESZ and by ourselves.

THEOREM 4.5. — *If $\sum a_n x^n$ is a power series whose radius of convergence is unity, and the function $f(x)$ which it represents satisfies the condition*

$$|f(x) - s| < A|1 - x|^\alpha,$$

where $\alpha > 0$, at all points inside a circle which lies inside the circle of convergence and touches it at the point $x = 1$, then $\sum a_n$ is summable (B) to sum s.

Cambridge, September 1915.

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J. E. LITTLEWOOD.

[17]) M. RIESZ, *Sur la représentation analytique des fonctions définies par des séries de* DIRICHLET [Acta Mathematica, t. XXXV (1912), pp. 253-270].

[18]) The proof depends on considerations of function theory and differs entirely in character from those given in this paper.

CORRECTIONS

p. 41, *line* 12. Read J_3.

p. 42, *line* 6. For Lemma 2.2 read Lemma 2.21.

p. 44, *line* 5. For $\sum_{\nu=1}^{n}$ read $\sum_{\nu=0}^{n}$.

—— *line 7 up*. For 'values of a' read 'values of a'.

p. 45, *line* 2 *of* § 3. For 1.3 read 3.1.

p. 46, *line* 5. For 'Cesàro's mean' read 'Cesàro's sum'.

—— *line* 2 *up*. For $\frac{1}{k!x^k}$ read $\frac{1}{x^k}$.

p. 47, *line* 7. For x^{n+k} read x^n.

p. 48, *line* 2 *up*. For $u+\alpha x$ read $\alpha u+x$.

p. 50, *line* 4. For $F(\nu)$ read $F(y)$.

—— *line* 14. For $\int_{kn}^{(k+1)n}$ read $\int_{kn}^{k(n+1)}$.

p. 51, *line* 5. For $1+\alpha$ read $1-\alpha$.

—— *lines* 5–6. Replace n by ν in the sums.

COMMENTS

Theorem 2.1, § 2, is the O-Tauberian theorem for Borel summability. The corresponding o-theorem is given in 1913, 1. The theorem was extended by Valiron,† with the Tauberian condition replaced by $s_n - s_m \to 0$ as $m \to \infty$, where $n > m$ and $(n-m)/\sqrt{m} \to 0$, and by Schmidt,‡ with $\underline{\lim}(s_n - s_m) \geqslant 0$ as $m \to \infty$ under the same conditions. A simplified proof was given by Vijayaraghavan.§ Another proof of the O-version is given in 1943, 4.

In 1943, 4, Hardy and Littlewood point out that there is a slip in the proof of Lemma 2.12. The conditions under which the asymptotic formula (2.125) is stated to hold should be replaced by $|h| \leqslant \mu^{\eta}$, where $\frac{1}{2} < \eta < \frac{2}{3}$, and there should be an extra term $O(1/\mu)$ in the bracket. Stirling's theorem and the logarithmic series give

$$\log(e^{-\mu}\mu^{h+\mu}/(h+\mu)!) = -\tfrac{1}{2}\log 2\pi\mu - \tfrac{1}{2}h^2/\mu + O(1/\mu) + O(|h|/\mu) + O(|h|^3/\mu^2),$$

and the subsequent argument requires $|h| = o(\mu^{\frac{2}{3}})$, i.e. $\eta < \frac{2}{3}$. There will then be sums over the ranges $(\mu - H\mu, \mu - \mu^{\eta})$ and $(\mu + \mu^{\eta}, \mu + H\mu)$ to be estimated, which require the condition $\eta > \frac{1}{2}$. For details, see D.S., Theorem 137, pp. 200–3. The rest of the proof of Theorem 2.1 is unaffected.

One-sided extensions of Theorems 3.11–3.12, as well as other extensions of Theorems 2–5 of 1913, 1 have been given by Lord (see the Comments on 1913, 1).

Theorem 3.2, § 3, corrects a statement in 1913, 1, p. 10, footnote. In the proof, the step from line 5 to line 6 is the known case, i.e. the case $k = 0$, with summability (B) in place of (B'). By Lemma 2 of 1913, 1, the formula on line 7 (see Corrections) states that the B-mean of s_n tends to $0\ (C, k)$, or alternatively, that the B'-integral of $a_1 + a_2 + \ldots$ is summable (C, k) to $-a_0$. This *implies* that the B'-integral of $a_0 + a_1 + \ldots$ is summable (C, k) to 0, but the last step can only be reversed if the B-mean of a_n tends to $0\ (C, k)$. For the case $k = 0$ of these relations, see 1904, 3, 1904, 4, and the Comments on them; for the general case, see D.S., § 10.10, and Bosanquet.||

† *Rend. del Circolo Mat. di Palermo* 42 (1917), 267–84.
‡ *Schriften Königsberger gelehrt. Ges.* 1 (1925), 205–56.
§ *Proc. London Math. Soc.* (2), 27 (1928), 316–26.
|| *Proc. London Math. Soc.* (3), 3 (1953), 267–304 (Theorem 5).

Some remarks in § 3.5 may be reconsidered in the light of later results. In D.S., p. 222, Hardy defines the (B, γ) sum, $\gamma > 0$, of $a_0+a_1+\ldots$ as the B-limit of the sequence $s_{[n/\gamma]}$, which is equivalent (Theorem 158) to the $(e, \frac{1}{2}\gamma)$ sum,†† whenever $a_n = o(1)$. In particular, for k an integer, the $(B, 1/k)$ sum is the B-limit of the sequence s_{kn}, which is therefore equivalent, when $a_n = o(1)$, to the $(e, 1/2k)$ sum. The $(B, 1/k)$ method was introduced by Borel as the J method corresponding to the integral function e^{x^k}, see Borel.‡‡ But this corresponds to (3.53) (in its corrected form) with $\alpha = \sqrt{k}$ (not $\alpha = k$). An alternative to (3.53) may be obtained by putting $t = u/2a$, $x = X/2a$ in (3.51), which gives

$$\sqrt{\left(\frac{a}{x}\right)} \int e^{-at^2/x} s(t+x)\, dt = \sqrt{\left(\frac{1}{2X}\right)} \int e^{-u^2/2X} s\left(\frac{u+X}{2a}\right) du.$$

This suggests the correct result that (e, a) corresponds to $(B, 2a)$. The final conjecture, substantiated by Theorems 4.2–4.4, is proved again in 1943, 4, Lemma 7. In D.S., p. 220, Hardy gives a better result for the *integral* version, that *the inference from* $a = b$ *to* $a = c$, $0 < c < b$, *holds whenever the integral in* (3.32) *is convergent for* $a = c$; see also the Comments on 1943, 4.

The *circle method* of radius k, say (Γ, k), defined in § 4, is analogous to the Euler method (compare D.S., p. 7). In both methods the circle of convergence is transformed to a half-plane, and the new sum function expanded in a power series; compare also Knopp's§§ extension of Euler's method. A second *circle method*, of radius $\beta = 1-\alpha$, called (γ, β) in D.S., p. 218, is defined in § 4.5 (ii), and used in 1943, 4. Here the preliminary transformation is omitted. This method was defined independently by Fekete‖‖ in 1916, and is also known as the *Taylor method*. Theorems 4.3–4.4 and Lemma 4 of 1943, 4, together show that, when $a_n = o(1)$, (Γ, k), $(e, \frac{1}{2}k)$ and $(\gamma, k/(k+1))$ are equivalent; see also D.S., Ch. IX. In Theorem 4.21 the words 'on its circle of convergence' may be omitted. Theorem 4.2 is then a corollary. The analogue of Theorem 4.2 for (γ, β) summability is given more explicitly in 1943, 4.

In the proof of Theorem 4.3, h should range between $-\mu^{\eta}$ and μ^{η} in (4.34), where $\frac{1}{2} < \eta < \frac{2}{3}$. The converse part of Theorem 4.31 depends on Theorem 4.4 (iii).

†† Here called $(E, \frac{1}{2}\gamma)$.

‡‡ 1st edn., pp. 129–30, 2nd edn., pp. 161–2. See also 1904, 4, § 5.

§§ *Math. Zeit.* 15 (1922), 226–53, and ibid. 18 (1923), 125–56; see D.S., p. 178.

‖‖ For references see Fekete, *J. Lond. math. Soc.* 33 (1958), 466–70.

On the convergence of certain multiple series. By G. H. HARDY, M.A., Trinity College.

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1. In a paper published in 1903 in the *Proceedings of the London Mathematical Society* *, and bearing the same title as this one, I proved a theorem concerning the convergence of multiple series, of the type

$$\Sigma a_{i_1, i_2, \dots i_k} u_{i_1, i_2, \dots i_k},$$

which is given (with an improvement in the conditions) on p. 89 of Dr Bromwich's *Theory of infinite series.* This theorem is one of a class of some importance; and I propose now to state and prove the leading theorems of this class in a form more systematic and general than has been given to them before. I shall begin by recapitulating, with certain changes of form, some known theorems concerning simply infinite series; and I shall then obtain the corresponding theorems for double series in a form as closely analogous as possible. The generalisation from double series to multiple series of any order may well be left to the reader.

Simply infinite series.

2. I shall say that a function a_m, real or complex, of a positive integral variable m is *of bounded variation* if

$$\sum_1^\infty |a_m - a_{m+1}|$$

is convergent. It is plain that this condition involves the existence of $a = \lim a_m$.

THEOREM 1. *The necessary and sufficient condition that a_m should be of bounded variation is that its real and imaginary parts should be of bounded variation.*

This follows at once from the inequalities

$$|\alpha_m - \alpha_{m+1}| \leqslant |a_m - a_{m+1}|, \quad |\beta_m - \beta_{m+1}| \leqslant |a_m - a_{m+1}|,$$

$$|a_m - a_{m+1}| \leqslant |\alpha_m - \alpha_{m+1}| + |\beta_m - \beta_{m+1}|,$$

where $$a_m = \alpha_m + i\beta_m.$$

* Ser. 2, vol. 1, pp. 124—128. See also 'Note in addition to a former paper on conditionally convergent multiple series', *ibid.*, vol. 2, 1904, pp. 190—191.

THEOREM 2. *The necessary and sufficient condition that a real function a_m should be of bounded variation is that it should be of the form $A_m - A_m'$, where A_m and A_m' are positive and decrease steadily as m increases.*

The sufficiency of the condition follows at once from the inequality

$$|a_m - a_{m+1}| \leqslant (A_m - A_{m+1}) + (A_m' - A'_{m+1}).$$

In order to prove that it is necessary, let us suppose that a_m is of bounded variation, and let us write

$$p_m = |a_m - a_{m+1}| \, (a_m - a_{m+1} \geqslant 0), \quad p_m = 0 \, (a_m - a_{m+1} < 0),$$
$$p_m' = |a_m - a_{m+1}| \, (a_m - a_{m+1} \leqslant 0), \quad p_m' = 0 \, (a_m - a_{m+1} > 0),$$
$$B_m = \sum_m^\infty p_n, \quad B_m' = \sum_m^\infty p_n'.$$

Then B_m and B_m' are positive and decrease steadily as m increases; and

$$B_m - B_m' = \sum_m^\infty (a_n - a_{n+1}) = a_m - a.$$

We may therefore take $A_m = B_m + C$ and $A_m' = B_m' + C'$, where C and C' are suitably chosen constants.

THEOREM 3. *If a_m is of bounded variation, and Σu_m is convergent, then $\Sigma a_m u_m$ is convergent.*

Theorem 1 shews that it is enough to prove this theorem when a_m is real. Theorem 2 shews that it is enough to prove it when a_m is positive and steadily decreasing. In this form the theorem is classical*.

LEMMA α. *If Σc_m is a divergent series of positive terms, we can find a sequence of positive numbers ϵ_m, tending steadily to the limit zero, such that $\Sigma \epsilon_m c_m$ is divergent.*

LEMMA β. *If Σc_m is a divergent series of positive terms, we can find a sequence of integers m_i such that the series $\Sigma c_m'$, where $c_m' = 0$ if $m = m_i$ and $c_m' = c_m$ otherwise, is divergent.*

Lemma α is due to Abel †. Lemma β is quite trivial, and the proof may be left to the reader.

* See Bromwich, *Infinite Series*, p. 48. Theorem 3 is given by Dedekind in his editions of Dirichlet's *Vorlesungen über Zahlentheorie*: see *e.g.* p. 255 of the third edition. The central idea of all such theorems is of course Abel's. The line of argument followed here is due substantially to Hadamard, 'Deux théorèmes d'Abel sur la convergence des séries', *Acta Mathematica*, vol. 27, 1903, pp. 177—184.

† 'Sur les séries', *Œuvres*, vol. 2, pp. 197—205.

THEOREM 4. *If $\Sigma a_m u_m$ is convergent whenever Σu_m is convergent, then a_m is of bounded variation.*

This theorem is due to Hadamard*. We have to shew that, if $\Sigma |a_m - a_{m+1}|$ is divergent, u_m can be so chosen that Σu_m is convergent and $\Sigma a_m u_m$ is not. By Lemma α, we can choose a sequence of positive and steadily decreasing numbers ϵ_m so that $\epsilon_m \to 0$ and Σc_m, where

$$c_m = \epsilon_m |a_m - a_{m+1}|,$$

is divergent. By Lemma β, we can then choose the sequence m_i so that $\Sigma c_m'$ is divergent. We take

$$u_1 = U_1, \quad u_m = U_m - U_{m-1} \qquad (m > 1),$$

where

$$U_{m_i} = 0,$$

and

$$U_m = \epsilon_m \frac{|a_m - a_{m+1}|}{a_m - a_{m+1}}$$

if $m \neq m_i$, the last expression being interpreted as meaning ϵ_m if $a_m = a_{m+1}$. We have then

$$\sum_1^{m_i} a_m u_m = \sum_1^{m_i - 1} (a_m - a_{m+1}) U_m + a_{m_i} U_{m_i} = \sum_1^{m_i - 1} c_m',$$

which tends to infinity with i. Thus $\Sigma a_m u_m$ is not convergent, while Σu_m converges to zero.

We may call a_m *a convergence factor* if $\Sigma a_m u_m$ is convergent whenever Σu_m is so. Theorems 3 and 4 may then be combined concisely in

THEOREM 5. *The necessary and sufficient condition that a_m should be a convergence factor is that it should be of bounded variation.*

Double series.

3. The convergence of a double series, in Pringsheim's sense†, does not necessarily involve the convergence of any of its rows or columns ‡. In this paper I shall confine my attention to convergent series whose rows and columns are convergent separately: in this case I shall say that the series is *regularly convergent.* A regularly convergent double series is also convergent when summed by rows or by columns, and its sum by rows or by columns is equal to its sum as a double series §.

Similarly I shall say that $a_{m,n}$ tends *regularly* to a limit if

$$\lim_{m \to \infty} a_{m,n} = a_n, \quad \lim_{n \to \infty} a_{m,n} = a_m,$$

* *l.c. supra.*

† Bromwich, *Infinite Series*, p. 72.

‡ Bromwich, *ibid.*, p. 74.

§ Bromwich, *ibid.*, p. 75.

and the double limit

$$\lim_{m,n\to\infty} a_{m,n} = a,$$

all exist. In this case a_m and a_n tend to a when m and n tend to infinity.

LEMMA γ. *If $\Sigma\Sigma u_{m,n}$ is regularly convergent, to the sum s, and*

$$s_{m,n} = \sum_1^m \sum_1^n u_{\mu,\nu},$$

then, given any positive number ϵ, we can find ω so that

$$|s_{m,n} - s| < \epsilon$$

if either m or n is greater than ω.

We may suppose $s = 0$ without loss of generality. Since the double limit exists, we can choose ω_1 so that $|s_{m,n}| < \epsilon$ if m and n are *both* greater than ω_1. When ω_1 is fixed we can choose ω_2 and ω_3 so that the inequality is satisfied for $1 \leqslant m \leqslant \omega_1$, $n > \omega_2$ and for $m > \omega_3$, $1 \leqslant n \leqslant \omega_1$. We can then take ω to be the greatest of ω_1, ω_2, and ω_3.

LEMMA δ. *In the same circumstances, we can choose ω so that*

$$\left| \sum_m^p \sum_n^q u_{\mu,\nu} \right| < \epsilon$$

if $p \geqslant m$, $q \geqslant n$, and either m or n is greater than ω.

This follows at once from Lemma γ and the identity

$$\sum_m^p \sum_n^q u_{\mu,\nu} = s_{p,q} - s_{p,n-1} - s_{m-1,q} + s_{m-1,n-1}.$$

4. I shall say that $a_{m,n}$ is *of bounded variation in* (m, n) if

(1) $a_{m,n}$ is, for every fixed value of m or n, of bounded variation in n or m,

(2) the series

$$\Sigma\Sigma\, |a_{m,n} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1}|$$

is convergent. And I shall say that $a_{m,n}$ is *a convergence factor* if $\Sigma\Sigma a_{m,n} u_{m,n}$ is regularly convergent whenever $\Sigma\Sigma u_{m,n}$ is regularly convergent. My main object is to prove the analogue of Theorem 5 for double series, *i.e.* to establish the equivalence of these two notions.

It will be convenient to write

$$\Delta_m a_{m,n} = a_{m,n} - a_{m+1,n}, \quad \Delta_n a_{m,n} = a_{m,n} - a_{m,n+1},$$
$$\Delta_{m,n} a_{m,n} = a_{m,n} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1}.$$

The condition that $a_{m,n}$ should be of bounded variation is then that the series $\Sigma \mid \Delta_m a_{m,n} \mid$, $\Sigma \mid \Delta_n a_{m,n} \mid$, and $\Sigma\Sigma \mid \Delta_{m,n} a_{m,n} \mid$ should all be convergent. It is clear that these conditions involve the regular convergence of $a_{m,n}$ to a limit a.

THEOREM 6. *If the condition (2) is satisfied, and $a_{m,1}$ and $a_{1,n}$ are of bounded variation in m and n respectively, then $a_{m,n}$ is of bounded variation in (m, n).*

For
$$\Delta_\mu a_{\mu,n} = \Delta_\mu a_{\mu,1} - \sum_{\nu=1}^{n-1} \Delta_{\mu,\nu} a_{\mu,\nu},$$
$$\sum_{\mu=1}^{m-1} \mid \Delta_\mu a_{\mu,n} \mid \leqslant \sum_{\mu=1}^{m-1} \mid \Delta_\mu a_{\mu,1} \mid + \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{n-1} \mid \Delta_{\mu,\nu} a_{\mu,\nu} \mid,$$
so that
$$\Sigma \mid \Delta_{\mu}, a_{\mu,n} \mid$$
is convergent.

THEOREM 7. *If $a_{m,n}$ is of bounded variation in (m, n), then*
$$a_m = \lim_{n\to\infty} a_{m,n}, \quad a_n = \lim_{m\to\infty} a_{m,n}$$
are of bounded variation in m and n respectively.

For
$$a_\nu = a_{1,\nu} - \sum_{\mu=1}^{\infty} \Delta_\mu a_{\mu,\nu},$$
$$a_\nu - a_{\nu+1} = \Delta_\nu a_{1,\nu} - \sum_{\mu=1}^{\infty} \Delta_{\mu,\nu} a_{\mu,\nu},$$
$$\sum_{\nu=1}^{n-1} \mid a_\nu - a_{\nu+1} \mid \leqslant \sum_{\nu=1}^{n-1} \mid \Delta_\nu a_{1,\nu} \mid + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{n-1} \mid \Delta_{\mu,\nu} a_{\mu,\nu} \mid,$$
and so
$$\Sigma \mid a_\nu - a_{\nu+1} \mid$$
is convergent.

THEOREM 8. *The necessary and sufficient condition that $a_{m,n}$ should be of bounded variation is that its real and imaginary parts should be of bounded variation.*

This follows from Theorem 1 and the inequalities
$$\mid \Delta_{m,n} \alpha_{m,n} \mid \leqslant \mid \Delta_{m,n} a_{m,n} \mid, \quad \mid \Delta_{m,n} \beta_{m,n} \mid \leqslant \mid \Delta_{m,n} a_{m,n} \mid,$$
$$\mid \Delta_{m,n} a_{m,n} \mid \leqslant \mid \Delta_{m,n} \alpha_{m,n} \mid + \mid \Delta_{m,n} \beta_{m,n} \mid,$$
where
$$a_{m,n} = \alpha_{m,n} + i\beta_{m,n}.$$

THEOREM 9. *The necessary and sufficient condition that a real function $a_{m,n}$ should be of bounded variation is that it should be of the form $A_{m,n} - A'_{m,n}$, where*

$$A_{m,n} \geqslant 0, \quad \Delta_m A_{m,n} \geqslant 0, \quad \Delta_n A_{m,n} \geqslant 0, \quad \Delta_{m,n} A_{m,n} \geqslant 0,$$

and $A'_{m,n}$ satisfies similar conditions.

Suppose first that $a_{m,n}$ is of the form indicated. It is plain that the series

$$\sum_m \Delta_m A_{m,n}, \quad \sum_n \Delta_n A_{m,n}, \quad \Sigma\Sigma \Delta_{m,n} A_{m,n},$$

and the corresponding series formed from $A'_{m,n}$, are all convergent. Further we have

$$|\Delta_m a_{m,n}| \leqslant \Delta_m A_{m,n} + \Delta_m A'_{m,n},$$

and similar inequalities for $\Delta_n a_{m,n}$ and $\Delta_{m,n} a_{m,n}$. Hence $a_{m,n}$ is of bounded variation.

Next suppose that $A_{m,n}$ is of bounded variation, and let

$$p_{m,n} = |\Delta_{m,n} a_{m,n}| \; (\Delta_{m,n} a_{m,n} \geqslant 0), \quad p_{m,n} = 0 \; (\Delta_{m,n} a_{m,n} < 0),$$
$$p'_{m,n} = |\Delta_{m,n} a_{m,n}| \; (\Delta_{m,n} a_{m,n} \leqslant 0), \quad p'_{m,n} = 0 \; (\Delta_{m,n} a_{m,n} > 0).$$

Suppose also that

$$B_{m,n} = \sum_m^\infty \sum_n^\infty p_{\mu,\nu}, \quad B'_{m,n} = \sum_m^\infty \sum_n^\infty p'_{\mu,\nu}.$$

Then it is plain that

$$\Delta_m B_{m,n} \geqslant 0, \quad \Delta_n B_{m,n} \geqslant 0, \quad \Delta_{m,n} B_{m,n} \geqslant 0,$$

and that $B'_{m,n}$ satisfies similar conditions.

Also

$$B_{m,n} - B'_{m,n} = \sum_m^\infty \sum_n^\infty \Delta_{\mu,\nu}\, a_{\mu,\nu} = a_{m,n} - a_m - a_n + a,$$

$$a_{m,n} = B_{m,n} - B'_{m,n} + a_m + a_n - a.$$

But, by Theorems 7 and 2, we have

$$a_m = C_m - C_m', \quad a_n = D_n - D_n',$$

where C_m, C_m', D_n, and D_n' are positive and steadily decreasing functions. Thus

$$a_{m,n} = A_{m,n} - A'_{m,n},$$

where

$$A_{m,n} = B_{m,n} + C_m + D_n + E, \quad A'_{m,n} = B'_{m,n} + C_m' + D_n' + E',$$

E and E' being suitably chosen constants; and it is clear that $A_{m,n}$ and $A'_{m,n}$ will satisfy the conditions of the theorem if E and E are sufficiently large.

Theorem 10. *If* $a_{m,n}$ *is of bounded variation, and* $\Sigma\Sigma u_{m,n}$ *is regularly convergent, then* $\Sigma\Sigma a_{m,n} u_{m,n}$ *is regularly convergent.*

In virtue of Theorem 8, it is enough to prove this when $a_{m,n}$ is real. In virtue of Theorem 9, it is enough to prove it when

$$a_{m,n} \geqslant 0, \quad \Delta_m a_{m,n} \geqslant 0, \quad \Delta_n a_{m,n} \geqslant 0, \quad \Delta_{m,n} a_{m,n} \geqslant 0.$$

In the first place, by Theorem 3, every row and column of the series $\Sigma\Sigma a_{m,n} u_{m,n}$ is convergent.

In the second place, we have

$$\sum_m^p \sum_n^q a_{\mu,\nu} u_{\mu,\nu} = \sum_m^{p-1} \sum_n^{q-1} \Delta_{\mu,\nu} a_{\mu,\nu} \sum_m^\mu \sum_n^\nu u_{i,j}$$
$$+ \sum_m^{p-1} \Delta_\mu a_{\mu,q} \sum_m^\mu \sum_n^q u_{i,j} + \sum_n^{q-1} \Delta_\nu a_{p,\nu} \sum_m^p \sum_n^\nu u_{i,j} + a_{p,q} \sum_m^p \sum_n^q u_{i,j}\text{*}.$$

It follows that, if $p \geqslant m$, $q \geqslant n$, we have

$$\left| \sum_m^p \sum_n^q a_{\mu,\nu} u_{\mu,\nu} \right| \leqslant a_{m,n} H_{m,n},$$

where $H_{m,n}$ is the upper bound of

$$\left| \sum_m^\mu \sum_n^\nu u_{i,j} \right| \quad (\mu \geqslant m, \ n \geqslant \nu).$$

Now

$$\sum_1^p \sum_1^q a_{\mu,\nu} u_{\mu,\nu} - \sum_1^m \sum_1^n a_{\mu,\nu} u_{\mu,\nu} = \left(\sum_{m+1}^p \sum_{n+1}^q + \sum_{m+1}^p \sum_1^n + \sum_1^m \sum_{n+1}^q \right) a_{\mu,\nu} u_{\mu,\nu},$$

and so

$$\left| \sum_1^p \sum_1^q a_{\mu,\nu} u_{\mu,\nu} - \sum_1^m \sum_1^n a_{\mu,\nu} u_{\mu,\nu} \right| \leqslant (a_{m+1,n+1} + a_{m+1,1} + a_{1,n+1}) h_{m,n},$$

where $h_{m,n}$ is the upper bound of

$$\left| \sum_k^\mu \sum_l^\nu u_{i,j} \right|,$$

for all values of k, l, μ, and ν such that $\mu \geqslant k$, $\nu \geqslant l$, and $k > m$ *or* $l > n$.

* See pp. 124—125 of my paper quoted in § 1, where the general form of this identity, for multiple series of any order, is given. Similar transformations of double series were given independently by M. Krause, 'Über Mittelwertsätze im Gebiete der Doppelsummen und Doppelintegrale', *Leipziger Berichte*, vol. 55, 1903, pp. 240—263. See also Bromwich, 'Various extensions of Abel's Lemma', *Proc. London Math. Soc.*, ser. 2, vol. 6, 1907, pp. 58—76, where further interesting applications of the identity are made.

Hence, by Lemma δ, we can choose ω so that

$$\left| \sum_1^p \sum_1^q a_{\mu,\nu} u_{\mu,\nu} - \sum_1^m \sum_1^n a_{\mu,\nu} u_{\mu,\nu} \right| \leq (a_{m+1,n+1} + a_{m+1,n} + a_{1,n+1})\,\epsilon < 3a_{1,1}\,\epsilon,$$

if m and n are greater than ω. Thus the double series is convergent, and, since its rows and columns are convergent, it is regularly convergent.

When $a_{m,n}$ and its various differences are positive, this theorem is nearly the same as that referred to in § 1. It is related to the latter theorem, in fact, as what Dr Bromwich calls 'Abel's test' for ordinary convergence is related to 'Dirichlet's test'.* The more direct generalisation is as follows.

THEOREM 11. *If $a_{m,n}$ is of bounded variation and tends regularly to zero, and*

$$\sum_1^m \sum_1^n u_{\mu,\nu}$$

is bounded, then $\Sigma\Sigma a_{m,n} u_{m,n}$ is regularly convergent.

The proof is similar to that of Theorem 10, and I need hardly write it out at length. The theorem shews, for example, that the series

$$\Sigma\Sigma \frac{\cos(m\theta + n\phi)}{(a + m\omega + n\omega')^s},$$

where θ and ϕ are real, ω'/ω is positive or complex, and the real part of s is positive, is regularly convergent except for certain special values of θ, ϕ, and a; or again that the series

$$\Sigma\Sigma \frac{\cos(m\theta + n\phi)}{(am^2 + 2bmn + cn^2)^s},$$

* Theorem 10 itself does not seem to have been enunciated before, even in the specialised form. The nearest theorem which I have been able to find is one given by C. N. Moore, 'On convergence factors in double series and the double Fourier's series', *Trans. Amer. Math. Soc.*, Vol. 14, 1913, pp. 73—104. Moore's theorem (a particular case of a theorem concerning Cesàro summability) is as follows: *if*

(1) $\Sigma\Sigma u_{m,n}$ is convergent as a double series in Pringsheim's sense,

(2) $$\left| \sum_1^m \sum_1^n u_{\mu,\nu} \right| < K,$$

(3) $$a_{m,n} \to 0,$$

(4) $$\lim_{m\to\infty} \sum_{n=1}^{\infty} |a_{m,n}| = 0, \quad \lim_{n\to\infty} \sum_{m=1}^{\infty} |a_{m,n}| = 0,$$

(5) $$\Sigma\Sigma |u_{m,n} a_{m,n}|$$

is convergent, then $\Sigma\Sigma a_{m,n} u_{m,n}$ is convergent.

where θ, ϕ, a, b, and c are real, and a, $ac-b^2$, and the real part of s are positive, is regularly convergent except for certain special values of θ and ϕ. In either of these series, of course, the cosine may be replaced by a sine.

In order to prove the converse of Theorem 10 we require two lemmas analogous to Lemmas α and β.

LEMMA ϵ. *If $\Sigma\Sigma c_{m,n}$ is a divergent series of positive terms, we can find $\epsilon_{m,n}$ so that (1) $\epsilon_{m,n}$ decreases when m or n increases, (2) $\epsilon_{m,n}$ tends regularly to zero, and (3) the series $\Sigma\Sigma\epsilon_{m,n}c_{m,n}$ is divergent.*

(1) Suppose first that at least one row or column of the original series, say the νth row $\Sigma c_{m,\nu}$, is divergent. By Lemma α, we can choose a steadily decreasing sequence η_m, with limit zero, so that $\Sigma\eta_m c_{m,\nu}$ is divergent. We take

$$\epsilon_{m,n}=\eta_m\,(n\leqslant\nu),\quad \epsilon_{m,n}=0\cdot(n>\nu),$$

and it is plain that the conditions of the lemma are satisfied.

(2) Suppose that every row and column is convergent; and let

$$\sum_{(m)} c_{m,n}=\gamma_n,\quad \sum_{(n)} c_{m,n}=\gamma_m.$$

Then $\Sigma\gamma_m$ is divergent. We choose a steadily decreasing sequence η_m so that $\Sigma\eta_m\gamma_m$ is divergent. Then $\Sigma\Sigma c'_{m,n}$, where

$$c'_{m,n}=\eta_m c_{m,n},$$

is divergent; and so $\Sigma\gamma_n'$, where

$$\gamma_n'=\sum_{(m)}\eta_m c_{m,n},$$

is divergent. We now choose a steadily decreasing sequence ζ_n, with limit zero, so that $\Sigma\zeta_n\gamma_n'$ is divergent. It is clear that, if we write

$$c''_{m,n}=\eta_m\zeta_n c_{m,n}=\epsilon_{m,n}c_{m,n},$$

all the conditions of the lemma will be satisfied.

LEMMA ζ. *If $\Sigma\Sigma c_{m,n}$ is a divergent series of positive terms, we can choose a sequence of pairs of integers (m_i, n_i), tending to infinity with i, so that the series $\Sigma\Sigma c'_{m,n}$, where $c'_{m,n}=0$ if $m=m_i$, $n\leqslant n_i$ or $m\leqslant m_i$, $n=n_i$, and $c'_{m,n}=c_{m,n}$ otherwise, is divergent.*

The modification to be made in the series is effected by drawing perpendiculars on to the axes from the points (m_i, n_i), and annulling all terms which correspond to points on these perpendiculars. Let σ_m denote the sum of the terms whose

representative points lie on the perpendiculars from (m, m) on to the axes. Then $\Sigma\sigma_m$ is divergent. Applying Lemma β to this series we obtain the construction required, m_i being in fact always equal to n_i.

THEOREM 12. *If $\Sigma\Sigma a_{m,n}u_{m,n}$ is regularly convergent whenever $\Sigma\Sigma u_{m,n}$ is regularly convergent, then $a_{m,n}$ is of bounded variation.*

In the first place it follows from Theorem 4 that $u_{m,n}$ is, for every value of n (or m), of bounded variation in m (or n). It remains only to shew that $\Sigma\Sigma\,|\,\Delta_{m,n}a_{m,n}\,|$ is convergent.

Suppose, on the contrary, that it is divergent. By Lemma ϵ, we can choose a sequence of positive numbers $\epsilon_{m,n}$, tending regularly to zero, so that $\Sigma\Sigma c_{m,n}$, where

$$c_{m,n} = \epsilon_{m,n}\,|\,\Delta_{m,n}a_{m,n}\,|,$$

is divergent. We can then modify this series as in Lemma ζ without destroying its divergence.

Now let

$$U_{m,n} = \sum_1^m\sum_1^n u_{\mu,\nu}$$

and suppose that

$$U_{m,n} = 0$$

if $m = m_i$, $n \leqslant n_i$ or $m \leqslant m_i$, $n = n_i$, and that otherwise

$$U_{m,n} = \epsilon_{m,n}\frac{|\,\Delta_{m,n}\,a_{m,n}\,|}{\Delta_{m,n}\,a_{m,n}}:$$

this last formula being interpreted as meaning $\epsilon_{m,n}$ if

$$\Delta_{m,n}\,a_{m,n} = 0.$$

These equations define $u_{m,n}$ uniquely for all values of m and n, and it is plain that $U_{m,n}$ tends regularly to zero, so that $\Sigma\Sigma\,u_{m,n}$ is regularly convergent. On the other hand

$$\sum_1^{m_i}\sum_1^{n_i} a_{m,n}\,u_{m,n} = \sum_1^{m_i-1}\sum_1^{n_i-1}\Delta_{m,n}\,a_{m,n}\,U_{m,n} = \sum_1^{m_i-1}\sum_1^{n_i-1} c'_{m,n},$$

which tends to infinity with i. Thus $\Sigma\Sigma\,a_{m,n}u_{m,n}$ is not convergent.

This proves Theorem 12. Combining it with Theorem 10 we obtain the analogue of Theorem 5, viz.

THEOREM 13. *The necessary and sufficient condition that $a_{m,n}$ should be a convergence factor is that it should be of bounded variation.*

CORRECTIONS

p. 91, *line* 14. For $A_{m,n}$ read $a_{m,n}$.

p. 93, *footnote*. In (5), for $u_{m,n}\,a_{m,n}$ read $\Delta_{m,n}\,a_{m,n}$.

COMMENTS

Theorems 3 and 4 (or 5) recapitulate the results of Dedekind and Hadamard giving necessary and sufficient conditions for a factor a_m to convert every convergent series into a convergent series (C–C); see the Comments on 1907, 2.

Theorems 10 and 12 (or 13) give necessary and sufficient conditions for a factor $a_{m,n}$ to convert every regularly convergent double series into a regularly convergent series (RC–RC). Hamilton† showed that the same conditions are necessary and sufficient for RC–C. On the other hand, Kojima‡ showed that the conditions for C–C for double series, are more stringent. Hamilton (loc. cit.) gave a number of other results for double series.

Theorem 11 and a theorem of Kojima (loc. cit.) give necessary and sufficient conditions for $a_{m,n}$ to convert every bounded double series into a regularly convergent series (B–RC). Hamilton showed that equivalent sets of conditions are:

$$\text{(i)}\quad \sum\sum |\Delta_{m,n}\,a_{m,n}| < \infty \quad \text{and}$$

$$\text{either}\qquad \text{(ii)}\quad \lim_{m\to\infty} a_{m,n} = 0 \quad\text{and}\quad \lim_{n\to\infty} a_{m,n} = 0$$

$$\text{or}\qquad \text{(ii)}'\quad a_{m,n} \to 0 \text{ regularly.}$$

Hamilton also showed that the conditions are necessary and sufficient for B–C; and so for the intermediate property B–BC.

The theorem stated in the Comments on 1904, 1, and completed in the Comments on 1905, 3, extends Hamilton's conditions for B–C, B–BC, and B–RC to multiple series. Extensions of the conditions for RC–RC to multiple series are given in Moore, § 1.20. Further results have been given by Hamilton.§

† *Bull. American Math. Soc.* 42 (1936), 275–83.
‡ *Tôhoku Math. J.* 17 (1920), 213–20.
§ *Duke Math. J.* 2 (1936), 29–60.

ABEL'S THEOREM AND ITS CONVERSE

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1.

INTRODUCTION AND SUMMARY.

1 . 1. Our main object in this paper is to obtain as far-reaching a generalisation as possible of "Tauber's Theorem", the well-known converse of Abel's famous theorem concerning power-series. It will be necessary to give a rapid summary of the results already known; and we can do this most shortly and clearly if we begin with a few definitions of a verbal character.

We shall always denote the power-series in question by $\Sigma a_n x^n$ or by S, and its sum by $f(x)$. We suppose that the radius of convergence of S is unity, and that the point on the circle of convergence which is in question is the point $x = 1$. The series Σa_n we shall call A, and we shall also use A to denote its sum, when it is convergent.

We shall use (K), (L), (O), and (o) as abbreviations for the propositions:

(K) *A is convergent,*

(L) $f(x) \to A$,

$$(O) \quad a_n = O\left(\frac{1}{n}\right),$$

$$(o) \quad a_n = o\left(\frac{1}{n}\right).$$

We shall be concerned in the sequel with certain classes of curves C along which x may approach the limit 1.* We shall call C a *path* if it is a simple Jordan curve which does not pass outside the circle: that

* We are, of course, concerned with the nature of C only in the neighbourhood of $x = 1$. It is therefore presupposed, in the definitions which follow, that their conditions need only be satisfied for values of x near enough to unity.

1920, 7 (with J. E. Littlewood) *Proceedings of the London Mathematical Society* (2), 18, 205–35.

is to say if it is defined by equations

$$x = \xi + i\eta, \quad \xi = \phi(t), \quad \eta = \psi(t) \quad (t_0 \leqslant t \leqslant T);$$

where ϕ and ψ are continuous for $t_0 \leqslant t \leqslant T$, $\phi(T) = 1$, $\psi(T) = 0$, $\phi^2 + \psi^2 \leqslant 1$, and $\phi(t_1) = \phi(t_2)$, $\psi(t_1) = \psi(t_2)$ are not both true unless $t_1 = t_2$.

If $\phi^2 + \psi^2 < 1$ except for $t = T$, we shall call C an *internal path*. If it lies entirely between two chords of the unit circle, meeting at $x = 1$, we shall call it a *Stolz-path*.

If C possesses a continuously turning tangent at every point except $x = 1$, and approaches $x = 1$ with a definite direction, so that am $(1-x)$ tends to a limit when $x \to 1$, we shall call it a *regular path*. If the limit of am $(1-x)$ is neither $\frac{1}{2}\pi$ nor $-\frac{1}{2}\pi$, C is a *regular Stolz-path*.

Thus a chord of the unit circle, or a segment of a circle which passes through $x = 0$ and $x = 1$, and contains an angle greater than a right angle, is a regular Stolz-path. An arc of a circle which touches the unit circle internally, or an arc of the unit circle itself, is regular, but not a Stolz-path. The curve

$$\eta = (1-\xi) \sin \frac{1}{1-\xi}$$

is a Stolz-path, but not regular. The curve

$$\eta = \sqrt{(1-\xi^2)} \sin \frac{1}{\sqrt{(1-\xi^2)}}$$

is a path, but neither regular nor a Stolz-path.

1 . 2. Abel's Theorem is

A. (K) *implies* (L) *when* C *is the radius* (0, 1).*

Stolz's generalisation is

B. (K) *implies* (L) *when* C *is any Stolz-path.*†

Proofs of **A** and **B** will be found in Bromwich's *Infinite Series*.‡ To

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* N. H. Abel, "Untersuchungen über die Reihe $1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \ldots$", *Journal für Math.*, Vol. 1, 1826, pp. 311–339 (*Œuvres*, Vol. 1, pp. 219–250).

† O. Stolz, "Beweis einiger Sätze über Potenzreihen", *Zeitschrift für Math.*, Jahrgang 20, 1875, pp. 369–376; "Nachtrag ...", *ibid.*, Jahrgang 29, 1884, pp. 127–128.

See pp. 130, 210–212.

these theorems should be added

C. *It is not true that* (K) *implies* (L) *whenever* C *is a regular path.**

Thus, for example, the series

$$\Sigma\, n^{-b} e^{Ain^a} \quad (A > 0,\ 0 < a < 1) \tag{1.21}$$

is convergent whenever $b > 1-a$; but the associated power-series $f(x)$ does not tend to a limit if $b < 1-\frac{1}{2}a$ and C is an arc of a circle touching the unit circle.

Tauber's Theorem is

D. (L) *and* (o) *imply* (K) *when* C *is the radius* $(0, 1)$.†

This theorem has been generalised in several directions. The generalisations with which we shall be most directly concerned are

E. (L) *and* (o) *imply* (K) *when* C *is any Stolz-path.*‡

F. (L) *and* (O) *imply* (K) *when* C *is the radius* $(0, 1)$.§

But we must also mention

G. *In either* **D** *or* **E**, (o) *may be replaced by the more general condition*

$$a_1+2a_2+\ldots+na_n = o(n).$$

This condition is also necessary for the truth of (K).‖

H. *In* **F**, (O) *may be replaced by the condition that* a_n *is real and* na_n *bounded above or below.*¶

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* G. H. Hardy and J. E. Littlewood, **1** (see the list of papers in 1.5), p. 475 (Theorem 47). The proof is not given, but the materials necessary for one will be found in a paper by Hardy, "A theorem concerning Taylor's series", *Quarterly Journal*, Vol. 44, 1913, pp. 147–160 (pp. 150 *et seq.*).

† A. Tauber, "Ein Satz aus der Theorie der unendlichen Reihen", *Monatshefte für Math.*, Vol. 8, 1897, pp. 273–277. See also Bromwich, *Infinite Series*, p. 251, or Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1916, p. 40.

‡ E. Landau, "Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande Konvergenzgebietes", *Monatshefte für Math.*, Vol. 18, 1907, pp. 8–28. See also Bromwich and Landau, *l.c. supra*.

§ J. E. Littlewood, "The converse of Abel's theorem on power-series", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1911, 434–448.

‖ Tauber (*l.c. supra*) proves this when C is the radius. We cannot refer to an explicit proof for the case in which C is an arbitrary Stolz-path; but the result is an immediate consequence of the arguments used by Tauber and by Landau.

¶ That is to say, $na_n < H$ or $na_n > -H$, where H is a constant. G. H. Hardy and J. E. Littlewood, **2**; see also Landau's book referred to above, pp. 45–56. The condition is plainly more general than (O) when a_n is real.

These theorems can only be appreciated if we bear in mind other results of a negative character. The trivial example

$$1-1+1-\ldots,$$

is enough to show that the existence of Abel's limit does not involve the convergence of the series; thus "(L) implies (K)", the straightforward converse of Abel's Theorem, is false. It is not quite so easy to find a similar example in which the terms of the series tend to zero. This was first done by Pringsheim*; but a more natural example is provided by the series (1.21) when $0 < b < 1-a$. Here $a_n = O(n^{-b})$, and a may be as small, and so b as nearly equal to 1, as we please. Thus no condition of this type, with $b < 1$, is sufficient to ensure the convergence of the series whenever Abel's limit exist. This suggests that **F** is really a "best possible" theorem of its kind; and this is shown by the theorem

K. *There is no function* $\phi(n)$, *such that* $\phi(n) \to \infty$ *and*

$$a_n = O\left\{\frac{\phi(n)}{n}\right\},$$

together with (L), *implies* (K).†

1.3. No extension of **F** to paths other than the radius has yet been published. The extension of theorems of the "o" character to paths other than Stolz-paths was first considered seriously in our paper **1.**‡ In this paper we confined ourselves to *regular* paths, and we found that, in order to obtain satisfactory results, it was essential to replace (L) by a different condition. This condition is

$$\text{(A)} \qquad \Phi(x) = \frac{1}{1-x}\sum \frac{a_n}{n+1}(1-x^{n+1}) \to A.$$

It is to be observed that, so long as C is regular, and

$$\sum \frac{a_n}{n+1}$$

is absolutely convergent, we have

$$(1.31) \qquad \Phi(x) = \frac{1}{1-x}\int_x^1 f(t)\,dt,$$

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* A. Pringsheim, "Über die Divergenz gewisser Potenzreihen an der Konvergenzgrenze", *Münchener Sitzungsberichte*, Vol. 31, 1901, pp. 505–524.

† J. E. Littlewood, *l.c.*, p. 444 (Theorem C).

‡ Pp. 475–477.

the integration being effected along C. It is an easy deduction that $\Phi(x) \to A$ in all cases in which $f(x) \to A$, whereas the converse is untrue. Thus (L) implies (Λ), and (Λ) is a generalisation of (L), at any rate in all such cases as we were considering before. This being so, we proved

L. (Λ) *and* (o) *imply* (K) *whenever* C *is a regular path.*

As a corollary we have

M. (L) *and* (o) *imply* (K) *whenever* C *is a regular path.*

This theorem includes **D** and **E** as special cases. We also proved* the direct converse of **L**, viz.

N. (K) *and* (o) *imply* (Λ) *whenever* C *is a regular path.*

This theorem becomes untrue if (Λ) is replaced by (L). It is an "Abelian" theorem, but differs fundamentally from the ordinary Abelian theorems **A** and **B** in that its truth depends upon a condition such as occurs in the "Tauberian" theorems. It is also unlike all the theorems which precede in being reversible: and, on combining it with **L**, we obtain

O. *If* A *satisfies* (o), *then the necessary and sufficient condition for its convergence is that* (Λ) *should be true when* x *tends to* 1, *either along any regular path, or along all.*

1.4. We begin our new investigations by a direct extension of **F** to a regular Stolz-path, viz.

P. (L) *and* (O) *imply* (K) *when* C *is a regular Stolz-path.*

This theorem is included in others which come after and are proved in a quite different way. But the method we use (in 2.1) seems to us of considerable interest in itself.

In 2.2 and the succeeding paragraphs we attack our main problem. Our object is to generalise **L**, (i) by replacing (o) by (O), and (ii) by getting rid of the restriction that C is a regular path; and the result is

Q. (Λ) *and* (O) *imply* (K), *for any path* C.

When C is regular, we can deduce as a corollary

R. (L) *and* (O) *imply* (K) *whenever* C *is a regular path.*

* The proof of this theorem (Theorem 50) is not stated explicitly, but is virtually contained in that of the preceding Theorem 49 (**L** of this paper).

We can also prove

S. *(L) and (O) imply (K) whenever C is a Stolz-path.*

But we cannot here get rid of *all* restrictions upon C.

In 3 we proceed to the corresponding Abelian theorems, and generalise **N** by proving

T. *(K) and (O) imply (Λ), for any path C.*

And by combining **Q** and **T**, we obtain

U. *If A satisfies (O), then the necessary and sufficient condition for its convergence is that (Λ) should be true when x tends to* 1, *either along any path C, or along all.*

This theorem affords the complete generalisation of **O** in each of the desired directions, and is far more comprehensive than any known theorem of its kind.

There are but few questions which remain to be answered. There is one to which we have already alluded and are unable to answer, namely whether **R** (or **S**) is true without any restriction on C. The others are connected with the question whether **T** is the "best possible" theorem of its kind. We prove

V. *(K) and (O) do not necessarily imply (L) for all paths C*:

so that the (Λ) of **T** certainly cannot be replaced by (*L*). And almost the same example suffices to prove

W. *(K) does not necessarily imply (Λ) for all paths C.*

Thus *T* certainly becomes untrue if the condition (*O*) is simply omitted. But it is desirable to prove more, viz. that (*O*) cannot be replaced by any less restrictive condition of the type which occurs in **K**, and we have not yet succeeded in establishing this by means of an example. If we could do this, and also remove the restriction on C in **R** (or **S**), we could fairly claim that our problem had been completely solved.

1.5. If, in Theorems **Q**–**U**, we suppose that C is an arc of the unit circle itself, we obtain a number of theorems concerning the convergence of a series

$$\Sigma(a_n+i\beta_n)\, e^{ni\theta} = \Sigma(a_n \cos n\theta+\beta_n \sin n\theta)+i\Sigma\,(a_n \sin n\theta-\beta_n \cos n\theta),$$

where $$a_n = O\left(\frac{1}{n}\right), \quad \beta_n = O\left(\frac{1}{n}\right);$$

theorems, that is to say, concerning the simultaneous convergence of a Fourier series and its *conjugate* or *allied* series. It is, however, more natural and more interesting to consider the two series independently. We prove first*

X. *If a_n and b_n satisfy (O), so that*

$$\Sigma(a_n \cos n\theta + b_n \sin n\theta)$$

is certainly the Fourier series of a summable function $f(\theta)$, then the necessary and sufficient condition that the series should converge to the sum A is that

$$\frac{1}{2a}\int_{\theta-a}^{\theta+a} f(t)\,dt \to A$$

when $a \to 0$.

When (*O*) is replaced by (*o*), Theorem **X** reduces to a theorem of Fatou.† These theorems correspond to **O** and **U**. The remaining theorems of the paper are of a somewhat different character: the most interesting of them are **Y** and **Z**, which are concerned with the Fourier series of bounded functions, and do not depend upon conditions such as (*o*) or (*O*).

We conclude these introductory remarks by giving a list of the papers of our own to which we shall have to refer. They are :—

1. "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1913, pp. 411–477.

2. "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *ibid.*, Vol. 13, 1914, pp. 174–191.

3. "Some theorems concerning Dirichlet's series", *Messenger of Math.*, Vol. 43, 1914, pp. 134–147.

4. "Theorems concerning the summability of series by Borel's exponential method", *Rendiconti del Circ. Mat. di Palermo*, Vol. 41, 1916, pp. 36–53.

5. "Sur la convergence des séries de Fourier et des séries de Taylor", *Comptes Rendus*, December 24, 1917.

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* We revert to the notation usual in the theory of Fourier series.

† P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Mathematica*, Vol. 30, 1906, pp. 335–400 (p. 385).

2.

THE TAUBERIAN THEOREMS.

Proof of Theorem **P.**

2.1. THEOREM **P.**—*If*

(*O*) $$a_n = O\left(\frac{1}{n}\right),$$

and (*L*) $$f(x) = \Sigma a_n x^n \to A,$$

when $x \to 1$ *along a regular Stolz-path* C, *then* Σa_n *converges to the sum* A.

We suppose, as obviously we may without loss of generality, that $|na_n| < 1$ and $A = 0$. We write

$$x = re^{i\theta} = e^{-\rho+i\theta},$$

and we suppose first that C is the particular curve

(2.11) $$\theta = k\rho.$$

Then, if $$F_k(\rho) = \Sigma a_n e^{-(1-ki)n\rho},$$

we have $$F_k(\rho) = o(1)$$

when $\rho \to 0$, and

$$\rho^p F_k^{(p)}(\rho) = (-1)^p (1-ki)^p \rho^p \Sigma n^p a_n e^{-(1-ki)n\rho}$$
$$= O(\rho^p \Sigma n^{p-1} e^{-n\rho}) = O(1),$$

for every positive integral value of p. It follows, from our fundamental theorems on derivatives*, that

(2.12) $$\rho^p F_k^{(p)}(\rho) = o(1),$$

for every such value of p.

We shall now prove that

(2.13) $$F_l(\rho) = o(1),$$

for any value of l such that $k-1 \leqslant l \leqslant k+1$.

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* The particular theorem required is obtained by supposing $\phi = \psi = 1$ in Case (*b*) of Theorems 6 and 8 of our paper **1.**

If $l = k+\delta$, so that $|\delta| \leqslant 1$, we have

$$F_l(\rho) = \Sigma\, a_n e^{-(1-ki)\,n\rho}\, e^{\delta i n\rho}$$

$$= \sum_{(n)} a_n e^{-(1-ki)\,n\rho} \left\{ \sum_{p=0}^{P-1} \frac{(\delta i n\rho)^p}{p!} + \Delta_P \right\}$$

$$= \sum_{p=0}^{P-1} \frac{(\delta\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho} + R_P,$$

where
$$|\Delta_P| < \frac{2(\delta n\rho)^P}{P!},$$

$$|R_P| < \frac{2(\delta\rho)^P}{P!}\, \Sigma n^P\, |a_n|\, e^{-n\rho}.$$

Now $|na_n| < 1$, and

$$\Sigma n^{P-1} e^{-n\rho} < \int_0^\infty \xi^{P-1} e^{-\xi\rho} d\xi + 2\,\mathrm{Max}\,(\xi^{P-1} e^{-\xi\rho})$$

$$= \frac{\Gamma(P)}{\rho^P} + 2\left(\frac{P-1}{\rho}\right)^{P-1} e^{-(P-1)}$$

$$< \frac{2\Gamma(P)}{\rho^P},$$

for $0 < \rho < \rho_0$, where ρ_0 is a number independent of P. Hence

$$|R_P| < \frac{4\delta^P}{P} \leqslant \frac{4}{P}. \tag{2.14}$$

And therefore
$$F_l(\rho) = \sum_{(p)} \frac{(\delta i\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho}$$

$$= \sum_{(p)} \frac{1}{p!} \left(-\frac{\delta i}{1-ki}\right)^p \rho^p F_k^{(p)}(\rho);$$

and this series is, in virtue of (2.14), *uniformly convergent for* $|\delta| \leqslant 1$ *and* $\rho > 0$. But, by (2.12), every term of the series tends to zero. Hence $F_l(\rho)$ tends to zero*: that is to say, we have proved (2.13).

It follows, by the repeated application of this argument, that if a_n satisfies (O) and $f(x)$ tends to zero along any Stolz-path of the type (2.11),

* Our argument here is the same in principle as that which we used in the proof of the "general Borel-Tauber" theorem: see our paper **4**, p. 44.

then it tends to zero along any other Stolz-path of the same type. In particular it tends to zero along the real axis, to which the path reduces when $k = 0$. And hence, by Theorem **F**, the series Σa_n converges to zero.

The theorem is thus proved for paths of the special type (2.11). In general, the equation of a regular Stolz-path may be written in the form

$$\theta = k\rho + o(\rho). \tag{2.15}$$

It is easy to see that, if $f(x)$ tends to zero along (2.15), it also tends to zero along (2.11). For, if (ρ, θ) and (ρ, θ') are corresponding points on the paths (2.11) and (2.15), we have

$$| e^{ni\theta} - e^{ni\theta'} | = | e^{ni(\theta-\theta')} - 1 | = o(n\rho)$$

when $\rho \to 0$, and so

$$f(e^{-\rho+i\theta}) - f(e^{-\rho+i\theta'}) = o(\rho)\Sigma n\,|a_n|\, e^{-n\rho} = o(1).$$

The truth of the theorem in its general form now follows from the argument which precedes.

Proof of Theorem **Q**.

2.2. THEOREM **Q**.—*If*

$$a_n = O\left(\frac{1}{n}\right), \tag{O}$$

and

$$\Phi(x) = \frac{1}{1-x}\Sigma\frac{a_n}{n+1}(1-x^{n+1}) \to A, \tag{Λ}$$

when $x \to 1$ along some path C, then Σa_n converges to the sum A.

We may suppose that $A = 0$, $a_0 = 0$, and $|na_n| < 1$. And we shall begin by proving

LEMMA α.—*(O) being satisfied, the necessary and sufficient condition that*

$$s_n = a_1 + a_2 + \ldots + a_n$$

should be bounded is that $\Phi(x)$ should be bounded for $|x| \leqslant 1$, $x \neq 1$; and this condition is satisfied if $\Phi(x)$ is bounded when $x \to 1$ along any particular path C.

We take
$$\nu = \left[\frac{1}{|1-x|}\right],$$
and we may suppose $|1-x| < 1$, so that
$$\tfrac{1}{2} < \nu\,|1-x| \leqslant 1.^*$$

Then

$$(2\,.\,21)\qquad \Phi(x) = \left(\sum_{0}^{\nu-1} + \sum_{\nu}^{\infty}\right) \frac{a_n}{n+1}\,\frac{1-x^{n+1}}{1-x} = \phi_1 + \phi_2,$$

say. In the first place we have

$$(2\,.\,22)\qquad |\phi_2| < \frac{2}{|1-x|}\sum_{\nu}^{\infty}\frac{1}{n(n+1)} = \frac{2}{\nu\,|1-x|} < 4.$$

Secondly,

$$1-\frac{1-x^{n+1}}{(n+1)(1-x)} = \frac{1}{n+1}\{(1-x)+(1-x^2)+\ldots+(1-x^n)\},$$

$$\left|1-\frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \frac{|1-x|}{n+1}(1+2+\ldots+n) = \tfrac{1}{2}n\,|1-x|,$$

$$(2\,.\,23)\qquad |s_{\nu-1}-\phi_1| \leqslant \sum_{1}^{\nu-1}|a_n|\left|1-\frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \tfrac{1}{2}|1-x|\sum_{1}^{\nu-1}|na_n|$$
$$< \nu\,|1-x| \leqslant 1.$$

From (2 . 21), (2 . 22), and (2 . 23), we obtain
$$|s_{\nu-1}-\Phi(x)| < 5\,;$$
which proves both parts of the lemma, since ν passes through an unbroken sequence of integral values when $x \to 1$ along C.

2 . 3. The remainder of the proof of Theorem **Q** depends upon certain lemmas in the theory of analytic functions, of the general type associated particularly with the names of Phragmén and Lindelöf.†

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* It is easily verified that $y\,[1/y] > \frac{1}{2}$ if $0 < y < 1$.

† See in particular the memoir "Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier", *Acta Mathematica*, Vol. 31, 1908, pp. 381–406. The chief results of this memoir may now be regarded as classical.

LEMMA β.*—*Suppose that T is the semi-infinite strip, in the plane of the complex variable $w = u+iv$, defined by the inequalities $0 \leqslant u \leqslant \pi$, $v \geqslant v_0$; that* **A** *and* **B** *are the left and right-hand edges of the strip, and that* **C** *is a simple Jordan curve which lies entirely inside T, extends to infinity, and divides T into two regions L and R. Suppose further that $f(w)$ is regular inside T and continuous and bounded throughout T.*

Finally, suppose that

$$\overline{\lim} \, |f(w)| \leqslant h, \tag{2.31}$$

where $h > 0$, when w tends to infinity along **A** *(or* **B***) and along* **C**. *Then* (2.31) *holds when w tends to infinity in any manner inside L (or R).*

Let us suppose that the data refer to **A** and **C**. We can choose $v_1(\epsilon)$ so that

$$|f| < h+\epsilon, \tag{2.33}$$

at all points of **A** and **C** for which $v > v_1$. We can then draw a cross-cut (*Querschnitt*) Q, cutting off from L an infinite region L_Q; and (2.33) will be satisfied at all points of $\mathbf{A}_Q$ and $\mathbf{C}_Q$, the parts of **A** and **C** which belong to the boundary of L_Q.

Let K be the upper bound of $|f|$ on Q. We can now choose

$$p(\epsilon, v_1, K) = p(\epsilon),$$

so that

$$\left| \frac{w}{w+p} \right| < 1,$$

---

* We state this and the following lemmas in the special forms in which they are required for our immediate purpose. All of them, naturally, are capable of wide generalisation. The most interesting of these generalisations is the following:

If (1) *T is the strip $\alpha \leqslant u \leqslant \beta$, $v \geqslant v_0$;* **A** *and* **B** *its edges;* $\mathbf{C}_\alpha$ *and* $\mathbf{C}_\beta$ *simple non-intersecting Jordan curves interior to T and asymptotic to* **A** *and* **B**; *and* **C** *a similar curve asymptotic to $u = \nu$, where $\alpha < \nu < \beta$:*

(2) *$f(w)$ is regular inside and continuous throughout the region T' formed by those points of T which lie to the right of* $\mathbf{C}_\alpha$ *and to the left of* $\mathbf{C}_\beta$:

(3) $\overline{\lim} \, |f(w)| \leqslant \mathbf{a}$ *when w tends to infinity along* $\mathbf{C}_\alpha$, *and* $\overline{\lim} \, |f(w)| \leqslant \mathbf{b}$ *when w tends to infinity along* $\mathbf{C}_\beta$:

(4) $$f(w) = O(e^{e^{cv}}),$$

where $c < \pi/(\beta-\alpha)$, uniformly throughout T': then

$$\overline{\lim} \, |f(w)| \leqslant \mathbf{a}^{(\beta-\nu)/(\beta-\alpha)} \mathbf{b}^{(\nu-\alpha)/(\beta-\alpha)},$$

when w tends to infinity along **C**. *This result still holds when* **a** *or* **b** *or both are zero.*

on $\mathbf{A}_Q$ and $\mathbf{B}_Q$, and
$$\left|\frac{w}{w+p}\right| < \frac{h+\epsilon}{K}$$
on Q. And if we write
$$F = \frac{wf}{w+p},$$
we have
$$|F| < h+\epsilon$$
at all points on the boundary of L_Q; and therefore* at all points of L_Q.

Hence
$$\overline{\lim}\, |F| \leqslant h,$$
when w tends to infinity in any manner inside L; and therefore
$$\overline{\lim}\, |f| \leqslant h,$$
which proves the lemma.

LEMMA γ.—*Suppose that* (2.31) *is satisfied on both* **A** *and* **B**, *and that*

(2.34)
$$\overline{\lim}\, |f(w)| \leqslant \delta,$$
where $0 < \delta < h$, *on* **C**. *Then*

(2.35)
$$\overline{\lim}\, |f(w)| \leqslant \sqrt{(\delta h)},$$
when w tends to infinity along **M**, *the straight line equidistant between* **A** *and* **B**.

We denote by M_L and M_R those parts of M which lie in L and R respectively.

Write
$$g = e^{kw} f,$$
where
$$e^{k\pi} = h/\delta.$$
Then
$$\overline{\lim}\, |g| \leqslant h,$$
on A, and
$$\overline{\lim}\, |g| \leqslant e^{k\pi}\delta = h,$$
on **C**. Hence
$$\overline{\lim}\, |g| \leqslant h,$$
on M_L; and so
$$\overline{\lim}\, |f| \leqslant he^{-\frac{1}{2}k\pi} = \sqrt{(\delta h)},$$
on M_L. Similarly, using an auxiliary function
$$g = e^{k(\pi - w)} f,$$
we can show that (2.35) holds on M_R, and so on the whole of M.

LEMMA δ.—*If $f(w) = O(1)$ throughout T, and $f(w) = o(1)$ on* **C**, *then $f(w) = o(1)$ on M.*

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* Phragmén and Lindelöf, *l.c.*, p. 388.

Write
$$g = f+\delta,$$
where $0<\delta<1$. Then we can choose an h independent of δ and such that the conditions of Lemma γ are satisfied; and
$$\overline{\lim}\,|g| \leqslant \sqrt{(\delta h)}$$
on M. Hence
$$\overline{\lim}\,|f| \leqslant \sqrt{(\delta h)}+\delta$$
on M; which proves the theorem, since δ is arbitrarily small.

2.4. We now transform Lemma δ, by means of the theory of conformal representation, into a proposition suitable for direct application to the theory of power-series.

The equation
$$\frac{x-1}{x+1} = ie^{iw}$$
transforms the strip $0 \leqslant v \leqslant \pi$ into the unit circle in the plane of x. The point $w=\frac{1}{2}\pi$ corresponds to $x=0$, the lines **A** and **B** to the upper and lower halves of the circle, and the line M to the real diameter. The upper and lower ends of the strip correspond to $x=1$ and $x=-1$ respectively. If, finally, we observe that
$$\left|\frac{1+x}{1-x}\right| = e^v,$$
we obtain

LEMMA ϵ.—*Suppose that $\Phi(x)$ is regular for $|x|<1$, and continuous and bounded for $|x|\leqslant 1$, $x\neq 1$. Suppose further that $\Phi(x)=o(1)$ when x tends to 1 along a certain internal path C. Then $\Phi(x)=o(1)$ when x tends to 1 by real values.*

2.5. We can now prove our main theorem. In the first place $\Phi(x)$ is continuous at all points in or on the circle, except perhaps the point $x=1$.* We may therefore suppose, without loss of generality, that C is an internal path.†

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* The series
$$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = \Sigma\, O\left(\frac{1}{n^2}\right)$$
is plainly uniformly convergent.

† We can replace C by an internal path C', which differs so little from C that $\Phi(x)$ tends to zero along C'.

Since Φ tends to a limit along C, s_n is bounded, by Lemma α; and therefore, by the same lemma, Φ is bounded for $|x| \leqslant 1$, $x \neq 1$. Hence Φ satisfies all the conditions of Lemma ϵ; and so

$$\Phi(x) = o(1) \tag{2.51}$$

when x tends to 1 along the real axis.

Suppose then that x is real. We have

$$\Phi'(x) = \frac{1}{1-x}\Phi(x) - \frac{1}{1-x}f(x),$$

$$\Phi''(x) = \frac{2}{(1-x)^2}\Phi(x) - \frac{2}{(1-x)^2}f(x) - \frac{1}{1-x}f'(x).$$

Now $\Phi(x)$ is bounded; $f(x)$ is bounded, since s_n is bounded; and

$$f'(x) = \Sigma n a_n x^{n-1} = \Sigma O(1)\, x^n = O\left(\frac{1}{1-x}\right).$$

Hence

$$\Phi''(x) = O\left\{\frac{1}{(1-x)^2}\right\}. \tag{2.52}$$

From (2.51) and (2.52), we deduce, by Theorem 8 of our paper **1**,

$$\Phi'(x) = o\left(\frac{1}{1-x}\right);$$

and so
$$f(x) = \Phi(x) - (1-x)\Phi'(x) = o(1).$$

That is to say, *Abel's limit for $f(x)$ exists when $x \to 1$ by real values;* and therefore, by Theorem **P**, the series Σa_n is convergent.

Theorems **R** *and* **S**.

2.6. When C is *regular*,

$$f(x) \to A \tag{L}$$

implies

$$\Phi(x) \to A. \tag{Λ}$$

We have therefore

THEOREM **R**.—*If (O) is satisfied, and $f(x) \to A$ when $x \to 1$ along some regular path C, then Σa_n converges to the sum A.*

This theorem of course includes **P** as a special case. It is in the most

essential respects as complete a generalisation of **F** as could be desired, for, although it involves a considerable limitation as to the nature of C, there is no limitation at all on the contact of C with the circle. The contact may be as close as we please, or C may be the circle itself: and it is questions of contact rather than of regularity which are of the first interest in these investigations.

It is, however, natural to suppose that *no* limitation on C is necessary; and, if this be so, it is very desirable that it should be proved. It would seem, however, that some important change in our argument would be needed for such a proof; for, unless C is subject to considerable restrictions, it is not possible to deduce the behaviour of Φ from that of f by means of its representation as an integral taken along C.

It is interesting to observe that, if we limit C to be a Stolz-path, we can get rid of the restriction that it is to be regular. We shall only sketch the proof, the general lines of which are as follows. We draw two chords of the circle through $x = 1$, including C between them, and we consider the region T formed by points which lie between these chords and within a certain distance of $x = 1$.

By a slight modification of the proof of Theorem **E** given by Landau, we can prove that s_n is bounded, and so that $f(x)$ is bounded in T.* And by an adaptation of the arguments used in §§ 2.3–4, we can show that f tends to a limit when $x \to 1$ along any regular Stolz-path inside T. If the real axis satisfies this condition, we can appeal to Theorem **F**; if not, to Theorem **P** or **R**. In any case, we obtain

THEOREM **S**.—*If* (O) *is satisfied and* $f(x) \to A$ *when* $x \to 1$ *along some Stolz-path* C, *then* Σa_n *converges to the sum* A.

3.

THE ABELIAN THEOREMS.

Proof of Theorem **T**.

3.1. THEOREM **T**.—*If*

(O) $$a_n = O\left(\frac{1}{n}\right)$$

and

(K) $$\Sigma a_n = A,$$

* The argument fails at this stage if C is not a Stolz-path. We could not prove that f is bounded in T if the boundaries of T touched the circle.

then

$$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1-x^{n+1}) \to A, \tag{A}$$

when $x \to 1$ *in any manner.*

One observation should be made before we proceed to the proof. The enunciation contains no reference to a *path*: we may say, of course, "when $x \to 1$ along any path C"; but the idea of approach along a continuous path is not really relevant. In this respect there is an essential difference between this theorem and the Tauberian theorems.

We may suppose, as before, that $A = 0$ and $|na_n| < 1$. That $\Phi(x)$ is *bounded* follows from Lemma α; but to prove convergence to a limit requires an argument of somewhat greater subtlety.

What we have to prove is that

$$S(x) = \Sigma \frac{a_n}{n+1}(1-x^{n+1}) = o(1-x). \tag{3.11}$$

We write

$$S = \sum_0^{m-1} + \sum_m^{\infty} = S_1 + S_2, \tag{3.12}$$

where

$$m = K\nu = K\left[\frac{1}{|1-x|}\right], \tag{3.13}$$

K being a (large) parameter. It is plain that we may suppose $|1-x|$ small enough to ensure that

$$\frac{K}{2|1-x|} < m \leqslant \frac{K}{|1-x|}. \tag{3.14}$$

As regards S_2, we have

$$|S_2| < 2\sum_m^{\infty} \frac{1}{n(n+1)} = \frac{2}{m} < \frac{4}{K}|1-x|. \tag{3.15}$$

In order to deal with S_1, we require a lemma.

3.2. **Lemma** ζ.—*If* Σa_n *is convergent, and*

$$t_n = \sum_n^{\infty} \frac{a_\mu}{\mu+1},$$

then (i) $$t_n = o\left(\frac{1}{n}\right),$$

and (ii) Σt_n *converges to the same sum as* Σa_n.

The proof of this lemma is very simple. We suppose, as before, that $A = 0$. We can choose n_0 so that

$$|a_n + a_{n+1} + \ldots + a_\mu| < \delta,$$

for $n_0 \leqslant n < \mu$, and then

$$(3.21) \qquad |t_n| = \left| \sum_n^\infty \frac{a_n + a_{n+1} + \ldots + a_\mu}{(\mu+1)(\mu+2)} \right| < \frac{\delta}{n+1}.$$

The same argument shows that

$$(3.22) \qquad |t_n| < \frac{\mathbf{m}}{n+1},$$

for all values of n, $\mathbf{m}$ being the maximum of $|a_n + a_{n+1} + \ldots + a_\mu|$ for all values of n and μ.

Finally, we have

$$\sum_0^n t_\mu = \sum_0^n a_\mu + (n+1) \sum_{n+1}^\infty \frac{a_\mu}{\mu+1},$$

and the last term tends to zero. We have thus

$$(3.23) \qquad \left| \sum_0^n t_\mu \right| < \delta,$$

for $n \geqslant n_1$.

3.3. Now

$$(3.31) \qquad S_1 = \sum_0^{m-1} \frac{a_n}{n+1}(1 - x^{n+1}) = \sum_0^{m-1} (t_n - t_{n+1})(1 - x^{n+1})$$

$$= (1-x) \sum_0^{m-1} t_n - (1-x) \sum_0^{m-1} t_n (1 - x^n) - t_m (1 - x^m)$$

$$= S_1' + S_1'' + S_1''',$$

say. Plainly

$$(3.32) \qquad |S_1'''| < \frac{2\delta}{m} < \frac{4\delta}{K} |1-x|,$$

if $m \geqslant n_0$, which is certainly so when $|1-x|$ is sufficiently small. Secondly

$$(3.33) \qquad |S_1'| < \delta |1-x|,$$

by (3.23), if $m \geqslant n_1$, which is certainly so when $|1-x|$ is sufficiently small.

Finally,

$$(3.34)\quad |S_1''| < |1-x|^2 \sum_0^{m-1} n\,|t_n| = |1-x|^2 \sum_0^{n_0-1} n\,|t_n| + |1-x|^2 \sum_{n_0}^{m-1} n\,|t_n|$$

$$< \mathbf{m} n_0 |1-x|^2 + \delta m |1-x|^2$$

$$< \mathbf{m} n_0 |1-x|^2 + 2K\delta |1-x|.$$

From (3.12), (3.15), and (3.31)–(3.34), it follows that

$$(3.35)\qquad |S| < |1-x| \left(\frac{4}{K} + 2K\delta + \delta + \frac{4\delta}{K} + \mathbf{m} n_0 |1-x|\right).$$

We can choose $K = K(\epsilon)$ so that $4/K < \frac{1}{3}\epsilon$; then $\delta = \delta(\epsilon, K) = \delta(\epsilon)$ so that

$$2K\delta + \delta + \frac{4\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) = \eta(\epsilon)$ so that (3.35) is satisfied, and

$$\mathbf{m} n_0 |1-x| < \tfrac{1}{3}\epsilon$$

if $|1-x| < \eta$. Thus $\qquad |S| < \epsilon |1-x|$

if $|1-x| < \eta$; which proves the theorem.

Combining Theorems **Q** and **T**, we obtain

THEOREM **U**.—*If the coefficients of the series Σa_n satisfy (O), then the necessary and sufficient condition for its convergence is that $\Phi(x)$ should tend to a limit, either along any particular path C, or along all.*

Proof of Theorems **V** *and* **W**.

3.4. Theorem **T** leaves an important question unanswered. Is it certain that the hypotheses do not imply, what is more* than the theorem asserts, that

$$(L)\qquad f(x) \to A$$

along C?

Theorem **B** shows that this is so when C is a Stolz-path: in this case

* (L) asserts more than (Λ) at any rate when C is regular, that is to say in all ordinary cases (cf. pp. 219–220).

indeed the convergence of the series ensures the existence of the limit without any hypothesis as to the order of a_n. Theorem **C** shows that this at any rate ceases to be true when C is allowed to touch the circle. In the example which we attached to our statement of Theorem **C**, we have to take

$$1-a < b \leqslant 1-\tfrac{1}{2}a;$$

and these inequalities allow b to exceed any number less than 1. It follows* that no condition of the type $a_n = O(n^{-b})$, where $b < 1$, is enough, in conjunction with the convergence of the series, to ensure the existence of Abel's limit along all tangential paths.

We shall now show that even the condition

$$(o) \qquad a_n = o\left(\frac{1}{n}\right)$$

is not enough for this purpose. It would be easy to modify our argument in such a manner as to show that *no* condition of the form

$$a_n = O(\chi_n),$$

where χ_n is a steadily decreasing function such that $\Sigma\chi_n$ is divergent, is enough: for simplicity, however, we take

$$\chi_n = \frac{1}{n \log n}.$$

THEOREM **V**.—*It is possible to find a convergent series Σa_n for which*

$$a_n = O\left(\frac{1}{n \log n}\right),$$

and a regular path C, such that $f(x)$ does not tend to a limit when x tends to 1 along C.

We take
$$a_n = \frac{1}{n \log n} \sin\frac{n\pi}{j},$$
if
$$n_j = e^{e^{e^j}} < n < n_{j+1} = e^{e^{e^{j+1}}}.$$

To prove that Σa_n is convergent, we write

$$w_{j,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n, \qquad w_j = \sum_{n_j < n < n_{j+1}} a_n,$$

---

* Compare p. 208, where we used the same series in a different manner to establish the same point about the Tauberian Theorem **F**

so that $$w_j = w_{j,\,[n_{j+1}]}.$$

Since $n \log n$ increases steadily with n, and $\Sigma \sin(n\pi/j)$, taken between any limits, is not greater in absolute value than $2 \operatorname{cosec}(\pi/j)$, we have

$$w_{j,\,k} = O\left(\frac{j}{n_j \log n_j}\right), \qquad w_j = O\left(\frac{j}{n_j \log n_j}\right).$$

Since $$\Sigma \frac{j}{n_j \log n_j}$$

is convergent, Σw_j is absolutely convergent; and since $w_{j,k} = o(1)$, Σa_n is convergent.

Now let $$f(x) = \Sigma a_n x^n,$$

$$\phi(r,\theta) = \mathbf{I}[f(re^{i\theta})] = \Sigma a_n r^n \sin n\theta.$$

We shall show that $\phi(r,\theta)$ does not tend to a limit when $x \to 1$ along a regular path which has sufficiently close contact with the upper half of the unit circle.

To prove this we observe first that the series

$$\Sigma a_n \sin n\theta \tag{3.41}$$

is convergent when θ is positive. In fact, if we write

$$u_{j,\,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n \sin n\theta, \qquad u_j = \sum_{n_j < n < n_{j+1}} a_n \sin n\theta,$$

and j is so large that $\pi/j < \frac{1}{2}\theta$, we find, by the same argument that was used above, that

$$u_{j,\,k} = \tfrac{1}{2} \Sigma \frac{1}{n \log n} \left\{ \cos n\left(\theta - \frac{\pi}{j}\right) - \cos n\left(\theta + \frac{\pi}{j}\right) \right\}$$

$$= O\left(\frac{1}{\theta n_j \log n_j}\right),$$

and in particular that $$u_j = O\left(\frac{1}{\theta n_j \log n_j}\right).$$

From these relations the convergence of (3.41) follows immediately.

If now we denote the sum of (3.41) by $\psi(\theta)$, we have

$$\psi(\theta) = \lim_{r \to 1} \phi(r, \theta),$$

for every positive θ. It is thus sufficient to show that $\psi(\theta)$ does not tend to a limit when $\theta \to 0$. Now

$$(3.42)\quad \psi\left(\frac{\pi}{j}\right) = \sum_{n_j}^{n_{j+1}} \frac{1}{n \log n} \sin^2 \frac{n\pi}{j} + \sum_{k \neq j} \sum_{n_k}^{n_{k+1}} \frac{1}{n \log n} \sin \frac{n\pi}{j} \sin \frac{n\pi}{k}$$

$$= \psi_1 + \psi_2,$$

say. Suppose j even. Then, if j is large, there are, between n_j and n_{j+1}, more than $n_{j+1}/2j$ numbers n of the form $(m+\frac{1}{2})j$, where m is an integer; and the sum

$$\Sigma \frac{1}{n \log n},$$

extended to these values of n, is greater than

$$\frac{1}{4j} \log \left(\frac{\log n_{j+1}}{\log n_j}\right) = \frac{e-1}{4j} e^j.$$

Thus

$$(3.43)\quad |\psi_1| > \frac{e-1}{4j} e^j.$$

On the other hand, the inner sum in ψ_2 is less than a constant multiple of

$$\frac{jk}{|j-k|} \frac{1}{n_k \log n_k} \leqslant \frac{jk}{n_k \log n_k},$$

and so

$$(3.44)\quad |\psi_2| = O\left(j \Sigma \frac{k}{n_k \log n_k}\right) = O(j).$$

Finally, from (3.42)–(3.44), it follows that

$$\psi\left(\frac{\pi}{j}\right) = \psi_1 + \psi_2 \to \infty,$$

which proves the theorem.

It is hardly necessary to point out that Theorem V greatly increases

the interest of Theorem **T**, and indeed of Theorem 50 of our paper **1**, of which Theorem **T** is a generalisation.

3.5. We can easily adapt the example by which we proved Theorem **V**, so as to prove

THEOREM **W**.—*It is possible to find a convergent series Σa_n and a regular path C, so that $\Phi(x)$ does not tend to a limit when x tends to 1 along C.*

We take $$a_n = (n+1)\, b_n,$$

where b_n is the a_n of the last paragraph. The argument by which we proved Σb_n convergent is sufficient to establish the convergence of Σa_n.*

Also $$\Phi(x) = \frac{1}{1-x}\,\Sigma b_n(1-x^{n+1}) = \frac{B-xg(x)}{1-x},$$

where $B = \Sigma b_n$ and $g(x) = \Sigma b_n x^n$. And as $g(x)$ does not tend to a limit, $\Phi(x)$ does not do so.

This proves the theorem. It will be observed that there is a great deal to spare in the conclusion: we have proved, in fact, that $\Phi(x)$ assumes values of order greater than that of $1/|1-x|$. The fact is that it ought to be possible to prove that series exist which satisfy the conditions of Theorem **W** and whose coefficients are of order $\phi(n)/n$, where $\phi(n)$ is any function which tends to infinity with n. Such an example would (in conjunction with Theorem **V**) prove that Theorem **T** is a best possible theorem in the same sense as (*e.g.*) Theorem **F**, and that neither hypotheses nor conclusion can be improved upon. We have no doubt that this is true, but we have not succeeded in finding an example to prove our point. In our last example the order of a_n is $1/\log n$, which is far from the limit desired. It is therefore not surprising that we should find that our argument carries us some distance over the mark.

At any rate, however, Theorem **W** is enough to show that, in Theorem **T**, some condition beyond that of mere convergence is essential.

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* The function $$\phi(n) = \frac{n+1}{n\log n}$$
has the properties (i) that it decreases steadily to the limit zero, and (ii) that
$$\Sigma j\phi(n_j)$$
is convergent. These were the only properties of $1/(n\log n)$ used in the proof of the convergence of Σb_n.

4.

FOURIER SERIES.

Proof of Theorem X.

4 . 1. It has been proved by Fatou* that, *if*

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right), \tag{4.11}$$

then the necessary and sufficient condition that the series

$$\tfrac{1}{2}A_0 + \Sigma A_n = \tfrac{1}{2}a_0 + \Sigma (a_n \cos n\theta + b_n \sin n\theta), \tag{4.12}$$

which is certainly the Fourier series of a summable function $f(\theta)$†, *should converge to the sum* A, *is that*

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A \tag{4.13}$$

when $\alpha \to 0$, *or* (what is the same thing) *that*

$$\Sigma A_n \frac{\sin n\alpha}{n\alpha} \to A \tag{4.14}$$

when $\alpha \to 0$.

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* Fatou, *l.c.* Fatou does not state the whole of this result explicitly as one theorem, but it is contained in pp. 345–7, 385–7 of his memoir. It is important to observe that, if the conditions (4 . 11) are satisfied and $F(\theta)$ is the integral of $f(\theta)$—or, what is the same thing, the sum of the series obtained by integrating (4 . 12) term-by-term—then

$$F(\theta+\alpha) + F(\theta-\alpha) - 2F(\theta) = o(\alpha)$$

for every value of θ, in virtue of a well known theorem of Riemann (quoted by Fatou, *l.c.*, p. 385). It follows that the three formulæ

$$\frac{1}{\alpha}\int_{\theta}^{\theta+\alpha} f(t)\,dt \to A, \qquad \frac{1}{\alpha}\int_{\theta-\alpha}^{\theta} f(t)\,dt \to A, \qquad \frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

are equivalent. This ceases to be true when the conditions (4 . 11) are replaced by the more general conditions (4 . 21), as appears at once from the simple example of the series

$$\Sigma \frac{\sin n\theta}{n}.$$

† By the "Riesz-Fischer Theorem", $\Sigma(a_n^2 + b_n^2)$ being convergent. In fact

$$\Sigma(|a_n|^{1+\delta} + |b_n|^{1+\delta})$$

is convergent for every positive δ, from which it follows that $|f(\theta)|^k$ is summable for all positive values of k. See W. H. Young, "On the determination of the summability of a function by means of its Fourier constants", *Proc. London Math. Soc.*. Ser. 2, Vol. 12, 1912, pp. 71–88.

The investigations of Sections 2 and 3 suggest very forcibly that it should be possible to replace the conditions (4 . 11) of Fatou's theorem by the corresponding conditions of the "O" type. We proceed to prove that this is so.

4 . 2. THEOREM **X**.*—*If a_n and b_n are the Fourier constants of a summable function $f(\theta)$, and*

(4 . 21) $$a_n = O\left(\frac{1}{n}\right), \quad b_n = O\left(\frac{1}{n}\right),$$

then the necessary and sufficient condition that the Fourier series of $f(\theta)$ should converge to the sum A is that

(4 . 22) $$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

when $\alpha \to 0$.

We may plainly take $A = 0$ and suppose that $|nA_n| < 1$.

In the first place, the condition is sufficient. This may be proved in a variety of manners by a mere combination of known theorems.

(*a*) We may prove, by using Poisson's integral in precisely the same way as Fatou, that
$$\Sigma A_n r^n \to 0,$$
when $r \to 1$. The convergence of ΣA_n then follows from Theorem **F**.†

(*b*) It was proved by Lebesgue‡ that the Fourier series of $f(\theta)$ is summable $(C, 2)$, to sum A, for any value of θ for which (4 . 22) is satisfied. The result then follows from the theorem that a series whose general term is of order $1/n$ cannot be summable by Cesàro's means unless it is convergent.§

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* We have already published a proof of this theorem, by a different method, in the *Comptes rendus* of December 24th, 1917 (our paper **5**).

† Fatou, of course, uses Theorem **D**.

‡ H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, Vol. 61, 1905, pp. 251–280 (p. 278).

§ G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320.

(*c*) It has been shown by W. H. Young* that if the Fourier series of

$$g(a) = \frac{1}{\sin \frac{1}{2}a} \int_{\theta-a}^{\theta+a} f(t)\,dt$$

is summable (C, r) for $a = 0$, then the Fourier series of $f(\theta)$ is summable $(C, r+1)$. If (4.23) is satisfied, $g(a)$ is continuous for $a = 0$, and we can take $r = 1$. The proof may then be completed as under (*b*).

4.3. We have now to prove that the condition is also necessary. We write†

(4.31) $$\Phi = \sum_1^\infty A_n \frac{\sin na}{na} = \sum_1^m + \sum_{m+1}^\infty = \Phi_1 + \Phi_2,$$

say, where $$m = [K/a],$$

so that $K/2a < m \leqslant K/a$ for all sufficiently small values of a. Then

(4.32) $$|\Phi_2| < \frac{1}{a} \sum_{m+1}^\infty \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K}.$$

Again, if we write $$t_n = \sum_n^\infty \frac{A_\mu}{\mu},$$

we have (as in § 3.2)

(4.33) $$|t_n| < \frac{\mathrm{m}}{n}, \quad |t_n| < \frac{\delta}{n} \ (n \geqslant n_0), \quad \left| \sum_1^n t_\mu \right| < \delta \ (n \geqslant n_1).$$

Now

(4.34) $$\Phi_1 = \sum_1^m A_n \frac{\sin na}{na} = \frac{1}{a} \sum_1^m (t_n - t_{n+1}) \sin na$$

$$= \frac{\sin a}{a} \sum_1^m t_n - \frac{\sin a}{a} \sum_1^m t_n \{1 - \cos(n-1)a\}$$

$$- \frac{1-\cos a}{a} \sum_1^m t_n \sin(n-1)a - \frac{1}{a} t_{m+1} \sin ma$$

$$= \Phi_1' + \Phi_1'' + \Phi_1''' + \Phi_1'''',$$

* W. H. Young, "On the Convergence of a Fourier series and its allied series", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 254–272 (pp. 262–266). Young's argument depends only on a series of elementary identities, and includes a new and greatly simplified proof of Lebesgue's theorem quoted above.

† The proof which we give here is (if hardly shorter) considerably simpler in principle than that which we gave in **5**. Naturally Theorem **T** can also be proved by an adaptation of our former method.

say. In the first place

(4 . 35) $$|\Phi_1''''| < \frac{\delta}{ma} < \frac{2\delta}{K},$$

if $m \geqslant n_0$, and

(4 . 36) $$|\Phi_1'| < \delta,$$

if $m > n_1$; and each of these conditions is satisfied when a is sufficiently small. Secondly,

(4 . 37) $$|\Phi_1'''| < \tfrac{1}{2}a \sum_1^m |t_n| < a\mathbf{m} \log m < a\mathbf{m} \log \frac{K}{a}.$$

Finally,

(4 . 38) $$|\Phi_1''| < \tfrac{1}{2}a^2 \sum_1^m n^2 |t_n| = \tfrac{1}{2}a^2 \left(\sum_1^{n_0-1} + \sum_{n_0}^m \right) n^2 |t_n|$$

$$< \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}a^2\delta m^2 < \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}K^2\delta.$$

From (4 . 31), (4 . 32), and (4 . 34)–(4 . 38), we deduce

(4 . 39) $$|\Phi| < \frac{2}{K} + \tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} + a\mathbf{m} \log \frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2.$$

Given ϵ, we choose $K(\epsilon)$ so that $2/K < \tfrac{1}{3}\epsilon$; then $\delta(\epsilon, K) = \delta(\epsilon)$ so that

$$\tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) \doteq \eta(\epsilon)$ so that (4 . 39) is satisfied, and

$$a\mathbf{m} \log \frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2 < \tfrac{1}{3}\epsilon,$$

if $0 < a < \eta$. We have then $|\Phi| < \epsilon$,

if $0 < a < \eta$, and the theorem is proved.

4 . 4. The condition (4 . 22) is certainly satisfied if $F(\theta)$ has a differential coefficient equal to A. But the convergence of ΣA_n does *not* involve the existence of such a differential coefficient. Thus, if

$$f(\theta) = \Sigma \frac{\sin n\theta}{n},$$

and we consider the particular value $\theta = 0$, the sum of the series is zero, and

$$\frac{1}{2a} \int_{-a}^{a} f(t)\, dt \to 0.$$

But, if a is positive,

$$\frac{1}{a}\int_0^a f(t)\,dt \to \tfrac{1}{2}\pi, \qquad \frac{1}{a}\int_{-a}^0 f(t)\,dt \to -\tfrac{1}{2}\pi.$$

In this respect Theorem **X** differs essentially from Fatou's theorem of which it is the generalisation.

Supplementary Remarks.

4 . 5. We shall conclude the paper with a few theorems of a slightly different character.

The characteristic properties of series which satisfy the condition (o) or (O) are shared to a great extent by series Σa_n such that

$$\Sigma n\,|a_n|^2, \tag{4.51}$$

or, more generally,

$$\Sigma n^p\,|a_n|^{p+1} \quad (p \geqslant 1), \tag{4.52}$$

are convergent.* It is easy to see that this is true of the properties which have been discussed in this paper.

Let us suppose, for example, that the series (4 . 51) is convergent; and let us return to the proof of Lemma α.

We can choose m so that

$$\sum_m^\infty n\,|a_n|^2 < \epsilon. \tag{4.53}$$

This being so, we have

$$|\phi_1| \leqslant \frac{2}{|1-x|}\sum_\nu^\infty \frac{|a_n|}{n+1} \leqslant \frac{2}{|1-x|}\sqrt{\left(\sum_\nu^\infty n\,|a_n|^2 \sum_\nu^\infty \frac{1}{n(n+1)^2}\right)}$$

$$\leqslant \frac{2\sqrt{\epsilon}}{\nu\,|1-x|} < 4\sqrt{\epsilon}.$$

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* See L. Fejér, "Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf der schlichte Ebene", *H. A. Schwarz Festschrift*, 1914, pp. 42–53; and our paper **3**. It is to be observed that the theorems which depend upon a condition of this type have all the simpler "o" character, and their proofs do not involve the peculiar difficulties of those of the "O" theorems.

Also

(4.54) $$|s_{\nu-1}-\phi_1| \leqslant \tfrac{1}{2}|1-x| \sum_1^m n|a_n| + \tfrac{1}{2}|1-x| \sum_{m+1}^{\nu-1} n|a_n|.$$

The first term on the right hand side of (4.54) is less than $\sqrt{\epsilon}$ if x is near enough to 1. The second does not exceed

$$\tfrac{1}{2}|1-x| \sqrt{\left(\sum_m^{\nu-1} n|a_n|^2 \sum_m^{\nu-1} n\right)} \leqslant \tfrac{1}{2}\nu|1-x|\sqrt{\epsilon} < \sqrt{\epsilon}.$$

Hence $$|s_{\nu-1}-\Phi(x)| < 6\sqrt{\epsilon},$$

if x is near enough to 1. As C is continuous, ν passes through an unbroken sequence of integral values as $x \to 1$. We thus obtain

THEOREM **U1**.—*If $\Sigma n|a_n|^2$ is convergent, then the necessary and sufficient condition that Σa_n should be convergent is that $\Phi(x)$ should tend to a limit when x tends to* 1, *either along any particular path C, or along all.*

There is no difficulty in proving the more general result which holds when the series (4.52) is convergent. It is only necessary to use the generalised form of the Cauchy-Schwarz inequality.*

Similarly we have

THEOREM **X1**.—*If $\Sigma na_n^2+b_n^2$ is convergent, then the necessary and sufficient condition for the convergence of the series*

$$\tfrac{1}{2}a_0+\Sigma(a_n\cos n\theta+b_n\sin n\theta) \sim f(\theta)$$

is that $$\frac{1}{2a}\int_{\theta-a}^{\theta+a} f(t)\,dt$$

should tend to a limit when $a \to 0$.

4.6. We have found that the condition (4.13) is both necessary and sufficient for the convergence of two important classes of Fourier series. There is a third class for whose convergence it is a *necessary*, though not a sufficient condition. This is the class of *Fourier series of bounded functions*.

* Cf. **3**, p. 136.

We have, in fact,

THEOREM Y.—*If $f(\theta)$ is a summable function, bounded in the neighbourhood of a particular value of θ, and if the Fourier series of $f(\theta)$ is convergent for that value of θ, then*

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt$$

tends to a limit when $\alpha \to 0$.

Suppose for simplicity that $a_0 = 0$, and let

$$F_1(\theta) = -\Sigma \frac{a_n \cos n\theta + b_n \sin n\theta}{n^2}$$

be the second integral of $f(\theta)$. Then, if the series converges to the sum zero,

$$\phi(\alpha) = F_1(\theta+\alpha)+F_1(\theta-\alpha)-2F_1(\theta) = o(\alpha^2) \tag{4.61}$$

when $\alpha \to 0$, in virtue of another well known theorem of Riemann.*

Now $$\phi'(\alpha) = F(\theta+\alpha)-F(\theta-\alpha) = \int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt.$$

If $f(\theta)$ were everywhere the differential coefficient of its integral, we should have

$$\phi''(\alpha) = f(\theta+\alpha)+f(\theta-\alpha) = O(1),$$

since $f(t)$ is bounded in the neighbourhood of θ; and from (4.61) and the last equation it would follow at once that

$$\phi'(\alpha) = o(\alpha),$$

proving our point. This is not now a valid proof. But it is easy to see that, in such a theorem as

"$\phi(\alpha) = o(\alpha^2)$ *and* $\phi''(\alpha) = O(1)$ *imply* $\phi'(\alpha) = o(\alpha)$",

the second condition may be replaced by the more general condition expressed by the inequality

$$|\phi'(\beta)-\phi'(\gamma)| < K|\beta-\gamma| \dagger :$$

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* See, for example, de la Vallée Poussin, *Cours d'Analyse*, ed. 2, Vol. 2, p. 172.

† See Landau, "Einige Ungleichungen für zweimal Differentiierbare Functionen", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1914, pp. 43–49.

a condition obviously fulfilled in this case, since ϕ' is the integral of a bounded function.

Thus Theorem **Y** is proved. It is, in fact, but a special case of

THEOREM **Z**.—*The Fourier series of a function $f(\theta)$, bounded in the neighbourhood of the particular value of θ under consideration, is either summable by Cesàro means of arbitrarily small positive order, or summable by no Cesàro mean of any order. The necessary and sufficient condition that it should be summable is expressed by the condition* (4.13).

The proof of this theorem would, however, carry us too far from the proper subject of the paper.*

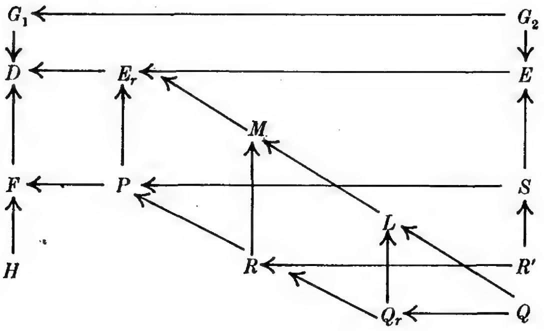
* We have published a sketch of the proof, under the title "On the Fourier series of a bounded function", in the *Records of Proceedings at Meetings* for December 6th, 1917 (*Proc. London Math. Soc.*, Vol. 17, p. xiii).

CORRECTIONS

p. 206, *line* 13. For 'a segment' read 'an arc'.
p. 207, *line* 6. For 'an arc of a circle' read 'an arc on the lower half of a circle'.
p. 209, *line* 8. For 'and **E**' read 'and **E** when C is a regular Stolz-path'.
p. 213, *line* 4. For $(\delta\rho)^p$ read $(\delta i\rho)^p$.
—— *lines* 5, 6, *and* 12. For δ read $|\delta|$.
—— *line* 4 *up*. For $\rho > 0$ read $0 < \rho < \rho_0$.
p. 215, *line* 9 *up*. For ϕ_2 read ϕ_1.
p. 217, *line* 1. For $\mathbf{B}_Q$ read $\mathbf{C}_Q$.
—— *line* 13. For **M** read M.
—— *line* 8 *up*. For A read **A**.
p. 218, *line* 11. For $0 \leqslant v \leqslant \pi$ read $0 \leqslant u \leqslant \pi$.
p. 223, *line* 10 *up*. For 'coefficients' read 'terms'.
p. 225, *line* 1. Read $w_j = w_{j,[n_j+1]}$.
—— *line* 8 *up*, and *p.* 226, *line* 3. For 'θ is positive' and 'every positive θ', read $0 < \theta < \pi$.

COMMENTS

The relations between the various Tauberian theorems discussed in this paper are indicated in the following diagram.



In the first column, D, G_1, F, and H are Tauber's 1st and 2nd theorems, Littlewood's O-theorem and Hardy and Littlewood's O_L-theorem. In the 2nd and 3rd horizontal lines, D and F are extended to regular and to arbitrary Stolz-paths. In the 1st column, the path is the radius $(0, 1)$. In E_r and P the path is regular (Stolz). In M, R, L, and Q_r it is regular. In the last column the path is arbitrary (Stolz) in E, G_2, and S, and arbitrary in R' and Q. In M, R, and R', the limit along C is an ordinary limit. In L, Q_r, and Q, it is the limit of a mean. E_r, which is E with a *regular* Stolz-path, is inserted to correct the statement in § 1.3 that M implies E. Q_r is Q with a *regular* path, and is inserted for completeness. The Tauberian condition in G_1 and G_2 is Tauber's 2nd condition (ω). In D, E_r, E, M, and L, the condition is Tauber's 1st condition (o). In F, P, S, R, R', Q_r, and Q, the condition is (O). In H it is (O_L).

R' is not in the present paper, but is taken from 1924, 7. It is the culminating Tauberian theorem in which the limit along C is an *ordinary limit*. Q has a companion Abelian theorem T. Together they show (Theorem U) that: *when* (O) *holds,* $\sum a_n$ *is convergent if and only if the mean tends to a limit along an arbitrary path.* This result extends Theorems 49 and 50 of 1913, 2; see Comments on 1913, 2, Section V. In 1924, 7, Hardy and Littlewood show (Theorem T′) that, in the Abelian part of U, the condition (O) cannot be weakened to $na_n = O(\phi(n))$, where $\phi(n)$ increases to ∞, however slowly. Littlewood† had shown the same in F.

The series
$$\sum n^{-b} e^{in^a} \quad (0 < a < 1,\ b > 0),$$
used in §§ 1.2 and 3.4, is investigated by Hardy in 1913, 6 (in Vol. IV). He proves that, if $f(z)$ is the analytic continuation of the associated power series throughout the complex plane cut along $(1, \infty)$, then as z approaches 1 from below, on a curve touching the unit circle at $z = 1$,
$$f(z) = (1+o(1))A+(1+o(1))B(\log 1/z)^{-(1-b-\frac{1}{2}a)/(1-a)}.\exp\{Pi^{1/(1-a)}(\log 1/z)^{-a/(1-a)}\},$$
where $P > 0$. He remarks that, in particular, if z approaches 1 along the lower arc of the unit circle, $z = e^{-i\theta}$, $0 < \theta < \pi$, and $(1-b-\frac{1}{2}a) \geqslant 0$, then
$$f(e^{-i\theta})-A \sim B(i\theta)^{-(1-b-\frac{1}{2}a)/(1-a)}.\exp\{Pi\theta^{-a/(1-a)}\} \quad \text{as } \theta \to +0,$$
so that $f(z)$ does not tend to a limit. On the other hand, suppose that z approaches 1 along the lower arc of a circle with centre on $(0, 1)$, say at $z = \frac{1}{2}$, so that the circle is $z = \cos\theta e^{-i\theta}$. Then $\log 1/z = i\theta+\frac{1}{2}\theta^2+o(\theta^2)$, and the behaviour of $f(z)$ is similar.

† *Proc. London Math. Soc.* (2), 9 (1911), 434–48; see also Ingham, *Proc. London Math. Soc.* (2), 23 (1925), 326–36.

A theorem concerning summable series. By Prof. G. H. HARDY.

[*Received* 23 December 1920. *Read* 7 February 1921.]

1. It is well known that *if the series* Σa_n *is summable* $(C, 1)$, that is to say if

$$s_n' \sim An \qquad (1),$$

where $s_n = a_0 + a_1 + \dots + a_n, \quad s_n' = s_0 + s_1 + \dots + s_n,$

then

$$\Sigma \frac{a_n}{n+1} = \frac{a_0}{1} + \frac{a_1}{2} + \dots \qquad (2)$$

*is convergent**. The converse is not true, as may be seen at once from trivial instances to the contrary. It is therefore interesting to frame a theorem of this kind which embodies a *necessary and sufficient* condition for summability. Such a theorem is the following.

Theorem. *The necessary and sufficient condition that* Σa_n *should be summable* $(C, 1)$ *to sum* A *is that*

$$s_n + (n+1) b_{n+1} \to A \qquad (3),$$

where

$$b_n = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \dots \qquad (4).$$

2. It is plain that we may (replacing a_0 by $a_0 - A$) suppose without loss of generality that $A = 0$. If

$$s'_n = o(n), \quad s_n = s'_n - s'_{n-1} = o(n).$$

Again, (3) and (4) involve the convergence of (2), *i.e.* involve $t_n \to B$, where $a_n = (n+1) c_n$ and $t_n = c_0 + c_1 + \dots + c_n$. And

$$s_n = \sum_0^n (\nu+1) c_\nu = (n+1) t_n - \sum_0^{n-1} t_\nu = o(n).$$

Hence we may suppose $s_n = o(n)$ in proving either part of the theorem.

This being so, we have

$$b_{n+1} = \sum_{n+1}^{\infty} \frac{s_\nu - s_{\nu-1}}{\nu+1} = \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s_\nu - s_{\nu-1}}{\nu+1}$$

$$= \lim_{m\to\infty} \left(\frac{s_m}{m+1} - \frac{s_n}{n+2} + \sum_{n+1}^{m-1} \frac{s_\nu}{(\nu+1)(\nu+2)} \right)$$

$$= -\frac{s_n}{n+2} + \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)},$$

$$s_n + (n+1) b_{n+1} = \frac{s_n}{n+2} + (n+1) \sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)}.$$

* H. Bohr, 'Bidrag til de Dirichlet'ske Rækkers Theori,' Inaugural Dissertation (Copenhagen, 1910), p. 100. The theorem follows at once from (1) and the identity

$$\sum_0^n \frac{a_\nu}{\nu+1} = 2 \sum_0^{n-2} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} + \frac{s'_{n-1}}{n(n+1)} + \frac{s'_n - s'_{n-1}}{n+1}.$$

1921, 4 Proceedings of the Cambridge Philosophical Society, 20 304–7.

The condition (3) is therefore equivalent to

$$\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} = o\left(\frac{1}{n}\right) \qquad \text{(5)}.$$

3. Suppose first that Σa_n is summable, to sum zero, *i.e.* that $s_n' = o(n)$. Then

$$\sum_{n+1}^{\infty} \frac{s_\nu}{(\nu+1)(\nu+2)} = \lim_{m\to\infty} \sum_{n+1}^{m} \frac{s'_\nu - s'_{\nu-1}}{(\nu+1)(\nu+2)}$$

$$= \lim_{m\to\infty} \left(\frac{s'_m}{(m+1)(m+2)} - \frac{s'_n}{(n+2)(n+3)} + 2\sum_{n+1}^{m-1} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)}\right)$$

$$= 2\sum_{n+1}^{\infty} \frac{s'_\nu}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s'_n}{(n+2)(n+3)}$$

$$= 2\sum_{n+1}^{\infty} o\left(\frac{1}{\nu^2}\right) - o\left(\frac{1}{\nu}\right) = o\left(\frac{1}{\nu}\right) \qquad \text{(6)}.$$

Hence (5), and therefore (3), is a *necessary* condition for summability.

4. Suppose now that (5) is satisfied. Then, by (6),

$$2\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{(n+2)(n+3)} = o\left(\frac{1}{n}\right) \quad \text{...(7)}.$$

Writing

$$\phi_n = (n+2)(n+3)\sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)},$$

we obtain

$$2\phi_n - s_n' = o(n) \qquad \text{(8)}.$$

But

$$\phi_n - \phi_{n-1} = 2(n+2)\,\phi_n \sum_{n+1}^{\infty} \frac{s_\nu'}{(\nu+1)(\nu+2)(\nu+3)} - \frac{s_n'}{n+3}$$

$$= \frac{2\phi_n - s_n'}{n+3} = o(1),$$

by (8); and therefore $\phi_n = o(n)$ and $s_n' = o(n)$, so that the series is summable to sum 0. Thus the theorem is proved.

5. In order to show that the theorem is not without application, I apply it to the deduction of two known convergence criteria*.

* See (for A) L. Fejér, 'La convergence sur son cercle de convergence d'une série de puissances effectuant une représentation conforme du cerc le sur le plan simple,' *Comptes Rendus*, 6 Jan. 1913, and 'Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene,' *H. A. Schwarz Festschrift*, 1914, pp. 42–53; G. H. Hardy and J. E. Littlewood, 'Some theorems concerning Dirichlet's series,' *Messenger of Mathematics*, vol. 43, 1914, pp. 134–147: and (for B) G. H. Hardy 'Theorems relating to the summability and convergence of slowly oscillating series,' *Proc. London Math. Soc.*, ser. 2, vol. 8, 1910, pp. 301–320; E. Landau 'Über die Bedeutung einiger neuerer Grenzwertsätze von Herrn Hardy und Axer,' *Prace Matematyczno-fizyczne*, vol. 21, 1910, pp. 97–177; M. Cipolla, 'Sul criterio di convergenza di Hardy,' *Rend. dell' Acc. di Napoli*, ser. 3, vol. 26, 1920, pp. 96–107, 151–160.

(A) *If* Σa_n *is summable* $(C, 1)$, *and* $\Sigma n^p\,|a_n|^{p+1}$ *is convergent for some positive* p, *then* Σa_n *is convergent.*

(B) *If* Σa_n *is summable* $(C, 1)$, *and either* (α) a_n *is real and*

$$a_n > -\frac{K}{n},$$

or (β)

$$a_n = O\left(\frac{1}{n}\right),$$

then Σa_n *is convergent.*

6. To prove (A) we observe that

$$b_{n+1} = \sum_{n+1}^{\infty} \frac{a_\nu}{\nu+1} = \sum_{n+1}^{\infty} \left((\nu+1)^{-\frac{2p+1}{p+1}} . (\nu+1)^{\frac{p}{p+1}} a_\nu\right),$$

and so*

$$|b_n| \leqq \left(\sum_{n+1}^{\infty} (\nu+1)^{-\frac{2p+1}{p}}\right)^{\frac{p}{p+1}} \left(\sum_{n+1}^{\infty} (\nu+1)^p |a_n|^{p+1}\right)^{-\frac{1}{p+1}}$$

$$= O\left(\left(n^{-\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\right) o(1) = o\left(\frac{1}{n}\right).$$

Hence $(n+1)\,b_{n+1} \to 0$, and so, by (3), $s_n \to A$.

7. To prove (B) we observe first that, if it is condition (β) that is given, we may suppose without loss of generality that a_n is *real*; for we may treat the real and the imaginary parts of the series separately. But then (β) becomes a special case of (α). It is therefore only necessary to consider condition (α). Further, we may plainly suppose that $A = 0$.

Suppose that
$$\overline{\lim}\, s_n = \lambda > 0,$$
and choose a sequence of values of n for which

$$s_n > \tfrac{1}{2}\lambda.$$

Let us denote a value of n, belonging to this sequence, by m; and choose H so that $0 < KH < \tfrac{1}{4}\lambda$. Then

$$s_\nu = s_m + a_{m+1} + a_{m+2} + \ldots + a_\nu$$
$$> \tfrac{1}{2}\lambda - K\left(\frac{1}{m+1} + \ldots + \frac{1}{\nu}\right) > \tfrac{1}{2}\lambda - K\frac{\nu - m}{m+1}$$
$$> \tfrac{1}{2}\lambda - KH > \tfrac{1}{4}\lambda,$$

* By the well known inequality

$$\Sigma ab \leqq (\Sigma a^k)^{\frac{1}{k}} (\Sigma b^l)^{\frac{1}{l}} \quad \left(a \geqq 0,\ b \geqq 0,\ k > 1,\ l > 1,\ \frac{1}{k} + \frac{1}{l} = 1\right).$$

if $m < \nu \leqq m + Hm$. Hence

$$\sum_{m+1}^{m+Hm} \frac{s_\nu}{(\nu+1)(\nu+2)} > \tfrac{1}{4}\lambda \sum_{m+1}^{m+Hm} \frac{1}{(\nu+1)(\nu+2)} \sim \frac{\lambda H}{4m}$$

when $m \to \infty$. But this plainly contradicts (5). Hence

$$\overline{\lim}\, s_n \leqq 0.$$

It may be shown in just the same way* that

$$\underline{\lim}\, s_n \geqq 0.$$

Hence $s_n \to 0$, and the theorem is proved.

* Using a sum $\sum_{m-Hm}^{m} \frac{s_\nu}{(\nu+1)(\nu+2)}$.

CORRECTIONS

p. 305, *line* 6 *up*. The factor ϕ_n before the sum on the right should be omitted.

p. 307, *line* 2. For the final expression read $\dfrac{\lambda H}{4m(1+H)}$.

COMMENTS

Hardy's theorem is the case $\lambda = 1$ of a result given by Knopp,† that: *if* $\lambda \geqslant 1$, *a necessary and sufficient condition for* $\sum a_n$ *to be summable* $(C, 1)$ *to* A *is that*

$$s_n + (n+1)^\lambda \sum_n^\infty a_r/(r+1)^\lambda \to A.$$

If a_n is written in the form $(n+1)(g_n - g_{n+1})$, Hardy's theorem becomes: *a necessary and sufficient condition for* $\sum (n+1)(g_n - g_{n+1})$ *to be summable* $(C, 1)$ *to* A *is that there should be a number* g *such that* $\sum (g_n - g)$ *converges to* A.

This form of Hardy's theorem is extended by Hardy and Littlewood in 1924, 1 (in Vol. III) and 1928, 1. Other extensions were obtained independently by Knopp‡ and Andersen.§

Bohr's|| theorem, referred to in § 1, showed that $\sum a_n/(n+1)^{1+\delta}$ $(\delta > 0)$ is summable $(C, k-1)$ whenever $\sum a_n$ is summable (C, k). The sharper result, with $\delta = 0$, was stated by Riesz,†† and is proved in H.R., Theorem 48 (for Riesz means). A proof of Riesz's theorem was given by Chapman.‡‡

† *Sitz. d. Berliner math. Ges.* 16 (1917), 45–50.

‡ *Math. Zeit.* 19 (1924), 97–113.

§ *Proc. London Math. Soc.* (2), 27 (1928), 39–71.

|| *Comptes rendus* 148 (1909), 75–80, and *Bidrag* . . ., p. 100, English translation, p. 87.

†† *Comptes rendus* 148 (1909), 1658–60. Bohr mentions Riesz's result in his Thesis.

‡‡ *Proc. London Math. Soc.* (2), 9 (1911), 369–409 (388). His statement of the theorem needs correcting.

The Equivalence of certain Integral Means

G. H. HARDY *and* J. E. LITTLEWOOD.

The equivalence of Hölder's and Cesàro's methods for the summation of divergent series was established by Knopp and Schnee, and later in a

more elegant manner by Schur;* and the corresponding theorem for integrals has been proved by Landau.† If $a > 0$, and $F(x)$ is integrable (in the sense of Lebesgue) in any finite interval (a, X), and

$$(1.1)\qquad F_r(x) = \frac{1}{x}\int_a^x \frac{dx_1}{x_1}\int_a^{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_{r-1}}{x_{r-1}}\int_a^{x_{r-1}} F(x_r)\,dx_r \to A$$

when $x \to \infty$, then we say that

$$(1.2)\qquad F(x) \to A \quad (H, r);$$

while if

$$(2.1)\qquad \mathrm{F}_r(x) = \frac{r}{x^r}\int_a^x F(y)(x-y)^{r-1}\,dy \to A,$$

we say that

$$(2.2)\qquad F(x) \to A \quad (C, r);$$

and Landau's theorem is substantially as follows: *if* $F(x) \to A$ (H, r), *then* $F(x) \to A$ (C, r), *and conversely.*‡

In the theory of trigonometrical series we are often concerned with integral mean values of a somewhat different kind, viz.,

$$(3.1)\qquad G_r(t) = \frac{1}{t}\int_0^t \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{r-1}}{t_{r-1}}\int_0^{t_{r-1}} G(t_r)\,dt_r$$

and

$$(4.1)\qquad \mathrm{G}_r(t) = \frac{r}{t^r}\int_0^t G(u)(t-u)^{r-1}\,du.$$

Here $t \to 0$, $G(t)$ is integrable (in the sense of Lebesgue) in any interval $0 < \tau \leqslant t \leqslant T$, and the integrals down to 0 exist either as Lebesgue integrals or as elementary generalized integrals of the type

$$\lim_{\epsilon \to 0}\int_\epsilon .$$

* K. Knopp, *Dissertation* (Berlin, 1907), pp. 19–22; W. Schnee, *Math. Annalen*, Vol. 67 (1909), pp. 110–125; I. Schur, *Math. Annalen*, Vol. 74 (1913), pp. 447–458 (see also Knopp, *ibid.*, pp. 459–461). Another proof was given by W. B. Ford, *American Journal of Math.*, Vol. 32 (1910), pp. 315–326.

† E. Landau, *Leipziger Berichte*, Vol. 65 (1913), pp. 131–138.

‡ We have modified Landau's enunciation in two respects. Landau takes $a = 0$, and he assumes more about the behaviour of $F(x)$ for finite values of x, viz. that it is the Riemann integral of a bounded function. What is essential is, of course, the behaviour of $F(x)$ when $x \to \infty$.

It would be natural to say that $G(t) \to A$ (H, r) if $G_r(t) \to 0$ and $G(t) \to A$ (C, r) if $\mathrm{G}_r(t) \to 0$; and analogy would lead us to expect these two generalizations of the notion of a limit also to be equivalent to one another.

The theorem thus suggested is in fact true, but a little consideration shows that it is not quite so obvious an analogue of Landau's theorem as we might have anticipated. In fact, if in (3.1) and (4.1) we write

$$t = \frac{1}{x},\ t_1 = \frac{1}{x_1},\ \ldots,\ G(t) = G\left(\frac{1}{x}\right) = F(x),$$

we obtain

$$(3.2) \qquad F_r(x) = x\int_x^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2} \cdots \frac{dx_{r-1}}{x_{r-1}}\int_{x_{r-1}}^\infty \frac{F(x_r)}{x_r^2}\,dx_r \to A$$

and

$$(4.2) \qquad \mathrm{F}_r(x) = rx\int_x^\infty F(y)\,(y-x)^{r-1}\frac{dy}{y^{r+1}} \to A\ ;$$

and these means are obviously of a different type from (1.1) and (2.1). If (3.2) is true, we say that

$$(3.3) \qquad F(x) \to A \quad (M, r),$$

and if (4.2) is true, we say that

$$(4.3) \qquad F(x) \to A \quad (N, r)\ ;$$

and we have to consider whether these definitions are equivalent to (1.2), (2.2), and to one another.

The results are as follows:—

(i) *the H means are equivalent to the C means*;
(ii) *the M means are equivalent to the N means*;
(iii) *the H (or C) means are never less general than the M (or N) means; if $F(x) \to A$ (M, r) then $F(x) \to A$ (H, r)*;
(iv) *if $r = 1$, all four sets of means are equivalent*; *but*
(v) *$F(x) \to A$ (H, r) does not necessarily imply $F(x) \to A$ (M, r) for any value of r greater than* 1.

We prove first that, if $F(x)$ is any integrable function, and $a > 0$, there are integrable functions $g(x)$ such that

$$5) \quad F(x) - F(a) = G(x) - G(a) - xg(x) + ag(a) = \int_a^x g(t)\,dt - xg(x) + ag(a)$$

for all values of x for which $F(x)$ is defined. We suppose, as we may do without loss of generality, that $G(a) = F(a) = 0$, and we then show that

the necessary and sufficient condition that $F(x) \to A$ (M, r) *is that there should be a* $g(x)$ *for which*

$$G(x) = o(x), \quad G(x) \to A \quad (M, r-1),$$

and that this remains true when M is replaced by N. From this the equivalence of the M and N means follows by induction. A similar proposition holds for H or C means; we have only to omit the provision that $G(x) = o(x)$, and our remaining theorems follow without difficulty.

Combining our theorem of equivalence with the necessary and sufficient criterion for the summability of a Fourier series which we enunciated recently in the *Proceedings*,* we obtain the theorem: *the necessary and sufficient condition that the Fourier series of* $f(x)$ *should be summable by Cesàro's means to sum* S *is that*

$$\frac{r}{t^r} \int_0^t \phi(u)\,(t-u)^{r-1}\,du,$$

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2S\},$$

should tend to zero with t *for some value of* r.

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* See *Records*, &c. for 8 June, 1922 (*Proc. Lond. Math. Soc.* (2), Vol. 21, pp. xxxi–xxxii). The detailed proofs of the theorems there stated will appear shortly in the *Math. Zeitschrift*.

COMMENTS

In the sketch of the proof of (ii), the statement that 'this remains true when M is replaced by N' is not immediately clear.

It follows from the definition of the (M, r) means that, for $r = 2, 3, \ldots$, (A): *necessary and sufficient conditions for* $F(x) \to A(M, r)$ *are that* (a_1) $\int F(t)/t^2\, dt$ *should converge, and*

$$(\mathrm{a}_2) \quad G(x) = x \int_x^\infty F(t)/t^2\, dt \to A \quad (M, r-1).$$

Alternative conditions are (I): *there is an integrable function* $g(x)$ *such that the differential equation* $F(x) = G(x) - xg(x)$ p.p., *where* $G(x)$ *is an integral of* $g(x)$, *has a solution such that* (1_1) $G(x) = o(x)$ *and* (1_2) $G(x) \to A$ $(M, r-1)$.

Similarly, it follows from the definition of the (N, k) means that, for $r = 2, 3, \ldots$, (B): *necessary and sufficient conditions for* $F(x) \to A$ (N, r) *are* (b_1) *that* (a_1) *should hold, and*

$$(\mathrm{b}_2) \quad H(x) = rx^r \int_x^\infty F(t)/t^{r+1}\, dt \to A \quad (N, r-1).$$

Here (b_1) implies that $H(x) = o(x)$. Alternative conditions are (II): *there is an integrable function* $h(x)$ *such that the differential equation* $F(x) = H(x) - r^{-1}xh(x)$ p.p., *where* $H(x)$ *is an integral of* $h(x)$, *has a solution such that* (2_1) $H(x) = o(x)$, *and* (2_2) $H(x) \to A$ $(N, r-1)$.

To show that (I) and (II) are equivalent, or what is the same thing, that (M, r) and (N, r) are equivalent, *we assume the inductive hypothesis that* $(M, r-1)$ *and* $(N, r-1)$ *are equivalent.*

Suppose first that (I) *holds*, i.e. that $F(x) \to A$ (M, r). Then the solution $G(x)$ of the differential equation in (I) satisfies (a_2). Now let $H(x)$ be defined by the integral in (b_2). Then $H(x)$ is a solution of the differential equation in (II), and (a_1) implies that $H(x) = o(x)$. To show that (II) holds, it remains to prove that $H(x) \to A$ $(N, r-1)$. From the definition of $H(x)$ we obtain

$$H(x) = rG(x) - (r-1)rx^r \int_x^\infty G(t)/t^{r+1}\, dt.$$

By the inductive hypothesis, (I) implies that $G(x) \to A$ $(N, r-1)$. This implies that $G(x) \to A$ (N, r), and hence

$$rx^r \int_x^\infty G(t)/t^{r+1}\, dt \to A \quad (N, r-1).$$

Thus $H(x) \to rA - (r-1)A$ $(N, r-1)$, and hence, by induction, (I) implies (II), i.e. (M, r) implies (N, r).

The converse is proved similarly, by means of the identity

$$G(x) = r^{-1}H(x) + r^{-1}(r-1)x \int_x^\infty H(t)/t^2\, dt.$$

Kuttner† has shown that (v) becomes true, for $r = 2, 3, \ldots$, if and only if the extra condition that $\int F(t)/t^2\, dt$ *should converge* is added. He has also considered the extension of (iii) and (v) to non-integral orders of summability. He finds that, in either case, if r is non-integral in the hypothesis, then the conclusion holds with r replaced by $r' > r$, but not with $r' = r$, provided the same extra condition is added in (v) when $r > 1$.

† *J. London Math. Soc.* 33 (1958), 107–18.

ABEL'S THEOREM AND ITS CONVERSE (II)

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. *Introduction.*

1. In an earlier paper bearing the same title as this one* we referred to two theorems which we were then unable to prove, and whose absence detracted from the completeness of our results.

In the first place, we proved† that *if*

(1 . 1) $$a_n = O\left(\frac{1}{n}\right)$$

and

(1 . 2) $$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1-x^{n+1}),$$

then

(1 . 3) $$\Sigma a_n = A$$

implies

(1 . 4) $$\Phi(x) \to A$$

when $x \to 1$ *along any path* C‡ *which does not pass outside the circle.* It is natural to suppose that this theorem, like other theorems of its kind, is a "best possible" theorem, *i.e.* that it becomes false if (1 . 1) is replaced by any less exacting condition of the same kind; but this we were at the time unable to prove. We give a proof now in § 2.

A second and more important gap in our work was this. We proved§

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* "Abel's Theorem and its Converse", *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1918), pp. 205–235. See p. 210. We refer to this paper as A.T.

† A.T., pp. 220–223 (Theorem **T**).

‡ See A.T., pp. 205–206, for precise definitions of *path*, *internal path*, *regular path*, *Stolz path*.

§ A.T., pp. 214–219 (Theorem **Q**).

that (1.1) and (1.4) involve (1.3), so that, if (1.1) is satisfied, (1.4) is the necessary and sufficient condition for the convergence of the series Σa_n. From this we deduced a theorem* which is a more direct extension of Tauber's Theorem, viz. that (1.1) *and*

$$f(x) = \Sigma a_n x^n \to A \tag{1.5}$$

involve the convergence of the series. But here we had to impose a restriction on C, viz. that it is what we called a regular path. We could also prove the result when C is a Stolz path†; but we were unable to eliminate *all* restrictions on C. This we can now do, and present the proof in § 3.

There is one point which demands a word of explanation. The series (1.2) is uniformly convergent for $|x| \leqslant 1$, and, so long as we are concerned with $\Phi(x)$, it is irrelevant whether our path C is restricted to be internal or may have points in common with the circle. The situation is different when we are concerned with $f(x)$, since $f(x)$ is not generally defined at points on the circle. We therefore state and prove our theorem on the hypothesis that C is internal.

The theorem remains true, however, if this restriction is removed, provided the hypotheses are properly stated. It may be possible to define $f(x)$, at a point common to C and the circle, by an Abelian limit

$$f(x) = f(1, \theta) = \lim_{\to 1} f(r, \theta),$$

and in this manner to complete the definition of $f(x)$ for all points of C; and it may then happen that $f(x)$, so defined, tends to a limit along C. Elementary considerations of continuity then show C may be replaced by a "sufficiently close" internal path C' along which also $f(x)$ tends to a limit. Our theorem then establishes the convergence of the series, the additional complication lying only in the interpretation of its hypotheses.

In § 4 we restate the final result of our work, which may now be regarded as completed, in the form of a comprehensive theorem.

2. *Addendum to Theorem* **T**.

2.1. THEOREM **T**′.—*It is not possible, in Theorem* **T**, *to replace the condition* (1.1) *by any less exacting condition of the same type. That is to say, if* $\phi(n)$ *is any positive function of* n, *tending steadily to infinity*

* A.T., pp. 219–220 (Theorem **R**).

† A.T., p 220 (Theorem **S**).

with n, it is possible to find a convergent series A, for which

$$a_n = O\left(\frac{\phi(n)}{n}\right), \tag{2.11}$$

and a regular internal path C for which (1.4) *is false.*

We may obviously suppose, without loss of generality, that the a's are real, that $a_0 = 0$, and that

$$\phi(n) < \sqrt{n} \quad (n > 0);$$

and we may replace
$$\frac{a_n}{n+1}(1-x^{n+1}),$$

in (1.2), by
$$\frac{a_n}{n}(1-x^n),$$

since

$$\frac{1}{1-x}\left\{\Sigma \frac{a_n}{n}(1-x^n) - \Sigma \frac{a_n}{n+1}(1-x^{n+1})\right\} = \Sigma \frac{a_n}{n(n+1)}\frac{1-x^n}{1-x} - \Sigma \frac{a_n}{n+1}x^n,$$

and each of these series is uniformly convergent for $|x| < 1$.

Let

$$\chi(a) = \frac{1}{1-e^{ia}}\Sigma \frac{a_n}{n}(1-e^{nia}), \tag{2.12}$$

where $0 < a < 2\pi$, and

$$\chi(r, a) = \frac{1}{1-re^{ia}}\Sigma \frac{a_n}{n}(1-r^n e^{nia}), \tag{2.121}$$

where $0 < r < 1$. For any fixed a, $\chi(r, a) \to \chi(a)$ when $r \to 1$; and we can define a regular internal path C, by an equation $r = r(a)$, in such a manner that $\chi(r, a) - \chi(a) \to 0$ when $a \to 0$. The theorem will therefore be proved if we can establish the existence of a function $\chi(a)$ which does not tend to a limit when $a \to 0$.

Again, if

$$\Phi(a) = \Sigma a_n \frac{\sin na}{na}, \tag{2.13}$$

$$\mathfrak{R}\left(e^{\frac{1}{2}ia}\chi(a)\right) = \frac{a}{2\sin\frac{1}{2}a}\Phi(a), \tag{2.14}$$

and $\chi(a)$ cannot tend to a limit unless $\Phi(a)$ does so. It is therefore sufficient to define a $\Phi(a)$ which does not tend to a limit.

2.2. LEMMA 1.—*If A is convergent, then*

$$\sum_{n \leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} = A + o(1). \tag{2.21}$$

Choose N so that

$$\Big| \sum_{\nu \leqslant n} a_\nu - A \Big| < \epsilon \quad (n \geqslant N). \tag{2.22}$$

Then

$$\left| \sum_{n \leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} - A \right| \tag{2.23}$$

$$\leqslant \left| \sum_{n \leqslant N} a_n \left(\frac{\sin n\alpha}{n\alpha} - 1 \right) \right| + \Big| \sum_{n \leqslant N} a_n - A \Big| + \left| \sum_{N < n \leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} \right|$$

$$= S_1 + S_2 + S_3,$$

say. We have

$$|S_2| < \epsilon, \tag{2.241}$$

by (2.22); and

$$|S_3| \leqslant \operatorname*{Max}_{N < n_1 \leqslant 1/\alpha} \Big| \sum_{N < n \leqslant n_1} a_n \Big| < 2\epsilon, \tag{2.242}$$

since $\sin u/u$ decreases steadily from 1 in the interval $0 < u \leqslant 1$. Finally, when N is fixed,

$$|S_1| < \epsilon \tag{2.243}$$

for all sufficiently small values of α. From (2.23)–(2.243) we deduce

$$\left| \sum_{n \leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} - A \right| < 4\epsilon$$

for all sufficiently small values of α, which proves the lemma.

2.3. LEMMA 2.—*If $\phi(x)$ is any function of x which tends steadily to infinity with x, we can find a sequence of positive integers n_ν such that*

$$m_\nu = \frac{n_{\nu+1}}{n_\nu} \to \infty, \qquad m_\nu < \phi(n_\nu). \tag{2.31}$$

We may clear our ideas by a preliminary remark. The important cases are those in which $\phi(x)$ tends to infinity very slowly, *e.g.* like $\log\log x$. The first condition on the m's means that n_ν tends to infinity more rapidly than any exponential $e^{a\nu}$, and the second that its increase is very little more rapid than that of such exponentials. Thus, when

$\phi(x) = \log\log x$, the conditions would be satisfied (except that the m's would not be integers) if we took

$$n_\nu = e^{\nu \log\log\log \nu}.$$

We begin by choosing non-integral m's such that

$$(2.32) \qquad m_\nu \to \infty, \qquad 1 < m_\nu < \tfrac{1}{2}\phi(n_\nu).$$

We can obviously find a function $\chi(x)$, with a positive, decreasing, and continuous derivative χ', such that

$$(2.33) \qquad \chi \to \infty, \qquad \chi > 1, \qquad \chi' = o\left(\frac{1}{x}\right), \qquad (x+1)\chi' < 1, \qquad e^{\chi} < \frac{\phi(e^\nu)}{2e}.$$

We take

$$(2.34) \qquad n_\nu = e^{\nu\chi(\nu)}.$$

Then $$e^{\chi(\nu)+\nu\chi'(\nu+1)} < e^{(\nu+1)\chi(\nu+1)-\nu\chi(\nu)} = m_\nu < e^{\chi(\nu)+(\nu+1)\chi'(\nu)},$$

and so, using the third of the conditions (2.33),

$$(2.35) \qquad 1 < m_\nu \sim e^{\chi(\nu)} \to \infty.$$

Also, using the fourth and fifth of (2.33),

$$(2.36) \qquad m_\nu < e^{\chi+(\nu+1)\chi'} < e^{\chi+1} < \tfrac{1}{2}\phi(e^\nu) < \tfrac{1}{2}\phi(n_\nu).$$

Thus the conditions (2.32) are satisfied.

We now replace every m_ν by the integer equal to or immediately above it. This increases n_ν and therefore $\phi(n_\nu)$, while m_ν is at most doubled. Hence the new set of m's satisfy (2.31).

We note that

$$(2.37) \qquad \sum_{\mu}^{\infty} \frac{1}{n_\nu^2} = O\left(\frac{1}{n_\mu^2}\right),$$

an obvious corollary of (2.31).

2.4. Lemma 3.—*When m_ν is defined as in Lemma 2, we can find a positive integer k_ν which tends steadily to infinity with ν in such a manner that*

$$(2.41) \qquad \frac{k_{\nu+1}}{k_\nu} \to 1, \qquad \frac{\log m_\nu}{k_\nu} \to \infty.$$

This is obvious; we have only to make the increase of k_ν sufficiently slow.

LEMMA 4.—*If*

$$a_\nu = 2\pi k_\nu/n_\nu, \tag{2.42}$$

then

$$a_{\nu+1}/a_\nu \to 0, \tag{2.43}$$

$$a_\nu - a_{\nu'} \sim a_\nu \quad (\nu' > \nu), \tag{2.44}$$

and

$$n_{\nu-1} < 1/a_\nu < n_\nu \tag{2.45}$$

for all sufficiently large values of ν.

Of these relations (2.43) follows from (2.31), (2.41), and (2.42), and (2.44) is an obvious corollary. Also

$$n_\nu a_\nu = 2\pi k_\nu \to \infty, \qquad n_{\nu-1}\, a_\nu = \frac{2\pi k_\nu}{m_{\nu-1}} \sim \frac{2\pi k_{\nu-1}}{m_{\nu-1}} \to 0,$$

by (2.41); which proves (2.45).

LEMMA 5.—*If* $0 < \theta < \frac{1}{2}\pi$ *and* $0 < N < N'$, *then*

$$\sum_N^{N'} \sin n\theta = O\left(\frac{1}{\theta}\right), \quad \sum_N^{N'} \frac{\cos n\theta}{n} = O\left(\frac{1}{N\theta}\right), \tag{2.46}$$

uniformly in the parameters. If also $0 < \phi < \frac{1}{2}\pi$, $\phi \neq \theta$, *then*

$$\sum_N^{N'} \frac{\sin n\theta \sin n\phi}{n} = O\left(\frac{1}{N\,|\,\theta - \phi\,|}\right). \tag{2.47}$$

These are familiar results, collected merely for purposes of reference.

2.5. We define our series A by

$$a_n = \frac{\sin n a_\nu}{n_\nu} = \frac{1}{n_\nu} \sin \frac{2n\pi k_\nu}{n_\nu} \quad (n_\nu \leqslant n < n_{\nu+1}). \tag{2.51}$$

Then
$$|\,a_n\,| \leqslant \frac{1}{n_\nu} = \frac{m_\nu}{n_{\nu+1}} \leqslant \frac{\phi(n_\nu)}{n_{\nu+1}} \leqslant \frac{\phi(n)}{n},$$

so that a_n satisfies (2.11). It remains to prove (1) that A is convergent, and (2) that $\Phi(a)$ does not tend to a limit.

We have first

$$\sum_{n_\nu}^{n_{\nu+1}-1} \sin n a_\nu = \frac{\sin \frac{1}{2}(n_{\nu+1} - n_\nu) a_\nu \sin \frac{1}{2}(n_\nu + n_{\nu+1} - 1) a_\nu}{\sin \frac{1}{2} a_\nu} = 0,$$

since
$$\tfrac{1}{2}(n_{\nu+1} - n_\nu) a_\nu = (m_\nu - 1) k_\nu \pi$$

is a multiple of π. Hence

$$\sum_{n_\nu}^{n_{\nu+1}-1} a_n = 0. \tag{2.52}$$

Also, if $n_\nu \leqslant \mu < n_{\nu+1}$, we have

$$\sum_{n_\nu}^{\mu} a_n = \frac{1}{n_\nu} \sum_{n_\nu}^{\mu} \sin n a_\nu = O\left(\frac{1}{n_\nu a_\nu}\right) = O\left(\frac{1}{k_\nu}\right) = o(1), \tag{2.53}$$

by (2.46) and (2.51), and uniformly in μ. From (2.52) and (2.53) it follows that A converges to the sum 0.

2.6. We write

$$\Phi(a) = \Sigma a_n \frac{\sin na}{na} = \sum_{\nu} \sum_{n_\nu}^{n_{\nu+1}-1} = \sum_{\nu} \psi_\nu(a) \tag{2.61}$$

$$= \sum_{1}^{\mu-1} \psi_\nu(a) + \psi_\mu(a) + \sum_{\mu+1}^{\infty} \psi_\nu(a) = S_1(a) + S_2(a) + \psi_\mu(a);$$

and we shall prove that

$$S_1(a_\mu) \to 0, \tag{2.62}$$

$$S_2(a_\mu) \to 0, \tag{2.63}$$

$$\psi_\mu(a_\mu) \to \infty \tag{2.64}$$

when $\mu \to \infty$. From these relations it will follow that $\Phi(a) \to \infty$ when $a \to 0$ through the particular sequence a_μ, and this will complete the proof of Theorem **T**′.

We have

$$S_1(a_\mu) = \sum_{n \leqslant n_\mu - 1} a_n \frac{\sin n a_\mu}{n a_\mu} = \sum_{n \leqslant 1/a_\mu} + \sum_{1/a_\mu < n < n_\mu} \tag{2.65}$$

$$= \sum_{1/a_\mu < n < n_\mu} + o(1) = S_1'(a_\mu) + o(1),$$

by Lemma 1. Since $1/a_\mu > n_{\mu-1}$, by (2.45), $\nu = \mu - 1$ throughout S_1'. Writing

$$M = 1 + \left[\frac{1}{a_\mu}\right], \tag{2.66}$$

we have

$$S_1'(a_\mu) = \frac{1}{n_{\mu-1} a_\mu} \sum_{M}^{n_\mu - 1} \frac{\sin n a_{\mu-1} \sin n a_\mu}{n} = O\left(\frac{1}{n_{\mu-1} a_\mu} \cdot \frac{1}{M(a_{\mu-1} - a_\mu)}\right)$$

by (2.47). But

$$M(a_{\mu-1} - a_\mu) \sim M a_{\mu-1} \sim a_{\mu-1}/a_\mu,$$

by (2.44) and (2.66); and so

$$S_1'(a_\mu) = O\left(\frac{1}{n_{\mu-1}\,a_{\mu-1}}\right) = O\left(\frac{1}{k_{\mu-1}}\right) = o(1). \tag{2.67}$$

From (2.65) and (2.67) we deduce (2.62).

Next, if $\nu > \mu$,

$$\psi_\nu(a_\mu) = \frac{1}{n_\nu a_\mu} \sum_{n_\nu}^{n_{\nu+1}-1} \frac{\sin n a_\mu \sin n a_\nu}{n} = \frac{1}{n_\nu a_\mu} O\left(\frac{1}{n_\nu(a_\mu - a_\nu)}\right) = O\left(\frac{1}{n_\nu^2 a_\mu^2}\right),$$

by (2.47) and (2.44); and hence, using (2.37), we obtain

$$S_2(a_\mu) = O\left(\frac{1}{a_\mu^2} \sum_{\mu+1}^{\infty} \frac{1}{n_\nu^2}\right) = O\left(\frac{1}{n_{\mu+1}^2 a_\mu^2}\right) = O\left(\frac{1}{n_\mu^2 a_\mu^2}\right) = O\left(\frac{1}{k_\mu^2}\right) = o(1),$$

which is (2.63).

Finally

$$\begin{aligned} \psi_\mu(a_\mu) &= \frac{1}{n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{\sin^2 n a_\mu}{n} \\ &= \frac{1}{2 n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{1}{n} - \frac{1}{2 n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{\cos 2 n a_\mu}{n}. \end{aligned} \tag{2.68}$$

The first term in (2.68) is greater than a constant multiple of

$$\frac{1}{n_\mu a_\mu} \log \frac{n_{\mu+1}}{n_\mu} = \frac{\log m_\mu}{2\pi k_\mu},$$

which tends to infinity, by (2.41). The second, by (2.46), is

$$O\left(\frac{1}{n_\mu a_\mu} \cdot \frac{1}{n_\mu a_\mu}\right) = O\left(\frac{1}{k_\mu^2}\right) = o(1).$$

This proves (2.64) and completes the proof of the theorem.

3. *The general Tauberian theorem.*

3.1. In this section we prove

THEOREM **R**′.—*If*

$$a_n = O\left(\frac{1}{n}\right), \tag{1.1}$$

and

$$f(x) \to A \tag{1.5}$$

when $x \to 1$ *along an internal path* C, *then*

(1.3) $$\Sigma a_n = A.$$

We may suppose $A = 0, \quad |na_n| \leqslant 1,$

and it is convenient to write e^{-z} for x, and then $z = x+iy$. We have then to prove: *if*

(3.11) $$|na_n| \leqslant 1,$$

and

(3.12) $$f(z) = \Sigma a_n e^{-nz} \to 0$$

when $z \to 0$ *along a Jordan curve* C *lying, except for the point* $z = 0$, *in the half-plane* $x > 0$, *then*

(3.13) $$\Sigma a_n = 0.$$

We suppose that C is

(3.14) $$z = Z(t),$$

the parameter t ranging from 0 to 1, and $t = 1$ corresponding to $z = 0$. There is no loss of generality in supposing that

(3.15) $$|f| < 1$$

at all points of C.

The letter K denotes an absolute constant, whose value varies from one occurrence to another. Occasionally we must preserve the identity of a K; in this case we distinguish it by a suffix, and such constants are the same wherever they occur. The constants of the O's which occur in our argument are also absolute.

3.2. LEMMA 6.—*If*

(3.21) $$|z| < 1, \quad |z'-z| < cx \quad (0 < c < 1),$$

then

(3.22) $$|f(z')-f(z)| < k(c),$$

where $k(c)$ *depends only on* c.

It follows from (3.21) that x is positive and

(3.23) $$x' > Kx.$$

Also $$|e^{-nz'}-e^{-nz}| = \left|n\int_z^{z'} e^{-nu}\,du\right| \leqslant n|z'-z| \operatorname{Max}(e^{-nx'}, e^{-nx})$$
$$\leqslant n|z'-z|(e^{-nx'}+e^{-nx}).$$

Hence

$$|f(z')-f(z)| = |\Sigma a_n(e^{-nz'}-e^{-nz})| \leqslant \Sigma n\,|a_n|\,|z'-z|\,(e^{-nx'}+e^{-nx})$$

$$\leqslant |z'-z| \left(\frac{1}{1-e^{-x'}}+\frac{1}{1-e^{-x}}\right) < K_1 x\left(\frac{1}{x'}+\frac{1}{x}\right) < K,$$

which is (3 . 22).

3 . 21. Lemma 6 leads to a very simple proof of our former Theorem **S**.* This theorem is required for the proof of **R**′, and we did not write out its proof in full: we therefore insert one here.

If C is a Stolz path, it is included in an angle **A**, whose vertex is at O, and whose sides include, and make acute angles with, the real axis. The angle **A**, in its turn, is included within a slightly larger angle **A**′ of the same type. It is plain that, if z is a point of C, and z' a point in **A**′ for which $x' \leqslant x$, then z' satisfies the condition (3 . 21), and so that $f(z')$ is bounded in **A**′. Since $f(z)$ is bounded in **A**′, and tends to a limit along C, it tends to a limit when $z \to 0$, uniformly throughout **A**.† In particular, it tends to a limit along the real axis, and therefore, by Theorem **F**, the series Σa_n is convergent.

3 . 3. The next lemma requires a word of explanation. Suppose that $z \to 0$ along C, so that $z = Z(t)$; we may denote the parameter of z by $t(z)$. If we write

$$\epsilon = \underset{t(z)\leqslant t' < 1}{\mathrm{Max}} \left| f\big(Z(t')\big) \right|, \tag{3 . 31}$$

then $\epsilon = \epsilon(t)$ is a function of t which tends steadily to zero when $t \to 1$. We write $\epsilon(t_1) = \epsilon_1$, and so on.

We denote generally by $\delta = \delta(\epsilon)$ a positive function of ϵ which tends to zero with ϵ.

LEMMA 7.—*Suppose that T is a point on the positive imaginary axis OY; TB_2B_1 a ray in the first quadrant making an angle $\theta < \frac{1}{4}\pi$ with TY; and TL the trisector, nearer to TB_1, of the angle B_1TY. Suppose that C cuts TB_2B_1 in B_1 and B_2, that t_1 and $t_2 > t_1$ are the parameters*

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* See p. 255 above, f.n. †.

† See E. Lindelöf, "Sur un principe général de l'analyse", *Acta Soc. Fennicae*, Vol. 46 (1915), pp. 1-35 (7-11); or P. Montel, "Sur les familles de fonctions analytiques qui admettent des valeurs exceptionelles dans une domaine", *Annales scientifiques de l'École Normale Supérieure* (3), Vol. 20 (1912), pp. 487-535 (519). Lemmas β–ϵ of A.T. embody a different proof of substantially the same theorem.

of B_1 and B_2, and that the part C' of C for which $t_1 < t < t_2$ lies entirely in the angle B_1TY. Then

(3.32) $$|f(z')| < K$$

at all points z' of the segment B_1B_2; and

(3.33) $$|f(z'')| < \delta(\epsilon_1)$$

*at all points z'' of any chord intercepted on TL by the arc B_1B_2.**

Let P' be a point of B_1B_2 (other than B_1 or B_2). We may suppose that P' is not a point of C'. The perpendicular to B_1B_2 at P' will meet TL in a point P''. It will also meet C'; we suppose that it does so first at P.

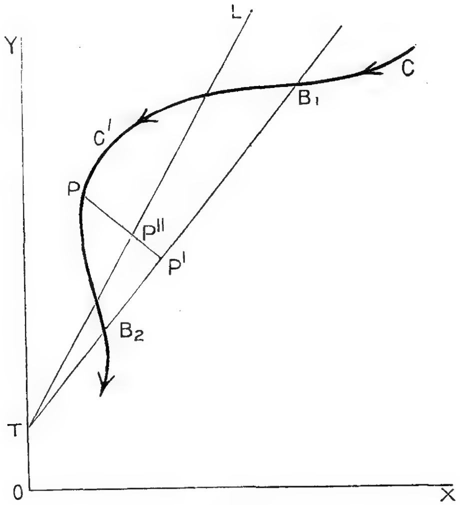

FIG. 1.

We define M as the greater of (i) ϵ_1 and (ii) the upper bound of $|f|$ on the segment B_1B_2, so that $M \geqslant \epsilon_1$. We denote by ζ a linear function of z which vanishes at T and is positive on TB_1†, so that the argument of ζ on TY is θ; and we write $F(\zeta)$ for $f(z)$. Finally we write

(3.34) $$g(\zeta) = \exp\{-ik(\log\zeta - i\theta)\},$$

where k is positive and

(3.341) $$e^{k\theta} = M/\epsilon_1.$$

It may be verified at once that

(3.351) $$|g| \leqslant 1$$

* See Fig. 1. There may be no points such as z'', in which case (3.33) is naturally trivial.

† ζ is thus fixed, except for a positive multiplier.

in B_1TY, and that $|g|$ assumes the values

(3.352) $$\frac{\epsilon_1}{M}, \quad \left(\frac{\epsilon_1}{M}\right)^{\frac{2}{3}}, \quad 1$$

on TB_1, TL, and TY respectively.

If z', z'', z refer to P', P'', P, we have

(3.36) $$|z''-z'| = P'P'' = \frac{\sin\frac{1}{3}\theta}{\sin\frac{2}{3}\theta}x'' \leqslant \tfrac{1}{2}\sec\frac{\pi}{12}\cdot x'' = K_1x'' < K_1x'.$$

This K_1 is less than unity, so that we may take $c = K_1$ in Lemma 6. Hence

(3.37) $$|f(z'')-f(z')| < K.$$

3.3. We must now distinguish two cases.

(*a*) *P lies between P' and P''* (or at P''). Then

(3.41) $$|z''-z| = PP'' \leqslant P'P'' \leqslant K_1x'' \leqslant K_1x.$$

In this case

(3.42) $$|f(z'')| < K$$

by Lemma 6, and

(3.43) $$|f(z')| < K+K < K$$

by (3.37).

(*b*) *P'' lies between P' and P.* Then P'' is a point of R, a region bounded by B_1B_2 and C' (or parts of them*). Now

$$|F| \leqslant M, \quad |g| \leqslant \epsilon_1/M$$

on B_1B_2, and

$$|F| \leqslant \epsilon_1, \quad |g| \leqslant 1$$

on C'. Hence $|Fg| \leqslant \epsilon_1$ on the boundary of R, and therefore in its interior; so that

(3.44) $$|f(z'')| = |F(\zeta'')| \leqslant \frac{\epsilon_1}{|g(\zeta'')|} = M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}},$$

by (3.352), and

(3.45) $$|f(z')| < K+M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}},$$

by (3.37) and (3.44). Comparing (3.43) and (3.45), we see that the latter holds in both cases (*a*) and (*b*).

Thus (3.45) holds for every P', and therefore

(3.46) $$M < K+M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}} < K+M^{\frac{2}{3}},$$

* R may be bounded by parts of C' only if the curve is not simple.

which is plainly impossible unless

$$M < K, \tag{3.47}$$

which proves (3.32).

To prove (3.33) we observe that, if P'' is a point of a chord intercepted by the curve on TL (which may or may not be the case), then P'' is certainly interior to an R bounded by B_1B_2 and C' (or parts of them), so that (3.44) is true. Hence

$$|f(z'')| \leqslant M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}} < K\epsilon_1^{\frac{1}{3}} < \delta(\epsilon_1), \tag{3.48}$$

which is (3.33).

3.5. LEMMA 8.—*Suppose that B is the point of C whose parameter is t; that OB makes with OY an angle $\theta < \frac{1}{4}\pi$; that B' is the last point of C, beyond B, such that no point of the arc BB' lies below OB; and that OL is the trisector, nearer OB, of the angle BOY.* Then

$$|f(z)| < K \tag{3.51}$$

at all points P of the segment BB', and

$$|f(z_1)| < \delta(\epsilon) \tag{3.52}$$

at all points P_1 of any chord intercepted on OL by the arc BB'.

It is to be observed that B' may, in extreme cases, fall at B or at O.

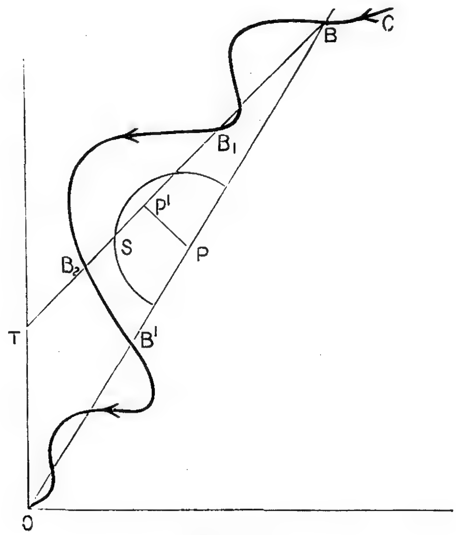

FIG. 2.

We may suppose that P is not on C'. There is then a semicircle S, with centre P and diameter along OB, which contains no point of the arc BB'. We draw a line BT, above BO, and making an angle less than $\frac{1}{4}\pi$ with OY, so as to intersect S; and we draw PP' perpendicular to BT.

There is an arc B_1B_2 of C, whose chord B_1B_2 contains P', which is related to B_1T as B_1B_2 was to B_1T in Lemma 7. Hence, by Lemma 7,

$$|f(z')| < K.$$

But BT may be drawn as close as we please to BO, so that P' is as close as we please to P. Hence, by continuity,

$$|f(z)| < K$$

which is (3.51).

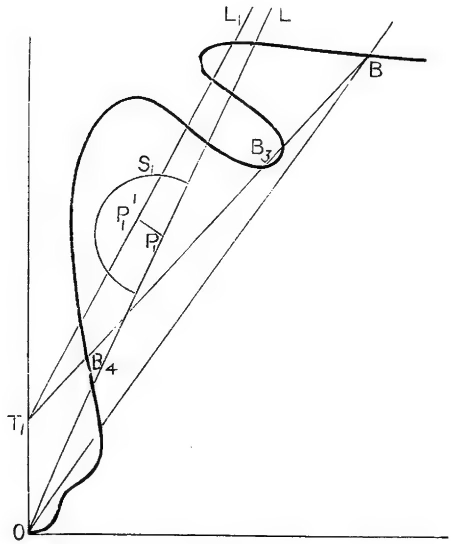

FIG. 3.

The proof of (3.52) is similar. We may suppose P_1 not a point of C'; there is then a semicircle S_1 related to P_1 and OL as S was to P and OB. We draw BT_1, above BO, making an angle less than $\frac{1}{4}\pi$ with OY, and so that T_1L_1, the trisector of BT_1Y nearer T_1B, intersects S_1; and P_1P_1' perpendicular to T_1L_1. There is an arc B_3B_4 of C', related to B_3T_1 in the manner of Lemma 7, and a chord intercepted by it on T_1L_1 which contains P_1'. Also $t_3 > t$, $\epsilon_3 \leqslant \epsilon$. Hence, by Lemma 7,

$$|f(z_1')| < \delta(\epsilon_3) < \delta(\epsilon).$$

But z_1' is as close as we please to z_1, and therefore

$$|f(z_1)| < \delta(\epsilon),$$

which is (3.52).

3.6. We can now prove the main theorem.

We denote by S, S' straight lines through the origin, in the first and fourth quadrants respectively, making with the positive or negative directions of the imaginary axis angles less than $\frac{1}{6}\pi$. We can distinguish, first, four possibilities :

(*a*) C cuts both an S and an S' arbitrarily near to O;

(*b*) C cuts an S, but no S', arbitrarily near to O;

(*c*) C cuts an S', but no S, arbitrarily near to O;

(*d*) C cuts no S or S' arbitrarily near to O.

Of these cases (*b*) and (*c*) are similar, and (*a*) requires only trivial changes in the argument which disposes of (*b*). It is sufficient then to discuss (*b*) and (*d*).

Case (*b*).—It is plain, first, that C lies entirely above some S', say S'_1. Let S_1 be an S cut by C arbitrarily near to O, and draw OB so that S_1 is the trisector of BOY nearer OB. Then OB makes with OY an angle less than $\frac{1}{4}\pi$.

Let C_0 be the curve derived from C by replacing every arc of C above S_1 by the corresponding chord. Then the parts of C_0 which are parts of S_1 correspond to arcs of C arbitrarily near to O; and so, by Lemma 8,

$$f(z) \to 0$$

when $z \to 0$ along C_0. But C_0 is a Stolz-path, since it lies between S_1 and S'_1, and our conclusion follows from our former Theorem **S.**

Case (*d*).—This case falls into three, according as (*d* 1) C lies ultimately between an S and an S_1 or (*d* 2) lies ultimately above every S or (*d* 3) lies ultimately below every S_1. In (*d* 1) C is a Stolz-path, and we can use Theorem **S.** It is enough then to consider (*d* 2).

We can draw an S (say OB) which meets C for the last time* in B; and the bisector OB' of BOY will meet C for the last time in B', say. Then, $|f| < K$ on OB, by Lemma 8; and so $|f| < K$ in the region R bounded by OB and C. But $f \to 0$ when $z \to 0$ along C, and therefore† $f \to 0$ when $z \to 0$ along OB'. As OB' is a Stolz-path, our conclusion follows once more from Theorem **S.**

* *I.e.* for the greatest t.

† See the memoirs of Lindelöf and Montel referred to on p. 263.

4. *Summary of conclusions.*

4.1. The results of our researches may be stated comprehensively in the following theorem.

THEOREM α.—*If*

(O) $$a_n = O\left(\frac{1}{n}\right),$$

then the necessary and sufficient condition that

(K) $$\Sigma a_n = A$$

is that

(Λ) $$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1-x^{n+1}) \to A$$

when $x \to 1$ *along some path* C. *This condition may also be written*

(Λ′) $$\Phi(x) = \frac{1}{1-x}\int_x^1 f(u)\,du \to A,$$

where $$f(x) = \Sigma a_n x^n,$$

and the path of integration is the straight line $(x, 1)$ *or, if* C *is regular,* C *itself.*

If (Λ) *is satisfied for any path* C, *it is satisfied for all paths* C.

If (O) *is replaced by any condition*

(O′) $$a_n = O\left(\frac{\phi(n)}{n}\right),$$

where $\phi(n)$ *tends to infinity with* n, *then* (Λ) *ceases to be either a necessary or a sufficient condition for* (K); *either proposition may be true and the other false.*

Assuming (O) *to be satisfied, the condition that*

(L) $$f(x) \to A,$$

along some path C, *is a sufficient condition for* (K). *But it is not necessary, and does not become necessary even when* (O) *is replaced by any more exacting condition of the type*

(O″) $$a_n = O(\chi_n)$$

where χ_n *is positive and* $\Sigma\chi_n$ *divergent. In any such case* (K) *may be true and* (L) *false.*

COMMENTS

In this paper the outstanding questions in 1920, 7 are answered.

In the proof of Lemma 2, there is no loss of generality in supposing that $\phi(n) > 2e^2$ for $n \geqslant 1$. Then the 4th condition in (2.33) can be satisfied. The value of n_0 is fixed by (2.34) as 1, and (2.36) holds for $\nu \geqslant 0$, provided the last $<$ in (2.36) is replaced by $\leqslant$.

In the new proof of Theorem S of 1920, 7, given in § 3.21, the statement 'if z is a point of C, and z' a point in $\mathbf{A}'$ for which $x' \leqslant x$, then z' satisfies the condition (3.21)' needs modification if the angle $\mathbf{A}'$ is large. But if $\mathbf{A}'$ is the angle $|z| \leqslant Kx$, then if $x' \leqslant x$, $|z-z'| \leqslant 2Kx$, and if also $x' \geqslant \frac{1}{2}x$, then the proof of Lemma 6 leads to the conclusion $|f(z')-f(z)| \leqslant k(K)$.

A FURTHER NOTE ON THE CONVERSE OF ABEL'S THEOREM

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. *Introduction.*

1.1. In this note we prove a theorem conjectured by Littlewood in a paper published in these *Proceedings* in 1910.*

The main result of Littlewood's paper was as follows. *Suppose that*

$$0 < \lambda_{n-1} < \lambda_n \quad (n = 2, 3, \ldots), \tag{1.11}$$

$$\lambda_n \to \infty, \tag{1.12}$$

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0; \tag{1.13}$$

and that

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \tag{1.14}$$

so that the series

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} \tag{1.15}$$

is convergent for $t > 0$. Then the existence of the limit

$$\lim_{t \to 0} f(t) = A \tag{1.16}$$

implies the convergence of the series Σa_n to sum A.

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* J. E. Littlewood, "The converse of Abel's theorem on power series", *Proc. London Math. Soc.* (2), 9 (1910), 434–448.

It was observed by Littlewood that the condition (1 . 13) is not essential to the truth of the theorem, but that it embodies, none the less, an important distinction. Suppose, first, that (1 . 13) is true, as in the standard case when $\lambda_n = n$. Then (1 . 14), in this particular case

$$a_n = O\left(\frac{1}{n}\right), \tag{1.17}$$

is not only a sufficient but also a "best possible" condition, which cannot be replaced by any less restrictive condition of the type $a_n = O(\phi_n)$. This ceases to be true when (1 . 13) is false, for example when $\lambda_n = 2^n$. In this case (1 . 14) becomes

$$a_n = O(1), \tag{1.18}$$

which is, in fact, a sufficient condition. But (1 . 18) is in no sense a best possible condition, and the conjecture to which we have referred, and which we now propose to substantiate, is this, that if

$$\frac{\lambda_n}{\lambda_{n-1}} \geqslant \theta > 1,$$

so that (1 . 14) becomes (1 . 18), or, in other words, if λ_n increases with sufficient rapidity and regularity, then the conclusion of the theorem is true *without any restriction whatever on* a_n.

1 . 2. Our main result, then, will be

THEOREM 1.—*If*

$$\lambda_1 > 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \geqslant \theta > 1 \quad (n = 1, 2, \ldots), \tag{1.21}$$

and

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} \to s, \tag{1.22}$$

when $t \to 0$, *then* Σa_n *converges to* s.

It should be observed that (1 . 22) is to be interpreted as implying the convergence of the series for $t > 0$. The condition for this is that

$$a_n = O(e^{\delta \lambda_n}) \tag{1.23}$$

for every positive δ.

The theorem at once shows that such functions as

$$e^{-at} - e^{-a^2 t} + e^{-a^3 t} - \ldots \quad (a > 1), \qquad e^{-1!t} - e^{-2!t} + e^{-3!t} - \ldots$$

cannot tend to limits when $t \to 0$. Such special oscillating functions have been considered by various writers.*

The proof of the theorem divides sharply into two stages. In § 2 we prove it when $a_n = O(1)$, the present form of (1 . 14). In §§ 3-4 we show that the general case may be reduced to that of § 2, and thus complete the proof. In § 5 we show how our analysis may be applied to the construction of non-differentiable functions.

It might seem natural that we should go further. We might suppose that $f(t)$ satisfies an asymptotic formula of the type

$$f(t) \sim At^{-\alpha}\left(\log\frac{1}{t}\right)^{\beta}\ldots,$$

and attempt to deduce corresponding asymptotic formulae for

$$s_n = a_1+a_2+\ldots+a_n,$$

on the lines of our corresponding researches for series subject to (1 . 13).† We leave such questions aside, because of the recent discovery by R. Schmidt of an alternative line of attack on these Tauberian theorems.‡ The method of Schmidt, involving, as it does, not only our machinery of repeated differentiation, but also the general theory of the "moment problem" of Stieltjes, is much less direct and elementary than ours, and the most important results should be proved in the simplest manner possible. But in view of the great power and elegance of Schmidt's methods, and the very comprehensive character of his results, we may well be content to leave our own researches in this field where they stand at present.

We must express our thanks to Prof. G. Pólya, who read this paper in manuscript, and made a series of suggestions which have led to a great simplification of the proof. A number of the lemmas, and, in particular, Lemmas 1, 2, 5, 6, and 10, in their present form, are due to him. We

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* See, for example, G. H. Hardy, "On certain oscillating series", *Quart. J. of Math.*, 38 (1907), 268-288, and "Some theorems concerning infinite series", *Math. Annalen*, 64 (1907), 77-94; J. Belifante, "On power-series of the form $x^{p_0}-x^{p_1}+x^{p_2}-\ldots$", *Proc. Acad. Sc. Amsterdam*, 26 (1924), 456-462.

† G. H. Hardy and J. E. Littlewood, "Some theorems concerning Dirichlet's series", *Messenger of Math.*, 43 (1914), 134-147. See also "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* (2), 13 (1913), 174-191.

‡ R. Schmidt, "Über divergente Reihen und lineare Mittelbildungen", *Math. Zeitschrift*, 22 (1925), 89-152. See also "Die Umkehrsätze des Borelschen Summierungsverfahrens", *Schriften d. Königsberger Gelehrten Gesellschaft*, 1 (1925), 205-256.

have also adopted his suggestion of enunciating Theorem 2 (§ 4 . 5) as an explicit theorem.

2. *Proof of Theorem* 1 *when* $a_n = O(1)$.

2 . 1. In what follows the letter A denotes always a positive number which is a function of θ only. O's and o's refer to the passage of n to infinity or t to zero, and are not, in general, uniform in the parameters of the analysis.* We suppose, as we may, that $|a_n| < 1$. We attribute to λ_0 the value 0.

LEMMA 1.—*If* $0 < \alpha < 1 < \beta$, $r = 2, 3, \ldots$, *then*

$$(2.11) \quad \int_0^{\alpha r} x^r e^{-x} dx < \left(\frac{\alpha}{1-\alpha}\right)^2 \frac{r!}{r-1}, \quad \int_{\beta r}^{\infty} x^r e^{-x} dx < \left(\frac{1}{\beta-1}\right)^2 \frac{r!\,(r+2)}{r^2}.$$

We have
$$\int_0^\infty \left(\frac{r}{x}-1\right)^2 x^r e^{-x} dx = \frac{r!}{r-1},$$

$$\left(\frac{1}{\alpha}-1\right)^2 \int_0^{\alpha r} x^r e^{-x} dx < \int_0^{\alpha r} \left(\frac{r}{x}-1\right)^2 x^r e^{-x} dx$$
$$< \int_0^\infty \left(\frac{r}{x}-1\right)^2 x^r e^{-x} dx = \frac{r!}{r-1},$$

which proves the first of the inequalities (2 . 11). The second follows similarly from the equation

$$\int_0^\infty (x-r)^2 x^r e^{-x} dx = r!\,(r+2).$$

2 . 2. LEMMA 2.—*If*

$$(2.21) \qquad t = \frac{r}{\sqrt{(\lambda_\nu \lambda_{\nu+1})}} \quad (\nu = 1, 2, \ldots),$$

$$(2.22) \qquad w_n = w_n(r, \nu) = \frac{1}{r!} \int_{\lambda_n t}^{\lambda_{n+1} t} x^r e^{-x} dx \quad (n = 0, 1, 2, \ldots),$$

then

$$(2.23) \quad \nu w_0 + (\nu-1) w_1 + \ldots + 2w_{\nu-2} + w_{\nu-1} + w_{\nu+1} + 2w_{\nu+2} + \ldots < \frac{A}{r}.$$

* Thus $f = O(\phi)$, where f and ϕ are functions of t and (say) θ and k, means $|f| < C\phi$, where C is independent of t but a function of θ and k.

We write

$$u_m = w_{\nu-m}+w_{\nu-m-1}+\ldots+w_0 = \frac{1}{r!}\int_0^{\lambda_{\nu-m+1}t} x^r e^{-x}\,dx,$$

$$v_m = w_{\nu+m}+w_{\nu+m+1}+\ldots = \frac{1}{r!}\int_{\lambda_{\nu+m}t}^{\infty} x^r e^{-x}\,dx.$$

Then
$$u_m = \frac{1}{r!}\int_0^{\alpha r} x^r e^{-x}\,dx,$$

where
$$\alpha = \frac{\lambda_{\nu-m+1}}{\lambda_\nu}\sqrt{\left(\frac{\lambda_\nu}{\lambda_{\nu+1}}\right)} < \theta^{\frac{1}{2}-m};$$

and so, by Lemma 1,

(2.24)
$$u_m < \frac{1}{r-1}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2.$$

Similarly
$$v_m = \frac{1}{r!}\int_{\beta r}^{\infty} x^r e^{-x}\,dx,$$

where
$$\beta = \frac{\lambda_{\nu+m}}{\lambda_{\nu+1}}\sqrt{\left(\frac{\lambda_{\nu+1}}{\lambda_\nu}\right)} > \theta^{m-\frac{1}{2}};$$

and so

(2.25)
$$v_m < \frac{r+2}{r^2}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2.$$

From (2.24) and (2.25) it follows that

$$\nu w_0+(\nu-1)w_1+\ldots+w_{\nu-1}+w_{\nu+1}+\ldots = u_1+u_2+\ldots+u_m+v_1+v_2+\ldots$$
$$< \left(\frac{1}{r-1}+\frac{r+2}{r^2}\right)\sum_{m=1}^{\infty}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2 < \frac{A}{r}.$$

2.3. LEMMA 3.—*If $r>0$, then*

(2.31)
$$\sum_1^{\infty} \lambda_n^r e^{-\lambda_n t} = O(t^{-r}).$$

Suppose that $\lambda = r/t,\quad \lambda_\mu \leqslant \lambda < \lambda_{\mu+1}.$

Then
$$t^r \sum_1^{\infty} \lambda_n^r e^{-\lambda_n t} = r^r e^{-r}\sum_1^{\infty}\left\{\frac{\lambda_n}{\lambda}\exp\left(1-\frac{\lambda_n}{\lambda}\right)\right\}^r = r^r e^{-r} S,$$

say. The function xe^{1-r} increases from $x=0$ to $x=1$, when it attains the maximum 1, and then decreases, so that no term of S exceeds 1. If $n \leqslant \mu-1$, $n=\mu-m$, where $1\leqslant m\leqslant \mu-1$, and

$$\frac{\lambda_n}{\lambda} = \frac{\lambda_{\mu-m}}{\lambda} \leqslant \frac{\lambda_{\mu-m}}{\lambda_\mu} < \theta^{-m} < 1;$$

and if $n \geqslant \mu+2$, $n = \mu+m+1$, where $m \geqslant 1$, and

$$\frac{\lambda_n}{\lambda} = \frac{\lambda_{\mu+m+1}}{\lambda} > \frac{\lambda_{\mu+m+1}}{\lambda_{\mu+1}} > \theta^m > 1.$$

Hence

$$S < 2+\sum_{m=1}^{\mu-1}\left\{\frac{\lambda_{\mu-m}}{\lambda_\mu}\exp\left(1-\frac{\lambda_{\mu-m}}{\lambda_\mu}\right)\right\}^r+\sum_{m=1}^{\infty}\left\{\frac{\lambda_{\mu+m+1}}{\lambda_\mu}\exp\left(1-\frac{\lambda_{\mu+m+1}}{\lambda_\mu}\right)\right\}^r *$$

$$< 2+\sum_{m=1}^{\infty}(\theta^{-m}e^{1-\theta^{-m}})^r+\sum_{m=1}^{\infty}(\theta^m e^{1-\theta^m})^r.$$

These series are greatest when $r=1$, so that $S < A$, which proves the lemma.

Lemma 4.—*If* $f(t) \to s$ *when* $t \to 0$, *and*

$$t^r f^{(r)}(t) = O(1)$$

for every r, *then*

$$t^r f^{(r)}(t) = o(1)$$

for every r.†

2 . 4. We can now prove Theorem 1 when $a_n = O(1)$. We may obviously suppose without loss of generality that $s = 0$. We have then $f(t) = o(1)$, and

$$t^r f^{(r)}(t) = (-1)^r t^r \sum_1^\infty a_n \lambda_n^r e^{-\lambda_n t} = O\left(t^r \sum_1^\infty \lambda_n^r e^{-\lambda_n t}\right) = O(1),$$

by Lemma 3, and therefore

$$t^r f^{(r)}(t) = o(1),$$

by Lemma 4.

Now

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} = \sum_1^\infty s_n(e^{-\lambda_n t}-e^{-\lambda_{n+1} t}) = t\sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-ut}\,du,$$

$$(-t)^r f^{(r)}(t) = t^{r+1}\sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} u^r e^{-ut}\,du - rt^r \sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} u^{r-1}e^{-ut}\,du$$

$$= V_r - rV_{r-1},$$

say. Hence

$$(2.41)\qquad V_r = rV_{r-1}+o(1) = r(r-1)V_{r-2}+o(1) = \ldots = r!\,V_0+o(1)$$
$$= r!\,f(t)+o(1) = o(1).$$

---

* The first of these sums is absent if $\mu < 2$.

† J. E. Littlewood, *loc. cit.*, 438.

Now $$r!\, s_\nu = s_\nu \int_0^\infty x^r e^{-x} dx = r!\, s_\nu \sum_0^\infty w_n\,;$$

and so, if $s_0 = 0$, $$\overline{\lim_{\nu\to\infty}} |s_\nu| = \overline{\lim_{\nu\to\infty}} \left| \sum_{n=0}^\infty s_\nu w_n \right|$$

$$\leqslant \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty s_n w_n \right| + \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty (s_n - s_\nu) w_n \right|.$$

But $$r! \sum_0^\infty s_n w_n = \sum_1^\infty s_n \int_{\lambda_n t}^{\lambda_{n+1} t} x^r e^{-x} dx = V_r = o(1),$$

by (2.41). Hence, since $|a_n| < 1$, we have

$$\overline{\lim_{\nu\to\infty}} |s_\nu| \leqslant \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty (s_n - s_\nu) w_n \right|$$

$$\leqslant \nu w_0 + (\nu-1) w_1 + \ldots + w_{\nu-1} + w_{\nu+1} + 2w_{\nu+2} + \ldots < \frac{A}{r},$$

by Lemma 2. This is only possible if $s_\nu \to 0$, which proves the theorem.

2.5. So far we have *assumed* the truth of (1.18). The remainder of the proof of Theorem 1 consists in showing that its hypotheses *imply* (1.18). For this, it is plainly sufficient to show that

(2.51) $$f(t) = O(1)$$

implies (1.18). We suppose, therefore, that

(2.52) $$a_n \neq O(1),$$

and we show that (2.51) is then impossible.

We write $$\overline{\lim} \frac{\log |a_n|}{\log \lambda_n} = \kappa,$$

so that $0 \leqslant \kappa \leqslant \infty$; and we may distinguish three cases: (i) $\kappa = \infty$, (ii) $0 < \kappa < \infty$, and (iii) $\kappa = 0$.

3. *Discussion of case* (i): $\kappa = \infty$.

3.1. The discussion of case (i) does not require the machinery of differentiation. We use two additional lemmas.

LEMMA 5.—*There is an $R = R(\theta)$* such that*

(3.11) $$\lambda_\nu^r e^{-\lambda_\nu t_\nu} > 2 \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu+1}^\infty \right) \lambda_n^r e^{-\lambda_n t_\nu}$$

* So that R is an A.

when

(3.12) $$r \geqslant R, \quad t_\nu = r/\lambda_\nu \quad (\nu = 1, 2, \ldots).$$

We have

$$S = \lambda_\nu^{-r} e^{\lambda_\nu t_\nu} \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu+1}^{\infty} \right) \lambda_n^r e^{-\lambda_n t_\nu}$$

$$= \sum_{m=1}^{\nu-1} \left\{ \frac{\lambda_{\nu-m}}{\lambda_\nu} \exp\left(1 - \frac{\lambda_{\nu-m}}{\lambda_\nu}\right) \right\}^r + \sum_{m=1}^{\infty} \left\{ \frac{\lambda_{\nu+m}}{\lambda_\nu} \exp\left(1 - \frac{\lambda_{\nu+m}}{\lambda_\nu}\right) \right\}^r$$

$$< \sum_{m=1}^{\infty} (\theta^{-m} e^{1-\theta^{-m}})^r + \sum_{m=1}^{\infty} (\theta^m e^{1-\theta^m})^r,$$

as in the proof of Lemma 3. These series are convergent when $r = 1$, and their terms decrease when r increases, so that they are uniformly convergent for $r \geqslant 1$. Finally, each term tends to zero when $r \to \infty$, so that $S \to 0$, which proves the lemma. It is plain that the 2 of the enunciation might be replaced by any larger constant.

3.2. LEMMA 6.—*Suppose that* $r \geqslant R$, *where* R *is defined as in Lemma* 5; *that the series*

(3.21) $$g(t) = \sum_1^{\infty} b_n e^{-\lambda_n t}$$

is convergent for $t > 0$, *and that*

(3.22) $$\varlimsup_{n\to\infty} \lambda_n^{-r} \, | b_n | = \infty.$$

Then

(3.23) $$\varlimsup_{t\to 0} t^r \, | g(t) | = \infty.$$

We denote by $\nu = \nu(t)$ the index and by $\mu = \mu(t)$ the value of the maximum term of the series $\Sigma | b_n | \lambda_n^{-r} e^{-\lambda_n t}$, so that

$$\mu(t) = \operatorname*{Max}_{1 \leqslant n < \infty} | b_n | \lambda_n^{-r} e^{-\lambda_n t} = | b_\nu | \lambda_\nu^{-r} e^{-\lambda_\nu t}.$$

If several terms have the same maximum value, ν is the highest of their indices. It is plain that μ increases steadily when t decreases. Also

$$\mu \geqslant | b_n | \lambda_n^{-r} e^{-\lambda_n t}$$

for every n. If H is given we can, by (3.22), determine n so that $| b_n | \lambda_n^{-r} > 2H$, and then choose t so that $e^{-\lambda_n t} > \frac{1}{2}$, $\mu > H$. It

follows that $\mu \to \infty$ when $t \to 0$, and from this that $\nu \to \infty$, since every term of the series is bounded. We can therefore, given any pair of positive numbers ϵ, g, choose τ so that

$$(3.24) \qquad 0 < \tau < \epsilon, \quad \nu = \nu(\tau) > g, \quad \mu = \mu(\tau) = |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau} > g.$$

We now take

$$(3.25) \qquad t = \tau + t_\nu = \tau + \frac{r}{\lambda_\nu}$$

(so that t_ν is defined as in Lemma 5). Then t tends to 0 with ϵ, since $\nu \to \infty$. Since ν defines the maximum term, we have

$$|b_n| \lambda_n^{-r} e^{-\lambda_n \tau} \leqslant |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau}$$

for all values of n, and so

$$\frac{|b_n| e^{-\lambda_n t}}{|b_\nu| e^{-\lambda_\nu t}} = \frac{|b_n| e^{-\lambda_n(\tau+t_\nu)}}{|b_\nu| e^{-\lambda_\nu(\tau+t_\nu)}} \leqslant \frac{\lambda_n^r e^{-\lambda_n t_\nu}}{\lambda_\nu^r e^{-\lambda_\nu t_\nu}}.$$

Hence, by Lemma 5,

$$\frac{1}{|b_\nu| e^{-\lambda_\nu t}} \left\{ \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) |b_n| e^{-\lambda_n t} \right\} \leqslant \frac{1}{\lambda_\nu^r e^{-\lambda_\nu t_\nu}} \left\{ \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) \lambda_n^r e^{-\lambda_n t_\nu} \right\} < \tfrac{1}{2};$$

and

$$(3.26) \qquad |g(t)| \geqslant |b_\nu| e^{-\lambda_\nu t} - \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) |b_n| e^{-\lambda_n t} > \tfrac{1}{2} |b_\nu| e^{-\lambda_\nu t}$$

$$= \tfrac{1}{2} \lambda_\nu^r e^{-\lambda_\nu t_\nu} . |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau} = \tfrac{1}{2} r^r e^{-r} t_\nu^{-r} \mu(\tau)$$

$$> \tfrac{1}{2} r^r e^{-r} t^{-r} \mu(t),$$

since τ and t_ν are each less than t. It is plain that (3.26) contains the result of Lemma 6, and more.

3.3. We can now dispose of case (i) of Theorem 1. If $\kappa = \infty$,

$$\overline{\lim}\, \lambda_n^{-r} |a_n| = \infty$$

for every r. We may therefore take $b_n = a_n$, $f(t) = g(t)$ in Lemma 6, and it appears at once that $f(t)$ is unbounded, and indeed that

$$\overline{\lim}\, t^r |f(t)| = \infty$$

for every r.

4. *Discussion of cases* (ii) *and* (iii).

4.1. The remaining cases, which we can discuss together, are rather more delicate. We require the machinery of repeated differentiation, and in particular the following lemma.

LEMMA 7.—*If* $\phi(t)$ *increases** *as* t *decreases,*

$$f(t) = o(\phi)$$

when $t \to 0$, *and*

$$t^r f^{(r)}(t) = O(\phi)$$

for every r, *then*

$$t^r f^{(r)}(t) = o(\phi)$$

for every r.†

4.2. We suppose now that

$$\overline{\lim_{n\to\infty}} \, |a_n| = \infty, \tag{4.21}$$

$$\overline{\lim} \frac{\log |a_n|}{\log \lambda_n} = \kappa < \infty, \tag{4.22}$$

and we have to prove that $f(t)$ is unbounded. We write

$$b_n = a_n \lambda_n^r, \quad g(t) = \Sigma b_n e^{-\lambda_n t} = (-1)^r f^{(r)}(t), \tag{4.23}$$

$$\mu(t) = |a_\nu| e^{-\lambda_\nu t} = \text{Max}\,(|a_n| e^{-\lambda_n t}) = \text{Max}\,(|b_n| \lambda_n^{-r} e^{-\lambda_n t}). \tag{4.24}$$

The definitions of $\mu(t)$ and $\nu(t)$ then agree with those of Lemma 6.

LEMMA 8.—*If* $\kappa < l$, *then* $t^l \mu(t) \to 0$.

For
$$t^l \mu(t) = |a_\nu| \lambda_\nu^{-l} (t\lambda_\nu)^l e^{-\lambda_\nu t} \leqslant l^l e^{-l} |a_\nu| \lambda_\nu^{-l},$$

which tends to zero, since $\nu \to \infty$ and $\kappa < l$.

LEMMA 9.—*There are arbitrarily small values of* t *for which*

$$\mu(\tfrac{1}{2}t) < 2^l \mu(t).$$

For otherwise there would be a positive δ such that

$$\mu(2^{-n}\delta) \geqslant 2^l \mu(2^{-n+1}\delta) \geqslant \ldots \geqslant 2^{nl} \mu(\delta),$$

$$(2^{-n}\delta)^l \mu(2^{-n}\delta) \geqslant \delta^l \mu(\delta),$$

for every n, and this would contradict Lemma 8.

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* In the wide sense.

† G. H. Hardy and J. E. Littlewood, "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1912), 411–478 (Theorem 6). We take this opportunity of observing that the proof given in this memoir of Theorem 2 is superfluous, since Theorem 2 can be deduced from Theorem 3 (which is independent of it) by making r tend to infinity.

4 . 3. LEMMA 10.—*We have*

$$\varlimsup_{t\to\infty} \frac{t^r\,|f^{(r)}(t)|}{\mu(\frac{1}{2}t)} < \infty \tag{4.31}$$

for every r, and

$$\varlimsup_{t\to\infty} \frac{t^r\,|f^{(r)}(t)|}{\mu(\frac{1}{2}t)} > 0 \tag{4.32}$$

for all sufficiently large values of r.

In the first place

$$|f^{(r)}(t)| \leqslant \Sigma(|a_n|\,e^{-\frac{1}{2}\lambda_n t}\,.\,\lambda_n^r e^{-\frac{1}{2}\lambda_n t}) \leqslant \mu(\tfrac{1}{2}t)\,\Sigma\,\lambda_n^r e^{-\frac{1}{2}\lambda_n t}.$$

The last series is $O(t^{-r})$, by Lemma 3, and this proves (4 . 31).

Next, we choose τ, as we may do after Lemma 9, so that $0<\tau<\epsilon$ and

$$\mu(\tfrac{1}{2}\tau) < 2^l\mu(\tau).$$

We take this τ as the τ of Lemma 6, and then choose $t=\tau+t_\nu$ as there. We have then

$$|f^{(r)}(t)| > C_r\mu(\tau)\,t^{-r},$$

by (3 . 26), C_r being a positive number depending only on r. But

$$\mu(\tau) > 2^{-l}\mu(\tfrac{1}{2}\tau) > 2^{-l}\mu(\tfrac{1}{2}t),$$

and so
$$|f^{(r)}(t)| > C_r\mu(\tfrac{1}{2}t)\,t^{-r};$$

which proves (4 . 32).

4 . 4. We can now complete the proof of the theorem. In fact, if $f(t)=O(1)$, we have *a fortiori*

$$f(t) = o\{\mu(\tfrac{1}{2}t)\},$$

and
$$t^r f^{(r)}(t) = O\{\mu(\tfrac{1}{2}t)\},$$

for every r, by (4 . 31). Hence, taking $\phi=\psi=\mu(\frac{1}{2}t)$ in Lemma 7, we have

$$t^r f^{(r)}(t) = o\{\mu(\tfrac{1}{2}t)\}$$

for every r. This plainly contradicts (4 . 32). Hence $f(t)$ cannot be bounded.

4 . 5. Our primary object in §§ 3–4 has been to prove that $f(t)$ cannot be bounded unless $|a_n|$ is bounded. It may be convenient that we should collect here, as an explicit theorem, and in a somewhat sharpened form, the principal results of our discussion, as these results have an interest independent of their application to the proof of Theorem 1.

THEOREM 2.—*Suppose that*

$$\lambda_1 > 0, \quad \theta > 1, \quad \lambda_{n+1} \geqslant \theta\lambda_n;$$

that $f(t) = \Sigma a_n e^{-\lambda_n t}$ *is convergent for* $t > 0$, *and that*

$$\mu(t) = \operatorname{Max}(|a_n| e^{-\lambda_n t}) \quad (n = 1, 2, \ldots).$$

Then, if

(4 . 51) $$\overline{\lim_{n\to\infty}}\, |a_n| = \infty,$$

we have

(4 . 521) $$\lim_{t\to 0} \mu(t) = \infty,$$

(4 . 522) $$\overline{\lim_{t\to 0}}\, |f(t)| = \infty,$$

(4 . 523) $$\overline{\lim_{t\to 0}}\, \frac{|f(t)|}{\mu(t)} > 0.$$

And if

(4 . 53) $$\overline{\lim_{n\to\infty}}\, \frac{\log |a_n|}{\log \lambda_n} = \infty,$$

we have

(4 . 541) $$\overline{\lim_{t\to 0}}\, \frac{\log \mu(t)}{\log (1/t)} = \infty,$$

(4 . 542) $$\overline{\lim_{t\to 0}}\, \frac{\log |f(t)|}{\log (1/t)} = \infty,$$

(4 . 543) $$\overline{\lim_{t\to 0}}\, \frac{|f(t)|}{\mu(t)} \geqslant 1.$$

Of the propositions (4 . 52), (4 . 521) was proved incidentally, and is in any case almost obvious*; (4 . 522) was our principal conclusion, and we deduced (4 . 522) from (4 . 523).†

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* Write a_n for $b_n\lambda_n^{-r}$ in the first sentences of the proof of Lemma 6.

† Strictly, from a slightly stronger proposition (with $\frac{1}{2}t$ for t).

The proofs of the propositions (4 . 54) are contained, in substance, in the proof of Lemma 6. Take there $b_n = a_n$, $g(t) = f(t)$; it will be observed that $\mu(t)$ has then no longer its meaning of § 3 . 2. In virtue of (4 . 53), we have $|a^n| > \lambda_n^r$ for every r and an appropriate sequence of values of n, and

$$t^r \mu(t) \geqslant (\lambda_n t)^r e^{-\lambda_n t} > e^{-1}$$

if $\lambda_n t = 1$. This proves (4 . 541), and (4 . 542) is equivalent to the main result of Lemma 6. Finally

$$|f(t)| \geqslant \tfrac{1}{2}\mu(t) \tag{4 . 57}$$

for an appropriate sequence of values of t, by (3 . 26).* This would give $\frac{1}{2}$ instead of the 1 of (4 . 543). But the $\frac{1}{2}$ arises only from the 2 of Lemma 5, and this 2 could obviously be replaced by any larger constant, so that the $\frac{1}{2}$ of (4 . 57) could be replaced by any number less than 1. This remark completes the proof of Theorem 2, which could naturally have been arranged more elegantly had this theorem been our goal.

5. *Examples of non-differentiable functions.*

5 . 1. We conclude by applying Theorem 1 to the construction of non-differentiable functions. We require two further lemmas.

LEMMA 11.—*Suppose that* $f(s) = f(\sigma+it)$ *is regular for* $\sigma > 0$, *continuous for* $\sigma \geqslant 0$, *uniformly in* t, *and tends uniformly to zero when* $\sigma \to \infty$; *and that* $f(iu) = \phi(u)$. *Then*

$$f(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma\phi(u)}{\sigma^2+(u-t)^2}\, du. \tag{5 . 11}$$

This is a very special case of well known results in the general theory of functions. All the conditions are satisfied if, as we shall suppose,

$$f(s) = \sum_1^{\infty} a_n e^{-\lambda_n s},$$

where λ_n satisfies (1 . 21) and Σa_n is absolutely convergent.

We write

$$f(s) = G(\sigma,\, t)+iH(\sigma,\, t), \quad \phi(t) = g(t)+ih(t).$$

* First line [noting the altered meaning of $\mu(t)$].

LEMMA 12.—*If $\phi(t)$ has a finite derivative $\phi'(t)$ for a particular value of t, then*

$$f'(s) \to \phi'(t) \tag{5.12}$$

when t is fixed and $\sigma \to 0$. If $g(t)$ has a finite derivative $g'(t)$, then

$$\frac{\partial G}{\partial t} \to g'(t); \tag{5.13}$$

and similarly for H and h.

It is plainly sufficient to prove (5.13); and we may suppose, without loss of generality, that the particular value of t in question, and the value of $g'(t)$, are both zero. We have then, on differentiating (5.11) and putting $t = 0$,

$$\frac{\partial G}{\partial t} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sigma u g(u)}{(\sigma^2+u^2)^2}\, du.$$

We can choose δ so that $|g(u)| < \epsilon |u|$ in $(-\delta, \delta)$. The integral outside these limits plainly tends to zero, and

$$\left| \frac{2}{\pi}\int_{-\delta}^{\delta} \frac{\sigma u g(u)}{(\sigma^2+u^2)^2}\, du \right| < \frac{4\epsilon}{\pi}\int_0^{\delta} \frac{\sigma u^2\, du}{(\sigma^2+u^2)^2} < \frac{4\epsilon}{\pi}\int_0^{\infty} \frac{w^2\, dw}{(1+w^2)^2} < A\epsilon;$$

which proves the lemma.

5.2. THEOREM 3.—*Suppose that λ_n satisfies* (1.21) *and that $\Sigma |a_n|$ is convergent. If then the function*

$$\phi(t) = \Sigma a_n e^{-\lambda_n it}$$

has a finite derivative for a particular value of t, the series

$$\Sigma a_n \lambda_n e^{-\lambda_n it} \tag{5.21}$$

is convergent, to sum $i\phi'(t)$. If a_n is real, and $g(t) = \Sigma a_n \cos \lambda_n t$ or $h(t) = \Sigma a_n \sin \lambda_n t$ has a finite derivative, then the corresponding one of the series

$$\Sigma a_n \lambda_n \sin \lambda_n t, \quad \Sigma a_n \lambda_n \cos \lambda_n t \tag{5.22}$$

is convergent, to sum $-g'(t)$ or $h'(t)$.

In other words, if the derivative exists, it is given by term-by-term differentiation.

To prove Theorem 3 we have only to observe that, by Lemma 12,

$$\Sigma a_n \lambda_n e^{-\lambda_n(\sigma+it)}$$

(or the corresponding real series) tends to a limit when $\sigma \to 0$. The result then follows from Theorem 1.

5 . 3. We can now construct very general classes of continuous non-differentiable functions.

In the first place, the series (5 . 21) cannot converge, for any t, if

$$a_n \lambda_n \neq o(1);$$

and in these circumstances the (complex) function $\phi(t)$ cannot be differentiable for any t. If $|a_n \lambda_n|$ has a positive lower bound, the real series are non-differentiable almost everywhere, since the set of points for which $\sin \lambda_n t$ or $\cos \lambda_n t$ tends to zero is in any case of measure zero;* but it is necessary to particularize further if we wish for results valid without exception.

Suppose that

$$a_n = \lambda_n^{-\alpha} b_n, \tag{5.31}$$

where $0 < \alpha \leqslant 1$ and $|b_n|$ has positive bounds. In these circumstances we can prove that the real series have no finite derivatives for any t. This is, however, not an immediate deduction from Theorem 1, and we need an additional lemma.

LEMMA 13.—*If $g(t)$ has a finite derivative $g'(t)$, then*

$$\frac{\partial^2 G}{\partial t^2} = o\left(\frac{1}{\sigma}\right).$$

The proof is similar to that of Lemma 12. Differentiating (5 . 11) twice, and making the same simplifications, we have

$$\frac{\partial^2 G}{\partial t^2} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(u^2-\sigma^2)}{(\sigma^2+u^2)^3} g(u)\, du.$$

The part of the integral outside $(-\delta, \delta)$ again tends to zero, and

$$\left| \frac{2}{\pi} \int_{-\delta}^{\delta} \frac{\sigma(u^2-\sigma^2)}{(\sigma^2+u^2)^3} g(u)\, du \right| < \frac{4\epsilon}{\pi} \int_0^{\delta} \frac{\sigma u(u^2-\sigma^2)}{(\sigma^2+u^2)^3}\, du < \frac{4\epsilon}{\pi\sigma} \int_0^{\infty} \frac{w(w^2-1)}{(1+w^2)^3}\, dw,$$

which proves the lemma.

* See F. Bernstein, "Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem", *Math. Annalen*, 71 (1912), 417–439 (421); G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Acta Math.*, 37 (1914), 155–190 (180).

5 . 4. If now

$$\psi(\sigma) = \frac{\partial G}{\partial t} = -\Sigma\, a_n \lambda_n \sin \lambda_n t\, e^{-\lambda_n \sigma},$$

$$\chi(\sigma) = \frac{\partial^2 G}{\partial t^2} = -\Sigma\, a_n \lambda_n^2 \cos \lambda_n t\, e^{-\lambda_n \sigma},$$

and $g'(t)$ exists, we have

$$\psi(\sigma) - g'(t) = o(1) = o(\sigma^{\alpha-1}), \tag{5 . 41}$$

by Lemma 12, and

$$\chi(\sigma) = o\left(\frac{1}{\sigma}\right) = o(\sigma^{\alpha-2}), \tag{5 . 42}$$

by Lemma 13. Also

$$\begin{aligned} \sigma^r \psi^{(r)}(\sigma) &= (-1)^{r+1} \sigma^r \Sigma\, a_n \lambda_n^{r+1} \sin \lambda_n t\, e^{-\lambda_n \sigma} \\ &\quad O(\sigma^r \Sigma \lambda_n^{r+1-\alpha} e^{-\lambda_n \sigma}) = O(\sigma^{\alpha-1}), \end{aligned} \tag{5 . 43}$$

by (5 . 31) and Lemma 3, and

$$\begin{aligned} \sigma^{r-1} \chi^{(r-1)}(\sigma) &= (-1)^r \sigma^{r-1} \Sigma\, a_n \lambda_n^{r+1} \cos \lambda_n t\, e^{-\lambda_n \sigma} \\ &= O(\sigma^{r-1} \Sigma \lambda_n^{r+1-\alpha} e^{-\lambda_n \sigma}) = O(\sigma^{\alpha-2}), \end{aligned} \tag{5 . 44}$$

for the same reasons. Applying Lemma 7 to (5 . 41) and (5 . 43), we obtain

$$\sigma^r \psi^{(r)}(\sigma) = o(\sigma^{\alpha-1}); \tag{5 . 45}$$

and similarly, from (5 . 42) and (5 . 44),

$$\sigma^{r-1} \chi^{(r-1)}(\sigma) = o(\sigma^{\alpha-2}). \tag{5 . 46}$$

From (5 . 45) and (5 . 46) we deduce

$$\begin{aligned} f^{(r+1)}(s) &= (-1)^{r+1} \Sigma\, a_n \lambda_n^{r+1} e^{-\lambda_n(\sigma+it)} \\ &= -\chi^{(r-1)}(\sigma) + i\psi^{(r)}(\sigma) = o(\sigma^{\alpha-r-1}). \end{aligned} \tag{5 . 47}$$

But this is impossible. For, by (4 . 523),

$$\overline{\lim} \frac{|f^{(r+1)}(s)|}{\mathbf{m}(\sigma)} > 0,$$

where $$\mathbf{m}(\sigma) = \operatorname{Max}(|a_n| \lambda_n^{r+1} e^{-\lambda_n \sigma}).$$

This is greater than a constant multiple of

$$\operatorname{Max}(\lambda_n^{r+1-a} e^{-\lambda_n \sigma}),$$

and therefore than a constant multiple of σ^{a-r-1}, which contradicts (5 . 47). It follows that $g'(t)$ cannot exist for any value of t.

5 . 5. An interesting case is that in which λ_{n+1}/λ_n is a bounded integer μ_{n+1}. Consider, for example, the cosine series $g(t)$. If $g(t)$ has a finite derivative, the first series (5 . 22) is convergent. If also $|a_n \lambda_n|$ has a positive lower bound, $\sin \lambda_n t$ must tend to zero. If then $t = \pi u$, we have

$$\lambda_n u = m_n + \epsilon_n,$$

where m_n is integral and $\epsilon_n \to 0$. Hence

$$(\mu_{n+1} m_n - m_{n+1}) + (\mu_{n+1} \epsilon_n - \epsilon_{n+1}) = 0,$$

which is only possible, for large n, if the two terms are separately zero. It follows that

$$\epsilon_{n+1} = \mu_{n+1} \epsilon_n \quad (n > n_0),$$

$$\epsilon_{n+k} = \mu_{n+1} \mu_{n+2} \ldots \mu_{n+k} \epsilon_n \quad (n > n_0).$$

As $\epsilon_{n+k} \to 0$ when $k \to \infty$, ϵ_n must be zero, and u is of the form m_n / λ_n. Thus $g(t)$ cannot possess a finite derivative except perhaps for values of t belonging to a certain enumerable set, viz. the set

$$t = \frac{M\pi}{\lambda_j} \quad (j = 1, 2, \ldots)$$

where M is integral.

Suppose, in particular, that a_n is positive, and that t has the value (5 . 51). We may suppose, without real loss of generality, that $j = 1$. Then if $g'(t)$ exists it is zero, by Theorem 3. But

$$\frac{g(t+h)-g(t)}{h} = -\frac{1}{h} \Sigma a_n (1 - \cos \lambda_n h) = -\frac{2}{h} \Sigma a_n \sin^2 \tfrac{1}{2} \lambda_n h.$$

The right-hand side has the sign opposite to that of h, and is numerically greater than a constant multiple of

$$\frac{1}{h} \sum_{\lambda_n h \leqslant 1} \frac{1}{\lambda_n} (\lambda_n h)^2 = \sum_{\lambda_n h \leqslant 1} \lambda_n h,$$

which is not less than 1 if h is of the form $1/\lambda_\nu$. Hence $g'(t)$ cannot be zero.

We may sum up the results of §§ 5 . 3–5 . 5 in

THEOREM 4.—*The continuous complex function* $\phi(t)$ *cannot possess a finite derivative for any value of* t *in any case in which* $a_n\lambda_n$ *does not tend to zero. If* a_n *is real, and* $|a_n\lambda_n|$ *has a positive lower bound, the continuous functions* $g(t)$ *and* $h(t)$ *can at most have finite derivatives at a set of points of measure zero. If* a_n *is of the form* $\lambda_n^{-\alpha} b_n$, *where* $0 < \alpha < 1$ *and* $|b_n|$ *has positive upper and lower bounds, then neither* $g(t)$ *nor* $h(t)$ *can have a finite derivative for any value of* t. *If* λ_{n+1}/λ_n *is a bounded integer, and* $|a_n\lambda_n|$ *has a positive lower bound, then* $g(t)$ *cannot have a finite derivative except perhaps when* t *is of the form* $M\pi/\lambda_j$, *where* M *is an integer. If in addition* a_n *is positive, then* $g(t)$ *cannot have a finite derivative for any value of* t.

For example, the series

$$\Sigma(-1)^{p_n} b^{-n\alpha} \cos b^n t, \quad \Sigma b^{-n\alpha} p_n \cos b^n t, \quad \Sigma(-1)^{p_n}(n!)^{-\alpha} \cos n!\, t$$

have never finite derivatives. Here $0 < \alpha \leqslant 1$, p_n is an arbitrary positive integer, and b is integral in the second series but not necessarily in the first.

A particular case of the theorem is that Weierstrass's function $\Sigma a^n \cos b^n t$ has no finite derivative for any value of t when $ab \geqslant 1$.*

* G. H. Hardy, "Weierstrass's non-differentiable function", *Trans. Amer. Math. Soc.*, 17 (1916), 301-325 (303).

CORRECTIONS

p. 221, 1*st footnote*. For Belifante read Belinfante.

p. 223, *line* 4 *up*. For xe^{1-r} read xe^{1-x}.

p. 231, *line* 4. For a^n read a_n.

p. 236, *line* 5 *up*. For 'never finite' read 'no finite'.

COMMENTS

Condition (1.13) was shown to be superfluous by Ananda-Rau.†

Theorem 1 is the 'high indices' theorem, which was conjectured by Littlewood fifteen years earlier. If the 'gap sequence' λ_n is given, the theorem may be regarded as a Mercerian theorem, in which the transformation from s_n to $f(x)$, and its inverse from $f(x)$ to s_n, are both regular; see the Comments on 1912, 5. On the other hand, if λ_n is an arbitrary increasing and unbounded sequence, the condition $\overline{\lim}\, |A(t)-A(\lambda_n) = 0$ as $n \to \infty$, where $t > \lambda_n$, $t/\lambda_n \to 1$, and

$$A(t) = \sum_{\lambda_\nu \leqslant t} a_\nu,$$

is a Tauberian condition. This is satisfied when λ_n is of the form (1.21), or when $a_n = O((\lambda_n - \lambda_{n-1})/\lambda_n)$; see D.S., p. 177. A simpler proof of Theorem 1 was given by Ingham.‡

In § 5, Hardy and Littlewood apply Theorem 1 to the study of conditions under which $\sum a_n e^{-\lambda_n iu}$, or its real or imaginary parts, are non-differentiable functions. In the final example, Weierstrass's function $\sum a^n \cos b^n t$, the conditions $0 < a < 1$, $b > 1$, $ab \geqslant 1$ were obtained by Hardy in 1916, 2 (in Vol. IV). Weierstrass's conditions were $0 < a < 1$, b an odd integer > 1, and $ab > 1+3\pi/2$. For references, see 1916, 2.

† *J. London Math. Soc.* 3 (1928), 200–5.

‡ *Quart. J. of Math.* (1st Oxford series), 8 (1937), 1–7.

NOTE ON THE MULTIPLICATION OF SERIES

G. H. HARDY†.

1. I proved in 1908‡ that if the series $A = \Sigma a_m$ and $B = \Sigma b_n$ are convergent, and ma_m and nb_n are bounded, then the series may be multiplied by Cauchy's rule. In 1911§ I proved a corresponding theorem for

† Received and read 10 March, 1927.

‡ G. H. Hardy, "The multiplication of conditionally convergent series", *Proc. London Math. Soc.* (2), 6 (1908), 410–423.

§ G. H. Hardy, "On the multiplication of Dirichlet's series", *Proc. London Math. Soc.* (2), 10 (1912), 396–405.

1927, 10 Journal of the London Mathematical Society, 2, 169–71.

"Dirichlet multiplication" of type λ_n, but assumed the type of the series to be subject to a restriction, viz. that $\lambda_n - \lambda_{n-1} = o(\lambda_n)$. This restriction was afterwards shown to be unnecessary by Rosenblatt†; and Landau, in 1920‡, gave an extremely concise proof of Rosenblatt's unrestricted theorem. Finally Neder, in 1923§, applied Landau's method to generalize the result further in various directions.

Concise as is Landau's proof, there is still, I think, a good deal to be said for my original method; and I prove here, by this method, the corresponding theorem in which a_m and b_n are real and subject to a "one-sided" condition only. This is hardly a new theorem (though it has not, so far as I know, been stated explicitly before), since it is easily deduced from Neder's generalization of the original theorem, and my object in proving it is mainly to show that the method which I followed before does, in fact, lead very simply to the most general results.

2. THEOREM. *Suppose* (1) *that λ_m and μ_n each increase steadily to infinity, and that ν_p is the sequence $\lambda_m + \mu_n$ arranged in order of magnitude*; (2) *that A and B are convergent*; *and* (3) *that a_m and b_n are real and*

$$\frac{\lambda_m a_m}{\lambda_m - \lambda_{m-1}} > -1, \quad \frac{\mu_n b_n}{\mu_n - \mu_{n-1}} > -1.$$

Then the series $C = \Sigma C_p$, where C_p is the sum of all products $a_m b_n$ for which $\lambda_m + \mu_n = \nu_p$, converges to AB.

We may plainly suppose, without loss of generality, that $\lambda_1 > 0$, $\mu_1 > 0$, $A = 0$, $B = 0$. It is then known|| that C is summable $(R, \nu, 1)$ to sum 0, *i.e.* that

$$(\nu_2 - \nu_1) C_1 + (\nu_3 - \nu_2) C_2 + \ldots + (\nu_p - \nu_{p-1}) C_{p-1} = o(\nu_p),$$

† A. Rosenblatt, "Über einen Satz des Herrn Hardy", *Jahresbericht d. Deutschen Math.-Vereinigung*, 23 (1914), 80–84. Rosenblatt had earlier extended the results of my first paper in other directions: see his memoir "Über die Multiplikation der unendlichen Reihen", *Bulletin de l'Acad. de Cracovie* (A), 1913, 603–631. For still other generalizations, see T. S. Broderick, "On Dirichlet multiplication of infinite series", *Proc. London Math. Soc.* (2), 22 (1923), 468–482.

‡ E. Landau, "Über einen Satz des Herrn Rosenblatt", *Jahresbericht d. Deutschen Math.-Vereinigung*, 29 (1920), 238.

§ L. Neder, "Über Taubersche Bedingungen", *Proc. London Math. Soc.*, 23 (1924), 172–184 (176–177).

|| See for example G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series* (Cambridge, 1915), 64. The essential difference between my method of proof and Landau's or Rosenblatt's is that I make use of this result and they do not.

where C_p is the sum of the first p of the c's. This is

$$\nu_p C_p - \nu_1 c_1 - \nu_2 c_2 - \ldots - \nu_p c_p = o(\nu_p),$$

so that the necessary and sufficient condition for the convergence of C to zero is

(1) $$C_p^* = \nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p = o(\nu_p).$$

If we write

$$A(x) = \sum_{\lambda_m \leqslant x} a_m, \quad B(x) = \sum_{\mu_n \leqslant x} b_n,$$

we have

$$C_p^* = \sum_{\lambda_m + \mu_n \leqslant \nu_p} (\lambda_m + \mu_n) a_m b_n$$

$$= \Sigma \lambda_m a_m B(\nu_p - \lambda_m) + \Sigma \mu_n b_n A(\nu_p - \mu_n) = X_p + Y_p,$$

say, the summations being bounded by $\lambda_m \leqslant \nu_p - \mu_1$ and $\mu_n \leqslant \nu_p - \lambda_1$ respectively. It is plainly sufficient to prove that X_p and Y_p are $o(\nu_p)$, and, on grounds of symmetry, we need only consider X_p.

I observe first that if $\mu_1 < H < \nu_p$ then

(2) $$\sum_{\nu_p - H < \lambda_m \leqslant \nu_p - \mu_1} \lambda_m |a_m| < 2H + o(\nu_p),$$

uniformly in H. For if we write $a_m = a_m^+ - a_m^-$, $|a_m| = a_m^+ + a_m^-$, we have

$$\Sigma \lambda_m |a_m| = \Sigma \lambda_m a_m + 2\Sigma \lambda_m a_m^- < o(\nu_p) + 2\Sigma(\lambda_m - \lambda_{m-1}),$$

since A is convergent, and this gives (2)†.

Since B converges to zero, we can choose $H = H(\epsilon) > \mu_1$ so that $|B(x)| < \epsilon$ for $x \geqslant H$. Then

$$|X_p| \leqslant \epsilon \sum_{\lambda_m \leqslant \nu_p - \mu_1} \lambda_m |a_m| + \sum_{\nu_p - H < \lambda_m \leqslant \nu_p - \mu_1} \lambda_m |a_m| |B(\nu_p - \lambda_m)|$$

$$\leqslant K\epsilon\nu_p + K\big(2H + o(\nu_p)\big) < 2K\epsilon\nu_p,$$

where K is a constant, for sufficiently large values of p; and so $X_p = o(\nu_p)$, which proves the theorem. In particular, the Cauchy product of A and B is convergent if $ma_m > -1$ and $nb_n > -1$.

† Cf. G. H. Hardy and J. E. Littlewood, "Two theorems concerning Fourier series", *Journal London Math Soc.*, 1 (1926), 19–25 (20, Lemma β). The first term in $\Sigma \lambda_m a_m^-$ here goes into the $o(\nu_p)$.

CORRECTION

p. 171, *line* 5 *up*. For $\sum_{\lambda_m \leqslant \nu_p - \mu_1}$ read $\sum_{\lambda_m \leqslant \nu_p - H}$.

COMMENTS

In Hardy's O-theorem of 1911 (Theorem II of 1912, 2), and its extensions by Rosenblatt (1914) and Neder (1923),† μ_n is taken equal to λ_n. To deduce Hardy's one-sided theorem, § 2, from Neder's theorem, we have, from hypotheses (2) and (3),

$$\sum_{\frac{1}{2}x \leqslant \lambda_n \leqslant x} |a_n| = \sum_{\frac{1}{2}x \leqslant \lambda_n \leqslant x} a_n + \sum_{\frac{1}{2}x \leqslant \lambda_n \leqslant x} (|a_n| - a_n)$$
$$< o(1) + \sum_{\frac{1}{2}x \leqslant \lambda_n \leqslant x} 2K(\lambda_n - \lambda_{n-1})/\lambda_n = O(1),$$

and similarly $\sum_{\frac{1}{2}x \leqslant \mu_n \leqslant x} |b_n| = O(1)$. The λ_n and μ_n may then be replaced by a single sequence by inserting zero terms in the series; see the Comments on 1912, 2.

† Date of presentation.

A THEOREM IN THE THEORY OF SUMMABLE DIVERGENT SERIES

By G. H. HARDY and J. E. LITTLEWOOD.

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1. *Introduction.*

1.1. This paper arises from a theorem which we published in 1924* and which proved to have important applications in the theory of Fourier series. The theorem ran as follows : *in order that the series*

$$\Sigma b_n = b_0 + b_1 + b_2 + \ldots \tag{1.11}$$

should be summable (C, r), *it is necessary and sufficient that the series*

$$\Sigma c_n = c_0 + c_1 + c_2 + \ldots, \tag{1.12}$$

where

$$c_n = \frac{b_n}{n+1} + \frac{b_{n+1}}{n+2} + \ldots, \tag{1.13}$$

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* G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power-series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67–96. See also "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1925), 211–246.

The substance of the present paper was written in the summer of 1925, and its publication has been delayed for various reasons. The subject is essentially the same as that of the independent researches of Andersen published recently in the *Proceedings*, though there are considerable differences in the point of view and the methods of proof. See A. F. Andersen, "Comparison theorems in the theory of Cesàro summability", *Proc. London Math. Soc.* (2), 27 (1927), 39–71. In particular one of our principal theorems (Theorem 3) is included as a particular case in Andersen's Theorem 2 (stated by him in the form of two theorems 2A and 2B), Andersen considering arbitrary real values of r where we consider only integral values.

We shall also have occasion to refer (under the short title of *Studier*) to Andersen's dissertation "Studier over Cesàro's Summabilitetsmetode", Copenhagen, 1921, which contains the most general account of the theory of summability yet published.

Finally we should add that all these theorems are very closely related to theorems proved earlier by Knopp. See K. Knopp, "Über die Oszillation einfach unbestimmter Reihen", *Sitzungsberichte der Berliner Math. Gesellschaft*, 16 (1917), 45-50, and "Zur Theorie der C- und H-Summierbarkeit", *Math. Zeitschrift*, 19 (1924), 97–113. See also G. H. Hardy, "A theorem concerning summable series", *Proc. Camb. Phil. Soc.*, 20 (1921), 304–307, and W. L. Ferrar, "Necessary and sufficient conditions for summability (C, r)", *Journal London Math. Soc.*, 1 (1926), 175–179.

should be summable $(C, r-1)$, *and the sums of the two series are the same.* Here r is a positive integer or zero, and the sum (1.13) by which c is defined is, for $r > 0$, a Cesàro sum of order $r-1$.

This theorem may be stated in a different form which is more illuminating and lends itself more naturally to extension. Suppose, first, that (1.11) is summable (C, r) to sum B, and write

$$b_n = (n+1)\,a_n.$$

Then Σa_n is summable $(C, r-1)$*, say to sum A, and

$$c_0 = A, \quad c_n = A - a_0 - a_1 - \ldots - a_{n-1} = A - A_{n-1} \quad (n > 0).$$

It follows that

$$A + (A - A_0) + (A - A_1) + \ldots$$

is summable $(C, r-1)$ to sum B, or that

$$\sum_{n=0}^{\infty} (A - A_n)$$

is summable $(C, r-1)$ to sum $B-A$. Conversely, if this series is, for any value of A, summable $(C, r-1)$ to sum $B-A$, then

$$A_n \rightarrow A \quad (C, r-1),$$

so that Σa_n is summable $(C, r-1)$ to sum A. If now we write

$$c_n = a_n + a_{n+1} + \ldots \quad (C, r-1),$$

then Σc_n is summable $(C, r-1)$ to sum $B - A + A = B$, and so Σb_n is summable (C, r) to the same sum.

We thus obtain

THEOREM 1. *In order that the series*

(1.14) $$\Sigma (n+1) a_n$$

should be summable (C, r) *to sum* B, *it is necessary and sufficient that*

(1.15) $$\Sigma (A - A_n),$$

where $A_n = a_0 + a_1 + \ldots + a_n$ *and* A *is the sum* $(C, r-1)$ *of* Σa_n, *should be summable* $(C, r-1)$ *to sum* $B-A$.

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* By a well known theorem due to M. Riesz. See S. Chapman, "On non-integral orders of summability of series and integrals", *Proc. London Math. Soc.* (2), 9 (1911), 369–409 (388); where a more general theorem is proved. A slightly less precise form of the theorem was proved independently by H. Bohr, *Bidrag til de Dirichlet'ske Raekkers Theori*, Copenhagen, 1910.

1.2. We observe now that, if we write generally

$$\Delta u_n = u_n - u_{n+1},$$

then $\Delta(n+1) = -1$. Thus Theorem 1 asserts that the series (1.14) is summable (C, r) if, and only if,

$$\Sigma(A_n - A)\,\Delta(n+1)$$

is summable $(C, r-1)$, *i.e.* if its order of summability is reduced by unity by an appropriate partial summation.

We are thus led to conjecture that if ϕ_n is any sufficiently regular function of a suitable rate of increase, the series

$$\Sigma a_n \phi_n \tag{1.21}$$

will be summable (C, r) if, and only if, the series

$$\Sigma(A_n - A)\Delta\phi_n, \tag{1.22}$$

where A is an appropriate Cesàro sum of Σa_n, is summable $(C, r-1)$. Similarly we are led to suppose that the integral

$$\int_0^\infty a(x)\,\phi(x)\,dx, \tag{1.23}$$

will be summable (C, r) if, and only if, the integral

$$\int_0^\infty \{A(x) - A\}\,\phi'(x)\,dx, \tag{1.24}$$

where

$$A(x) = \int_0^x a(t)\,dt \tag{1.25}$$

and A is an appropriate Cesàro sum of

$$\int_0^\infty a(x)\,dx, \tag{1.26}$$

is summable $(C, r-1)$. Our object here is to investigate how far these conjectures are true.

1.3. Our conclusions, which it is convenient to state for integrals in the first instance, may be summarized as follows. Our theorem, that the summability $(C, r-1)$ of (1.24) is a necessary and sufficient condition for the summability of (C, r) of (1.23), is true, in the first instance, for

$$\phi(x) = x^a \quad (a > 0).$$

This we prove in § 3. More generally, it is true if $\phi(x)$ is a sufficiently

regular function whose rate of increase is approximately that of a positive power of x. The conditions to be imposed upon $\phi(x)$ could be stated, for any given value of r, as inequalities to be satisfied by $\phi(x)$ and its first $r+1$ derivatives; but it is more convenient to adopt a different point of view. We suppose that $\phi(x)$ is, in the language of Hardy's *Orders of infinity**, a "logarithmico-exponential" function or "L-function", that is to say a function definable by some finite combination of algebraic, logarithmic, or exponential symbols. This being so, the theorem is true if $\phi(x)$ is an L-function of the region

$$x^{\delta} \prec \phi \prec x^{\Delta},$$

that is to say an L-function which increases neither more nor less rapidly than all positive powers of x. If $\phi(x)$ lies outside this region, the theorem is false; thus when $\phi(x) = \log x$ our condition is necessary, but not sufficient, while if $\phi(x) = e^{x}$ it is sufficient, but not necessary.

This we prove in § 5. In § 4 we prove the theorem for series which corresponds to that of § 3; here ϕ_n has the special form

$$\phi_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)},$$

which is more convenient than the substantially equivalent form $(n+1)^{\alpha}$. The more general theorem corresponding to that of § 5 is discussed briefly in § 6.

There is naturally no really important difference between the proofs of the theorems for series and for integrals, but partial integration is formally a little simpler than partial summation, and we have therefore put integrals first throughout the paper. The formal difficulties are then reduced to a minimum, and the kernel of the argument is clearer, while it is easy to indicate the modifications necessary in the other case.

2. *Definitions and lemmas.*

2.1. We suppose that $f(x)$ is integrable† over any finite interval $(0, X)$, and we write

(2.11)

$$F_0(x) = F(x) = \int_0^x f(t)\,dt, \quad F_1(x) = \int_0^x F_0(t)\,dt, \quad F_2(x) = \int_0^x F_1(t)\,dt, \quad \ldots.$$

* G. H. Hardy, *Orders of infinity*, Cambridge Tracts in Mathematics, 12 (2nd ed., 1924), 17.

† In the sense of Lebesgue.

Then

$$F_k(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)\,dt \tag{2.12}$$

if $k > 0$. If

$$F_k(x) \sim F\frac{x^k}{k!} \tag{2.13}$$

when $x \to \infty$, we say that

$$\int_0^\infty f(x)\,dx = F \quad (C,\ k), \tag{2.14}$$

or that the integral (2 . 14) is summable (C, k) to sum F. We also say that

$$F_0(x) \to F \quad (C,\ k). \tag{2.15}$$

We shall say that the integral (2 . 14) is summable $(C, -1)$ if it is convergent, *i.e.* summable $(C, 0)$, and

$$f(x) = o\left(\frac{1}{x}\right). \tag{2.16}$$

We shall also write

$$F_0(x) \sim Kx^l, \quad F_0(x) = O(x^l), \quad F_0(x) = o(x^l) \quad (C,\ k), \tag{2.17}$$

where $l > -1$, meaning thereby

$$F_k(x) \sim K\frac{x^{l+k}}{(l+1)(l+2)\dots(l+k)}, \quad F_k(x) = O(x^{l+k}), \quad F_k(x) = o(x^{l+k}) \tag{2.18}$$

respectively*.

We shall use F generally to denote either the integral (2 . 14) or its value or sum, if it has one, and F_0 to denote $F_0(x)$ or $F(x)$, or the same function of some other variable. And we shall apply the same system of notation, without further explanation, to letters other than f, F.

2 . 2. In dealing with series we write

$$F_n^0 = F_n = f_0 + f_1 + \dots + f_n, \quad F_n^1 = F_0^0 + F_1^0 + \dots + F_n^0, \quad \dots . \tag{2.21}$$

so that

$$F_n^k = \sum_{\nu=0}^{n} P_{n-\nu}^k f_\nu, \tag{2.22}$$

where

$$P_n^k = \binom{n+k}{n} = \frac{(n+k)!}{n!\,k!} = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\,\Gamma(k+1)}. \tag{2.221}$$

---

* The first approximation is to mean Kx^l when $k = 0$.

The last equation may be regarded as defining P_n^k for all real values of k greater than -1.

We shall say that

$$\Sigma f_n = f_0+f_1+f_2+\ldots = F \quad (C, k), \tag{2.23}$$

or that Σf_n, or the series F, is summable (C, k) to sum F, if

$$F_n^k \sim FP_n^k, \tag{2.24}$$

or (what is the same thing) if

$$F_n^k \sim F\frac{n^k}{k!}. \tag{2.241}$$

We shall also say in these circumstances that

$$F_n^0 \rightarrow F \quad (C, k). \tag{2.25}$$

If F is convergent, *i.e.* summable $(C, 0)$, and

$$f_n = o\left(\frac{1}{n}\right), \tag{2.26}$$

we shall say that F is summable $(C, -1)$.

Finally, by

$$F_n^0 \sim KP_n^l, \quad F_n^0 = O(P_n^l), \quad F_n^0 = o(P_n^l) \quad (C, k) \tag{2.27}$$

we mean

$$F_n^k \sim KP_n^{l+k}, \quad F_n^k = O(P_n^{l+k}), \quad F_n^k = o(P_n^{l+k}). \tag{2.28}$$

respectively. Here l is any real number greater than -1.

2.3. LEMMA 1. *If*

$$F_0(x) \sim Kx^l \quad (C, k), \tag{2.31}$$

where $l > -1$, *then*

$$x^{-s}F_s(x) \sim K\frac{x^l}{(l+1)(l+2)\ldots(l+s)} \quad (C, k-s) \tag{2.32}$$

for $s = 0, 1, 2, \ldots, k$. *In particular, if*

$$F_0(x) \rightarrow F \quad (C, k), \tag{2.33}$$

then

$$x^{-s}F_s(x) \rightarrow \frac{F}{s!} \quad (C, k-s). \tag{2.34}$$

If

$$F_0(x) = O(x^l) \quad (C,\ k), \tag{2.35}$$

then

$$x^{-s} F_s(x) = O(x^l) \quad (C,\ k-s); \tag{2.36}$$

and similarly with $o(x^l)$.

It is sufficient to prove that (2 . 32) follows from (2 . 31), and, since the conclusion is obvious when $k=0$ or $k=s$, we have to prove that

$$\frac{1}{(k-s-1)!}\int_0^x F_s(t)\, t^{-s}(x-t)^{k-s-1}\, dt \sim \frac{K}{(l+1)(l+2)\ldots(l+s)}\, \frac{x^{l+k-s}}{(l+1)(l+2)\ldots(l+k-s)} \tag{2.37}$$

for $1 \leqslant s \leqslant k-1$. Integrating the left-hand side $k-s$ times by parts we obtain

$$x^{-s}F_k(x) + \frac{(-1)^{k-s}}{(k-s-1)!}\int_0^x F_k(t)\left(\frac{d}{dt}\right)^{k-s}\left\{\frac{(x-t)^{k-s-1}}{t^s}\right\} dt, \tag{2.38}$$

the terms integrated out vanishing until the last partial integration. If we effect the differentiations, we obtain a sum of multiples of integrals of the type

$$\int_0^x F_k(t)(x-t)^{k-s-1-\lambda}\, t^{-k+\lambda}\, dt,$$

where $0 \leqslant \lambda \leqslant k-s-1$, and each of these is asymptotically equivalent to the integral obtained by replacing $F_k(t)$ by

$$K\frac{t^{l+k}}{(l+1)(l+2)\ldots(l+k)}.$$

It follows that our original integral is asymptotically equivalent to that obtained by replacing $F_k(t)$ by its asymptotic value; and this, on reversing the process of partial integration, becomes

$$\frac{1}{(k-s-1)!}\int_0^x K\frac{t^{l+s}}{(l+1)\ldots(l+s)}\, t^{-s}(x-t)^{k-s-1}\,dt = \frac{K}{(l+1)\ldots(l+s)}\, \frac{x^{l+k-s}}{(l+1)\ldots(l+k-s)}.$$

2 . 4. We shall say that $\psi(x)$ is a *regular convergence factor* in $(\xi,\ \infty)$ if (*a*) it is indefinitely differentiable in (ξ, ∞), and (*b*) $(-1)^s\,\psi^{(s)}(x)$, where $s \geqslant 0$ and $\psi^{(s)}$ is the s-th derivative of ψ, is positive from a certain

point onwards (this point in general depending upon s) and tends to zero when $x \to \infty$. Thus x^{-a} $(a > 0)$ is a regular convergence factor in (ξ, ∞) if $\xi > 0$, but not in $(0, \infty)$*.

LEMMA 2. *If*

$$F_0(x) = O(x^l) \quad (C, k), \tag{2.41}$$

where $l > -1$, *and* $\psi(x)$ *is a regular convergence factor in* $(0, \infty)$, *then*

$$F_0(x)\,\psi(x) = o(x^l) \quad (C, k). \tag{2.42}$$

We shall be concerned in the argument only with a finite number of terms $(-)^s\,\psi^{(s)}(x)$, and these will be positive from a certain point on, the same for all. There is, therefore, no real loss of generality in supposing that each derivative which occurs is of fixed sign for all positive x.

We may suppose $k > 0$. We have

$$\int_0^x F_0\psi(x-t)^{k-1}\,dt = (k-1)!\,F_k\psi + (-1)^k \int_0^x F_k \left(\frac{d}{dt}\right)^k \{(x-t)^{k-1}\psi\}\,dt, \tag{2.43}$$

by k partial integrations†, or

$$\int_0^x F_0\,\psi(x-t)^{k-1}\,dt = (k-1)!\,F_k\psi + \sum_{s=1}^{k} \binom{k}{s}(k-1)(k-2)\ldots s \int_0^x F_k(-1)^s\psi^{(s)}(x-t)^{s-1}\,dt. \tag{2.44}$$

Every function which occurs here as a coefficient of F_k is positive. Hence the order of magnitude of the right-hand side is not greater than that of the function obtained by replacing F_k by x^{l+k} or t^{l+k}; and this function is transformed, by reversing the process of partial integration, into a constant multiple of

$$\int_0^x \psi t^l (x-t)^{k-1}\,dt = o(x^{l+k}),$$

which proves the lemma.

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* [The fact that x^{-a} has a singularity at the origin introduces some small complications which have no bearing on the essence of the problem; see, for example, Lemma 3 and its application in § 3.3. For this reason we now regret that we did not work throughout with $(x+1)^a$ instead of x^a; the latter factor is more convenient for printing, but its disadvantages outweigh its advantages. It is not possible to make the change now without great inconvenience. (*Added Dec.* 1926).]

† It is to be understood that when a function, such as F_k, occurs without an argument under the integral sign, the argument is the variable of integration.

We add a corollary, which will be useful:

LEMMA 3. *We may suppose, in Lemma 2, that $\psi(x) = x^{-\beta}$, where $\beta > 0$, provided that $l > \beta - 1$ and that $F_0(x) = o(x^{\beta-1})$ when $x \to 0$.*

We have only to observe that

$$F_1\psi = o(1), \quad F_2\psi' = o(1), \quad F_3\psi'' = o(1), \quad \ldots,$$

when $x \to 0$, so that the partial integrations are still legitimate. The rest of the proof goes as before, provided that $l - \beta > -1$.

2.5. The corresponding lemmas for series are as follows.

LEMMA 4. *If*

$$F_n^0 \sim KP_n^l \quad (C, k), \tag{2.51}$$

where $l > -1$, then

$$F_n^s/P_n^s \sim K(P_n^l/P_s^l) \quad (C, k-s) \tag{2.52}$$

for $s = 0, 1, \ldots, k$. In particular, if

$$F_n^0 \to F \quad (C, k), \tag{2.53}$$

then

$$F_n^s/P_n^s \to F \quad (C, k-s). \tag{2.54}$$

There are also results for O and o corresponding to those of Lemma 1.

We say that ψ_n is a *regular convergence factor* if it and each of its differences is positive from a certain point onwards and tends to zero when $n \to \infty$.

LEMMA 5. *If*

$$F_n^0 = O(P_n^l) \quad (C, k), \tag{2.55}$$

where $l > -1$, and ψ_n is a regular convergence factor, then

$$F_n^0 \psi_n = o(P_n^l) \quad (C, k). \tag{2.56}$$

2.6. The proofs of Lemmas 4 and 5 are the same in principle as those of Lemmas 1 and 2, and we need only indicate the formal differences in the argument. If

$$G_n^0 = F_n^s/P_n^s,$$

we have
$$G_n^{k-s} = \sum_{j=0}^{n} P_{n-j}^{k-s-1} G_j^0 = \sum_{j=0}^{n} F_j^s (P_{n-j}^{k-s-1}/P_j^s).$$

Now Abel's formula for partial summation may be written

$$\sum_{j=0}^{n} u_j v_j = \sum_{j=0}^{n} U_j^0 \Delta v_j,$$

if we adopt the convention that any v whose suffix exceeds n is to be regarded as zero, so that Δv_n means v_n. Hence, with this convention,

$$G_n^{k-s} = \sum_{j=0}^{n} F_j^k \Delta^{k-s}(P_{n-j}^{k-s-1}/P_j^s). \tag{2.61}$$

This sum corresponds to (2 . 38). It follows, as in the proof of Lemma 1, that G_n^{k-s} is asymptotic to the sum obtained by replacing, in (2 . 61), F_j^k by KP_j^{l+k}. We thus obtain a dominant sum which, on reversing the process of partial summation, reduces to

$$K \sum_{j=0}^{n} P_j^{l+s}(P_{n-j}^{k-s-1}/P_j^s) \sim K(P_n^{k-s+l}/P_s^l);$$

and this proves (2 . 52).

We should add that the most important part of Lemma 4, that is to say that (2 . 54) follows from (2 . 53), has been proved, in a more general form, by Andersen*.

To prove Lemma 5 we observe that, if

$$G_n^0 = F_n^0 \psi_n,$$

we have $$G_n^k = \sum_{j=0}^{n} P_{n-j}^{k-1} G_j^0 = \sum_{j=0}^{n} F_j^0 P_{n-j}^{k-1} \psi_j = \sum_{j=0}^{n} F_j^k \Delta^k (P_{n-j}^{k-1} \psi_j),$$

with the same convention as before regarding ψ's whose rank exceeds n. This formula may be written

$$G_n^k = \sum_{j=0}^{n} F_j^k \sum_{s=1}^{k} P_s^{k-s} P_{n-j}^{s-1} \Delta^s \psi_{j+k-s}. \tag{2.62}$$

The formula (2 . 62) is, in the first instance, subject to the convention; but the convention may be ignored, since the terms which it suppresses are $o(n^{l+k})$; and from this point the argument proceeds as before.

3. *Integrals: the case $\phi(x) = x^a$.*

3 . 1. THEOREM 2. *If*

$$\phi(x) = x^a \quad (a > 0), \tag{3.11}$$

* See A. F. Andersen, *Studier*, 64 (where the result is stated without proof).

and $a(x)$ *is integrable over any finite interval* $(0, X)$, *then the necessary and sufficient condition that*

$$\int_0^\infty a(x)\,\phi(x)\,dx \tag{3.12}$$

should be summable (C, r), *to sum* B, *is that*

$$\int_0^\infty \{A_0(x)-A\}\,\phi'(x)\,dx \tag{3.13}$$

should be summable $(C, r-1)$ *to sum* $-B$. *Here* r *is a positive integer or zero;* A *is the sum* (C, r) *of the integral*

$$\int_0^\infty a(x)\,dx\,; \tag{3.14}$$

and the sufficiency of the criterion is to be understood as implying that, if (3.13) *is summable* $(C, r-1)$ *for any value of* A, *then* (3.12) *and* (3.14) *are summable* (C, r), *and* A *is necessarily the value of the latter integral.*

It is to be observed that (3.14) is certainly summable (C, r) if (3.12) is so*.

There is no real loss of generality in supposing throughout that $A = 0$. If this were not so, we could replace $a(x)$ by the function

$$\bar{a}(x) = a(x)-A \quad (0 \leqslant x \leqslant 1), \quad \bar{a}(x) = a(x) \quad (x > 1).$$

We therefore adopt this simplification.

We write

$$b(x) = a(x)\,\phi(x), \tag{3.15}$$

$$p(x) = A_0(x)\,\phi'(x), \tag{3.16}$$

$$q(x) = -B_0(x)\frac{d}{dx}\left\{\frac{1}{\phi(x)}\right\}, \tag{3.17}$$

so that when, as here, $\phi(x) = x^a$, we have

$$b = x^a a, \quad p = ax^{a-1}A_0, \quad q = ax^{-a-1}B_0.$$

It may be verified at once that

$$P_0(x) = Q_0(x)\,\phi(x) = A_0(x)\,\phi(x)-B_0(x). \tag{3.18}$$

* See G. H. Hardy, "Notes on some points in the integral calculus (30)", *Messenger of Math.*, 40 (1911), 108–112. The theorem there proved is the analogue for integrals of what Andersen calls the "Bohr-Hardy'ske Saetning".

3.2. We dispose first of the case $r = 0$. We have then to show that (3.12) is convergent if and only if (i) (3.13) is convergent and (ii)

(3.21) $$A_0\phi' = o\left(\frac{1}{x}\right).$$

Suppose first that (3.12) is convergent, so that $B_0 \to B$. Then $A_0 \to A = 0$, and so

$$Q_0 = A_0 - \frac{B_0}{\phi} \to 0.$$

Now

(3.22) $$P_0 = x^a Q_0 = ax^a \int_0^x t^{-a-1} B_0\, dt,$$

and $$Q = a\int_0^\infty t^{-a-1} B_0\, dt$$

is convergent, so that $Q_0 \to Q$. It follows that $Q = 0$, and that we may write (3.22) in the form

(3.23) $$P_0 = -ax^a \int_x^\infty t^{-a-1} B_0\, dt.$$

Thus $$P_0 \sim -aBx^a \int_x^\infty t^{-a-1}\, dt = -B,$$

which proves (i). Also

$$A_0\phi' = ax^{a-1} A_0 = \frac{a}{x}(B_0 + P_0) = o\left(\frac{1}{x}\right),$$

which proves (ii).

The converse implication is trivial, since (ii) gives $A_0 = o(x^{-a})$ and

$$B_0 + P_0 = x^a A_0 = o(1).$$

3.3. Passing to the general case, we suppose first that (3.12) is summable (C, r), so that

(3.31) $$B_r(x) \sim B\frac{x^r}{r!}.$$

Then

(3.32) $$Q_0 = -\int_0^x B_0 \frac{d}{dt}\left(\frac{1}{\phi}\right) dt = -B_1 \frac{d}{dx}\left(\frac{1}{\phi}\right) + \int_0^x B_1 \frac{d^2}{dt^2}\left(\frac{1}{\phi}\right) dt$$

$$= \ldots = \sum_{s=1}^{r} (-1)^s B_s \frac{d^s}{dx^s}\left(\frac{1}{\phi}\right) + (-1)^{r+1}\int_0^x B_r \frac{d^{r+1}}{dt^{r+1}}\left(\frac{1}{\phi}\right) dt$$

$$= \sum_{s=1}^{r} j_s + J_{r+1},$$

say. Hence, by (3 . 18),

(3 . 33) $$P_0 = \sum_{s=1}^{r} \phi j_s + \phi J_{r+1}.$$

Now

(3 . 34) $$x^{-s} B_s \to \frac{B}{s\,!} \quad (C,\, r-s),$$

by Lemma 1, and

(3 . 35) $$j_s = a(a+1) \dots (a+s-1) x^{-a-s} B_s \to 0 \quad (C,\, r-s),$$

by Lemmas 2 and 3*. Also the integral

(3 . 36) $$Q = (-1)^{r+1} \int_0^{\infty} B^r \frac{d^{r+1}}{dx^{r+1}} \left(\frac{1}{\phi}\right) dx$$

is convergent. It follows from (3 . 32), (3 . 35), and (3 . 36) that

$$Q_0 \to Q \quad (C,\, r-1).$$

But $Q_0 = A_0 - B_0/\phi$, and A_0 and B_0/ϕ tend to zero $(C,\, r)$, the first by hypothesis and the second by Lemmas 2 and 3. Thus Q must be zero, and we may write (3 . 33) in the form

(3 . 37) $$P_0 = \sum_{s=1}^{r} \phi j_s + \phi I_{r+1},$$

where

(3 . 38) $$I_{r+1} = (-1)^r \int_x^{\infty} B_r \frac{d^{r+1}}{dt^{r+1}} \left(\frac{1}{\phi}\right) dt.$$

3 . 4. Now

(3 . 41) $$\phi j_s = a(a+1) \dots (a+s-1) x^{-s} B_s \to \frac{a(a+1) \dots (a+s-1)}{s\,!} B \quad (C,\, r-s),$$

by (3 . 34), and

(3 . 42) $$\phi I_{r+1} = -a(a+1) \dots (a+r) x^a \int_x^{\infty} B_r t^{-a-r-1} dt$$

$$\sim -\frac{a(a+1) \dots (a+r)}{r\,!} B x^a \int_x^{\infty} t^{-a-1}\, dt = -\frac{(a+1) \dots (a+r)}{r\,!} B.$$

---

* The application of Lemma 3 requires a word of explanation. We have to take

$$F_0 = x^{-s} B_s, \quad l = 0, \quad k = r-s, \quad \beta = a.$$

Since $b = x^a a$, and a is integrable down to 0, $B_0 = o(x^a)$ and $B_s = o(x^{a+s})$, so that $F_0 = o(x^a) = o(x^{a-1})$. Hence the second special condition of Lemma 3 is satisfied.

The first condition ($l > \beta - 1$) is only satisfied if $a < 1$. If $a > 1$ we must use the lemma several times in succession. If *e.g.* $1 \leqslant a < 2$, we may use the lemma first, with $\beta = \frac{1}{2}a$, to show that $x^{-\frac{1}{2}a-s} B_s \to 0$ $(C,\, r-s)$. We then take this function as a new F_0, and apply the lemma a second time, again with $\beta = \frac{1}{2}a$. The first condition is evidently satisfied in both applications.

Finally, from (3 . 37), (3 . 41), and (3 . 42),

$$P_0 \to \left\{a + \frac{a(a+1)}{2!} + \ldots + \frac{a(a+1)\ldots(a+r-1)}{r!} - \frac{(a+1)\ldots(a+r)}{r!}\right\} B$$

$$= -B \quad (C,\, r-1).$$

3 . 5. Next, we suppose that (3 . 13), with $A = 0$, is summable $(C,\, r-1)$, so that

(3 . 51) $$P_{r-1} \sim P \frac{x^{r-1}}{(r-1)!}.$$

We have $$B_0 = q \frac{\phi^2}{\phi'},$$

(3 . 52) $$B_1 = \int_0^x q \frac{\phi^2}{\phi'} dt = Q_0 \frac{\phi^2}{\phi'} - \int_0^x Q_0 \frac{d}{dt}\left(\frac{\phi^2}{\phi'}\right) dt$$

$$= P_0 \frac{\phi}{\phi'} - \int_0^x P_0 \psi\, dt,$$

where

(3 . 53) $$\psi = \frac{1}{\phi} \frac{d}{dt}\left(\frac{\phi^2}{\phi'}\right).$$

In the present case this formula takes a particularly simple form, since

(3 . 54) $$\frac{\phi}{\phi'} = \frac{x}{a}, \quad \psi = \frac{1+a}{a},$$

(3 . 55) $$B_1 = \frac{x}{a} P_0 - \frac{1+a}{a} P_1;$$

from which we deduce

(3 . 56) $$B_r = \frac{x}{a} P_{r-1} - \frac{r+a}{a} P_r$$

$$\sim \frac{P}{(r-1)!\, a}\left\{x^r - (r+a)\int_0^x t^{r-1} dt\right\} = -P \frac{x^r}{r!}.$$

This completes the proof of the theorem.

4. *Series: the case* $\phi_n = \dfrac{\Gamma(n+1+a)}{\Gamma(n+1)}$.

4 . 1. THEOREM 3. *If*

(4 . 11) $$\phi_n = \frac{\Gamma(n+1+a)}{\Gamma(n+1)} \quad (a > 0),$$

then the necessary and sufficient condition that

(4 . 12) $$\sum_0^\infty a_n \phi_n$$

should be summable (C, r), *to sum* B, *is that*

(4 . 13) $$\sum_0^\infty (A_n^0 - A)\,\Delta\phi_n$$

should be summable $(C, r-1)$ *to sum* $B - A\phi_0$. *Here* A *is the sum* (C, r) *of* Σa_n; *and the sufficiency of the criterion is to be interpreted as in Theorem* 1.

It is to be observed, as in § 3 . 1, that Σa_n is certainly summable (C, r) if $\Sigma a_n \phi_n$ is so, and that there is no loss of generality in supposing $A = 0$.

4 . 2. The proof is substantially the same as that of Theorem 2. We write

(4 . 21) $$b_n = a_n \phi_n, \quad p_n = A_n^0 \Delta\phi_n, \quad q_n = -B_n^0 \Delta\left(\frac{1}{\phi_n}\right),$$

so that when, as here, $\phi_n = \dfrac{\Gamma(n+1+a)}{\Gamma(n+1)}$, we have

$$b_n = a_n \frac{\Gamma(n+1+a)}{\Gamma(n+1)}, \quad p_n = -aA_n^0 \frac{\Gamma(n+1+a)}{\Gamma(n+2)}, \quad q_n = -aB_n^0 \frac{\Gamma(n+1)}{\Gamma(n+2+a)}.$$

Then

(4 . 22) $$P_n^0 = Q_n^0 \phi_{n+1} = -A_n^0 \phi_{n+1} + B_n^0.$$

We dispose first of the case $r = 0$; the proof in this case is so like that of § 3 . 2 that we need hardly repeat it. Passing to the general case, we suppose first that (4 . 12) is summable (C, r), so that

(4 . 23) $$B_n^r \sim BP_n^r = B\binom{n+r}{n}.$$

Then

(4 . 24) $$Q_n^0 = -\sum_0^n B_j^0 \Delta\left(\frac{1}{\phi_j}\right) = -B_n^1 \Delta\left(\frac{1}{\phi_n}\right) - \sum_0^{n-1} B_j^1 \Delta^2\left(\frac{1}{\phi_j}\right)$$

$$= \ldots = -\sum_{s=1}^r B_{n-s+1}^s \Delta^s\left(\frac{1}{\phi_{n-s+1}}\right) - \sum_0^{n-r} B_j^r \Delta^{r+1}\left(\frac{1}{\phi}\right)$$

$$= -\sum_{s=1}^r \sigma_s - S_{r+1},$$

say.

We now prove, much as in § 3 . 3, that the series

$$\sum_0^\infty B_n^r \Delta^{r+1}\left(\frac{1}{\phi_n}\right)$$

converges to zero. The essential point is that

$$\sigma_s \to 0 \quad (C,\ r-s);$$

and

$$\sigma_s = B_{n-s+1}^s \Delta^s\left(\frac{1}{\phi_{n-s+1}}\right) = \frac{a(a+1)\dots(a+s-1)}{s!}\,\frac{\Gamma(n+2)}{\Gamma(n+2+a)}\,\frac{B_{n-s+1}^s}{P_{n-s+1}^s},$$

which tends to zero $(C,\ r-s)$ by Lemmas 4 and 5.

We have therefore

$$P_n^0 = Q_n^0\phi_{n+1} = -\sum_{s=1}^{r}\phi_{n+1}\sigma_s + \phi_{n+1}T_{r+1}, \tag{4.25}$$

where

$$T_{r+1} = \sum_{n-r+1}^{\infty} B_j^r \Delta^{r+1}\left(\frac{1}{\phi_j}\right),$$

and the proof may now be completed as in § 3 . 4.

4 . 3. The converse inference is (as with integrals) easier. We have

$$q_n = -B_n^0\Delta\left(\frac{1}{\phi_n}\right) = B_n^0\frac{\Delta\phi_n}{\phi_n\phi_{n+1}},$$

$$\begin{aligned} B_n^1 &= \sum_0^n B_j^0 = \sum_0^n q_j\frac{\phi_j\phi_{j+1}}{\Delta\phi_j} \\ &= Q_n^0\frac{\phi_n\phi_{n+1}}{\Delta\phi_n} + \sum_0^{n-1} Q_j^0\Delta\left(\frac{\phi_j\phi_{j+1}}{\Delta\phi_j}\right) \\ &= P_n^0\frac{\phi_n}{\Delta\phi_n} + \sum_0^{n-1} P_j^0\psi_j, \end{aligned} \tag{4.31}$$

where

$$\psi_n = \frac{1}{\phi_{n+1}}\Delta\left(\frac{\phi_n\phi_{n+1}}{\Delta\phi_n}\right). \tag{4.32}$$

But here

$$\frac{\phi_n}{\Delta\phi_n} = -\frac{n+1}{a}, \quad \psi_n = \frac{1+a}{a}, \tag{4.33}$$

so that (4 . 31) becomes

$$B_n^1 = -\frac{n+1}{a}P_n^0 + \frac{1+a}{a}P_{n-1}^1. \tag{4.34}$$

From this it follows, as in § 3 . 5, that $P_n^0 \to P$ $(C, r-1)$ implies $B_n^0 \to P$ (C, r). This completes the proof of Theorem 3.

4 . 4. We have already mentioned the particular case of Theorem 3 in which $a = 1$, $\phi_n = n+1$. Suppose now that we specialize by making $r = 1$. Then the necessary and sufficient condition that

$$\Sigma b_n = B \quad (C, 1)$$

is that

$$\sum_{j=0}^{n-1} (A_j^0 - A)\Delta\phi_j \to B - A\phi_0. \tag{4.41}$$

But $$B_n^0 = \sum_{j=0}^{n} a_j \phi_j = \sum_{j=0}^{n-1} (A_j^0 - A)\Delta\phi_j + A\phi_0 + (A_n^0 - A)\phi_n.$$

Hence (4 . 41) takes the form

$$B_n^0 + (A - A_n^0)\phi_n \to B, \tag{4.42}$$

or

$$B_n^0 + \phi_n \left(\frac{b_{n+1}}{\phi_{n+1}} + \frac{b_{n+2}}{\phi_{n+2}} + \ldots\right) \to B, \tag{4.43}$$

where the series in brackets is a $(C, 1)$ sum.

If $a \geqslant 1$, and Σb_n is summable $(C, 1)$, the series in brackets is convergent. We thus obtain

THEOREM 4. *The necessary and sufficient condition that Σb_n should be summable $(C, 1)$, to sum B, is that*

$$b_0 + b_1 + b_2 + \ldots + b_n + \phi_n \left(\frac{b_{n+1}}{\phi_{n+1}} + \frac{b_{n+2}}{\phi_{n+2}} + \ldots\right) \to B,$$

where $a \geqslant 1$ and $$\phi_n = \frac{\Gamma(n+1+a)}{\Gamma(n+1)}.$$

This theorem is due to Knopp. It is to be observed that it remains true for $0 < a < 1$, provided that the series in brackets is regarded as a summable series only.

5. *Integrals : the general case.*

5 . 1. We suppose now that $\phi(x)$ is an L-function* of the range (x^δ, x^Δ). Then $\phi(x)$ possesses derivatives of every order, all of which

* For an account of the principal properties of L-functions see Hardy's tract quoted on p. 330. We refer to this tract as *O.I.*

are continuous and monotonic from a certain value of x onwards*, and we may suppose, as we are at any moment considering a finite number of derivatives only, that these conditions are satisfied for $x > 0$. Then†

$$\phi = x^a \chi,$$

where χ is an L-function for which

$$x^{-\delta} \prec \chi \prec x^{\delta}.$$

We shall suppose, in order to avoid trivial complications, that χ and its derivatives are continuous also for $x = 0$. In these circumstances we shall say that ϕ is an L-function of the range (x^{δ}, x^{Δ}) regular for $x > 0$.

The orders of magnitude of the derivatives of ϕ and χ, or of any elementary combinations of them, may be calculated by the rules laid down in Chap. 5 of Hardy's tract‡.

If η is an L-function which tends to zero, then either η or $-\eta$ is a regular convergence factor in the sense of § 2 . 4. If, *e.g.*, η is ultimately positive, then η' is ultimately negative and tends to zero, and the argument may evidently be repeated.

THEOREM 5. *If $\phi(x)$ is any L-function of the range (x^{δ}, x^{Δ}), regular for $x > 0$, then the necessary and sufficient condition that the integral* (3 . 12) *should be summable (C, r), to sum B, is that the integral* (3 . 13) *should be summable $(C, r-1)$ to sum $-B + A\phi(0)$.*

We may suppose, as before, that $A = 0$; and we may confine ourselves to pointing out where the argument of §§ 3 . 3–3 . 5 requires modification.

Very little change is necessary in §§ 3 . 3–3 . 4. The formulae (3 . 32) and (3 . 33) are unaltered, but we must reconsider the behaviour of the various derivatives of $1/\phi$. We have

$$\frac{1}{\phi} = x^{-a} \omega,$$

where ω is an L-function of the range $(x^{-\delta}, x^{\delta})$; and

$$\frac{d}{dx}\left(\frac{1}{\phi}\right) = -ax^{-a-1}\omega\left(1 - \frac{x\omega'}{a\omega}\right) = -ax^{-a-1}\omega(1+\eta),$$

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* *O.I.*, 17–20.
† *O.I.*, 21–22.
‡ *O.I.*, 32–43.

where η is an L-function which tends to zero*. Similarly

$$\left(\frac{d}{dx}\right)^s\left(\frac{1}{\phi}\right) = (-1)^s a(a+1)\ldots(a+s-1)x^{-a-s}\omega(1+\eta),$$

where η has the same significance. Thus every function which arises is asymptotically equivalent to the corresponding function of §§ 3 . 3–3 . 4, and its dominant term leads to an identical result. The error term differs from the dominant term in every case by a factor η, and the corresponding Cesàro limit is zero, by Lemma 2. The conclusion is therefore unaltered.

5 . 2. Rather more alteration is required in the argument of § 3 . 5, since the function there denoted by ψ is no longer a constant. We have now

$$(5\,.\,21)\qquad B_1 = P_0\frac{\phi}{\phi'} - \int_0^x P_0\psi\,dt = P_0\frac{\phi}{\phi'} - P_1\psi + \int_0^x P_1\psi'\,dt$$

$$= P_0\frac{\phi}{\phi'} + \sum_{s=1}^{r-1}(-1)^s P_s\psi^{(s-1)} + (-1)^r\int_0^x P_{r-1}\psi^{(r-1)}dt.$$

Now

$$(5\,.\,22)\qquad \frac{\phi}{\phi'} = \frac{x}{a}(1+\eta),\quad \frac{\phi^2}{\phi'} = \frac{x^{a+1}\chi}{a}(1+\eta),$$

$$(5\,.\,23)\qquad \psi = \frac{1}{\phi}\frac{d}{dx}\left(\frac{\phi^2}{\phi'}\right) = \frac{1+a}{a}+\eta,$$

$$(5\,.\,24)\qquad \psi^{(s-1)} = x^{-s+1}\eta \quad (s>1),$$

where η is in each case an L-function which tends to zero. Thus we may write (5 . 21) in the form

$$(5.\,25)\qquad B_1 = \frac{x}{a}P_0 - \frac{1+a}{a}P_1 + \sum_{s=1}^{r-1}\mu_s + \nu_r,$$

where

$$(5\,.\,26)\qquad \mu_s = x^{-s+1}\eta P_s,\quad \nu_r = (-1)^r\int_0^x P_{r-1}t^{-r+1}\eta\,dt.$$

5 . 3. Now

$$P_{r-1} \sim P\frac{x^{r-1}}{(r-1)!},$$

so that

$$(5\,.\,31)\qquad P_1 \sim Px \quad (C,\ r-2).$$

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* *O.I.*, 36.

It follows from Lemma 1, with $l = 1$, that

$$\frac{P_2}{x} \sim \tfrac{1}{2}Px \quad (C, r-3),$$

and so
$$\frac{1}{x}\int_0^x tP_0\,dt = P_1 - \frac{P_2}{x} \sim \tfrac{1}{2}Px \quad (C, r-2),$$

$$xP_0 \sim Px \quad (C, r-1),$$

(5 . 32)
$$\frac{x}{a}P_0 - \frac{1+a}{a}P_1 \sim -Px \quad (C, r-1).$$

It also follows from (5 . 31) and Lemma 1 that

$$x^{-s+1}P_s \sim \frac{Px}{s!} \quad (C, r-s-1)$$

for $s = 1, 2, \ldots, r-1$; and from Lemma 2 that

(5 . 33)
$$\mu_s = o(x) \quad (C, r-s-1).$$

Finally

(5 . 34)
$$\nu_r = \int_0^x O(t^{r-1})\, o(t^{-r+1})\,dt = o(x)\,;$$

and from (5 . 25), (5 . 32), (5 . 33), and (5 . 34) we obtain

$$B_1 \sim -Px \quad (C, r-1)$$

or
$$B_0 \to -P \quad (C, r).$$

5 . 4. It is natural to ask whether the result of Theorem 5 is valid when $\phi(x)$ is an L-function of the ranges $(1, x^\delta)$ or (x^Δ, ∞). An examination of the argument shows that the hypothesis $\phi \prec x^\delta$ is used only in the argument from P to B, and the hypothesis $\phi \prec x^\Delta$ only in the argument from B to P. If $\phi \prec x^\delta$

$$\phi\frac{d}{dx}\left(\frac{1}{\phi}\right) = -\frac{\phi'}{\phi} \prec \frac{1}{x},$$

and generally
$$\phi\left(\frac{d}{dx}\right)^s\left(\frac{1}{\phi}\right) \prec \frac{1}{x^s}.$$

In this case the argument of §§ 3 . 3–3 . 4 succeeds as though a were zero. The only point of the proof which requires special comment is that involved in (3 . 38) and (3 . 42). We have to show that $\phi I_{r+1} \to -B_r$

and this reduces to showing that

$$\frac{(-1)^{r-1}}{r!}\,\phi\int_x^\infty t^r\left(\frac{d}{dt}\right)^{r+1}\left(\frac{1}{\phi}\right)dt \to 1.$$

But, if $\psi = 1/\phi$, we have*

$$\frac{(-1)^r}{r!}\,\psi^{(r+1)} \sim x^{-r}\psi' = -x^{-r}\frac{\phi'}{\phi^2},$$

and so $$\frac{(-1)^{r-1}}{r!}\,\phi\int_x^\infty t^r\left(\frac{d}{dt}\right)^{r+1}\left(\frac{1}{\phi}\right)dt \sim \phi\int_x^\infty \frac{\phi'}{\phi^2}\,dt = 1.$$

If $\phi \succ x^\Delta$ this argument fails; but that of § 3.5, which depends essentially on the fact that ψ tends to a limit when $x\to\infty$, succeeds, since

$$\psi = 2-\frac{\phi\phi''}{\phi'^2}\to 1.$$

We have, in fact,

THEOREM 6. *If ϕ is an L-function of the range $1 \prec \phi \prec x^\delta$, then the summability $(C, r-1)$ of (3.13) is a necessary, but not a sufficient, condition for the summability (C, r) of (3.12). If ϕ is an L-function of the range $\phi \succ x^\Delta$, then the condition is sufficient but not necessary.*

It is easy to give examples to prove the negative assertions of the theorem. If

$$\phi(x) = e^x, \quad a(x) = e^{aix-x} \quad (a > 0),$$

then $$\int_0^\infty a(x)\,\phi(x)\,dx = \int_0^\infty e^{aix}\,dx$$

is summable $(C, 1)$, but

$$\int_0^\infty \{A_0(x)-A\}\,\phi'(x)\,dx = \frac{1}{ai-1}\int_0^\infty e^{aix}\,dx$$

is not convergent. On the other hand, if

$$\phi(x) = \log x, \quad a(x) = \frac{x^{-1-ai}}{\log x}\quad (x \geqslant 2), \quad a(x) = 0 \quad (x < 2),$$

then $$A_0(x) = \int_2^x \frac{t^{-1-ai}}{\log t}\,dt = A-\frac{x^{-ai}}{ia\log x}+O\left(\frac{1}{(\log x)^2}\right),$$

---

* *O.I.*, 36–37 (see especially 37, top).

and $$\int_2^\infty \{A_0(x)-A\}\frac{dx}{x} = -\frac{1}{ia}\int_2^\infty \frac{x^{-1-ai}}{\log x}\,dx+\int_2^\infty O\left(\frac{1}{x(\log x)^2}\right)dx$$

is convergent. But

$$\int_2^\infty a(x)\log x\,dx = \int_2^\infty x^{-1-ai}\,dx$$

is not summable by any Cesàro mean.

6. *Series: the general case.*

6.1. It will hardly be necessary now that we should enter into details concerning the proof of the theorem for series corresponding to Theorem 5. Suppose first that n, in Theorem 3, is replaced by $n+1$, so that n runs over 1, 2, 3, ... Then ϕ_n may be replaced by n^a or by any function of the form

$$n^a\left(1+\frac{u_1}{n}+\frac{u_2}{n^2}+\ldots\right),$$

where the series is convergent or asymptotic. In fact, if

$$\psi_n = \phi_n\left(1+\frac{u_1}{n}+\frac{u_2}{n^2}+\ldots\right),$$

then the series $$\Sigma a_n\phi_n,\quad \Sigma a_n\psi_n$$

are summable or non-summable, whatever the order, together*.

Finally, if we observe that, when ϕ_n is an L-function of n, of the range $n^\delta \prec \phi_n \prec n^\Delta$, then all the differences of ϕ_n are also L-functions, and are asymptotically equivalent to multiples of the corresponding derivatives, we may at once enunciate

THEOREM 7. *The result of Theorem 3 remains valid when ϕ_n is any L-function of the range $n^\delta \prec \phi_n \prec n^\Delta$.*

* The materials necessary for a formal proof will be found in Andersen, *Studier*, Kap. 3.

CORRECTIONS

p. 331, *line* 2. For f read F.
p. 335, *last line*. For G^0 read G^0_j.
p. 339, *line* 8. For B^r read B_r.
—— *footnote, line* 5 *up*. For > 1 read $\alpha \geqslant 1$.
p. 346, *line* 8 *up*. For $\phi \prec x^\delta$ read $\phi \succ x^\delta$.

COMMENTS

The paper develops further the results initiated independently by Knopp† and Hardy, 1921, 4, and extended, with applications, by Knopp‡ and Hardy and Littlewood, 1924, 1 (in Vol. III). An independent paper by Andersen§ overlaps the present one.

Lemma 4 and its converse (not included) constitute an equivalence theorem for iterated Cesàro means, of the form $(C,k) \sim (C,k-s)\ (C,s)$. For $l = 0$, this was obtained by Hausdorff|| and Kogbetliantz,†† for all real k, s such that $k > -1, s > -1$, $k-s > -1$. Zygmund‡‡ gave another proof, and an extension which includes Lemma 4 and its converse. He also stated, independently of Hardy and Littlewood, that his proof 'lässt sich leicht auf summierbare Integrale übertragen'. Under the same conditions at the origin as in Lemma 1, Zygmund's method establishes Lemma 1 and its converse for all real k, s such that $0 \leqslant s \leqslant k$. For a remark about conditions at the origin, see D.S., p. 119.

In Lemma 3, the condition $F_0(x) = o(x^{\beta-1})$ may be replaced by $F_1(x) = o(x^{\beta})$. This is enough to show that the integral on the left in (2.44) converges at the origin if and only if the last integral in the sum on the right does. The role of the condition $l > \beta-1$ is to establish the convergence at the origin§§ of the last integral on the right. On the other hand, if the integral on the left is known to converge,|||| the condition $l > \beta-1$ is not required, and the result holds for $l > -1$. It follows that the instructions in the 2nd part of the footnote to § 3 may be omitted.

Conditions involving only $\phi(x)$ and $\phi^{(r+1)}(x)$, in the necessity part of Theorem 2, have been given by Borwein.†††

The equisummability of $\sum a_n \phi_n$ and $\sum a_n \psi_n$, § 6, follows from the Bohr–Hardy theorem, 1908, 1, by taking f_n to be ψ_n/ϕ_n or ϕ_n/ψ_n.

† *Sitz. d. Berliner math. Ges.* 16 (1917), 45–50.
‡ *Math. Zeit.* 19 (1924), 97–113.
§ *Proc. London Math. Soc.* (2), 27 (1928), 39–71.
|| *Math. Zeit.* 9 (1921), 74–109.
†† Kogbetliantz (1), *Comptes rendus* 176 (1923), 224–7; (2) *Ann. de l'École norm. sup.* (3), 42 (1925), 193–216. He had $0 \leqslant s \leqslant k$.
‡‡ *Bull. de l'Acad. Polonaise* (A), 1927, 309–31.
§§ With the apparent proviso that $F_k(x) = O(x^{k+l})$ for small x.
|||| e.g. if $F_0\ (x) = o(x^\beta)$ as $x \to 0$.
††† *J. London Math. Soc.* 29 (1954), 276–92.

NOTES ON THE THEORY OF SERIES (XI): ON TAUBERIAN THEOREMS

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. This note originated from our desire to prove a theorem for *moment constants* corresponding to our "Tauberian" theorem concerning power series with positive coefficients*. We quote here, for reasons which will appear in the sequel, not this theorem, but the corresponding theorem for integrals (from which it is easily deduced).

THEOREM 1. *If $f(t)$ is positive† and integrable‡ over every finite range $(0, T)$ and $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for every $x>0$; and if*

$$g(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim Hx^{-a}, \tag{1.1}$$

where $a>0$, $H>0$, when $x\to 0$; then

$$F(t) = \int_0^t f(u)\,du \sim \frac{H}{\Gamma(1+a)}\,t^a \tag{1.2}$$

when $t\to\infty$. The result is still true when $a>0$, $H=0$ if we interpret (1.1) to mean $g(x)=o(x^{-a})$ and (1.2) similarly; and it is still true when $a=0$, $H>0$§.

It should be observed that the theorem is trivial when $a=0$.

The corresponding theorem for moment constants is (in the form in which it first presents itself)

* Hardy and Littlewood, **1** and **2**. See also Hobson, **3**, 185, or Landau, **5**, 50.

† In the wider sense. We use the word thus throughout.

‡ In the sense of Lebesgue.

§ If a and H were both zero, $f(t)$ would necessarily be null.

THEOREM 2. *If $f(t)$ is positive and integrable over $(\delta, 1)$, for every $\delta > 0$, and $t^n f(t)$ is integrable over $(0, 1)$ for sufficiently large n; and if*

$$(1.3) \qquad g_n = \int_0^1 t^n f(t)\,dt \sim Hn^{-a},$$

where $a > 0$, $H > 0$, when $n \to \infty$ by integral values; then

$$(1.4) \qquad F(t) = \int_{1-t}^1 f(u)\,du \sim \frac{H}{\Gamma(1+a)}\, t^a$$

when $t \to 0$. The result is still true when $a > 0$, $H = 0$, if (1.3) and (1.4) be then interpreted as in Theorem 1.

This theorem has no significance when $a = 0$, since $F(t) \to 0$ for any integrable $f(t)$. Its relation to Theorem 1 is better appreciated if we reduce it to a different form. It is plain first, since g_n decreases when n increases, that we may replace n by a continuous x. If then we write $t = e^{-w}$, $e^{-w} f(e^{-w}) = h(w)$, the hypothesis becomes

$$(1.31) \qquad g(x) = \int_0^\infty e^{-xw}\, h(w)\,dw \sim Hx^{-a}.$$

When we make a similar transformation in the conclusion, and observe that

$$\log \frac{1}{1-t} \sim t,$$

it becomes

$$(1.41) \qquad H(t) = \int_0^t h(w)\,dw \sim \frac{H}{\Gamma(1+a)}\, t^a.$$

Hence Theorem 2 is equivalent to

THEOREM 3. *If $f(t)$ is positive and integrable over every finite range $(0, T)$, and $e^{-xt} f(t)$ is integrable over $(0, \infty)$ for sufficiently large x; and if (1.1) holds, with $a > 0$, when $x \to \infty$; then (1.2) holds when $t \to 0$.*

This theorem is almost identical verbally with Theorem 1, differing from it only in an exchange of rôles between large and small values of x and t.

2. Theorems 1 and 3 are of just the same "depth" and Theorem 3 may be proved by an argument which runs parallel to that of Theorem 1. In spite of this similarity, it does not seem possible to deduce either, in any obvious manner, from the other. There is, however, another theorem

which is in essentials symmetrical as between large and small values of the variables and from which both Theorems 1 and 3 may be deduced.

THEOREM 4. *Suppose that $f(t)$ is positive and integrable over every finite range $(0, T)$, and that*

$$\frac{f(t)}{(t+x)^\rho},$$

where $\rho > 0$, is integrable over $(0, \infty)$ for some (and so for all) $x > 0$. Suppose further that

$$h(x) = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\sigma}, \tag{2.1}$$

where $0 < \sigma < \rho$, $H > 0$, when $x \to \infty$. Then

$$F(t) = \int_0^t f(u)\,du \sim \frac{H\Gamma(\rho)}{\Gamma(\sigma)\,\Gamma(\rho-\sigma+1)}\, t^{\rho-\sigma} \tag{2.2}$$

when $t \to \infty$.

Further, if (2.1) *holds when $x \to 0$, then* (2.2) *holds when $t \to 0$.*

We have stated the theorem with $0 < \sigma < \rho$, $H > 0$. The results are, in fact, true for $0 \leqslant \sigma \leqslant \rho$, $H \geqslant 0$, if we pay regard to the following glosses :—

(i) equations with $H = 0$ are to be interpreted as in Theorem 1;

(ii) when $0 < \sigma = \rho$, we are to suppose $H > 0$ in the first part and $H = 0$ in the second part of the theorem;

(iii) when $0 = \sigma < \rho$, we are to suppose $H = 0$ in the hypothesis of the first part, and $H > 0$ in the hypothesis of the second part, while the conclusion of each part is to be replaced by $F(t) = o(t^\rho)$;

(iv) if $0 = \sigma = \rho$, we are to suppose $H > 0$ in both hypotheses, and interpret both conclusions as meaning $F \to H$.

The results in these special cases are all trivial, and there is only one of them (the case $0 < \sigma = \rho$, $H > 0$, $x \to \infty$, $t \to \infty$) to which we shall refer later. We therefore simplify the discussion by dismissing them at the outset.

3. We observe first that, when $\rho = 1$, $0 < \sigma < 1$, and x and t tend to infinity, Theorem 4 is substantially the same as a theorem of Valiron rediscovered by Titchmarsh*.

* Valiron, **9**, 121–127; Titchmarsh, **7**, **8**.

Valiron's theorem* asserts that, *if $f(t)$ is positive and increasing, and vanishes for sufficiently small t, and*

$$\int_0^\infty \frac{f(t)\,dt}{t(t+x)} \sim Ax^{\lambda-1}, \tag{3.1}$$

where $0<\lambda<1$, when $x\to\infty$, then

$$f(t) \sim At^{\lambda}, \tag{3.2}$$

when $t\to\infty$ †. We shall show that this theorem is deducible from the first part of Theorem 4 (with $\rho=1$), and conversely. Our proof depends on a theorem of Landau‡, which we shall require again later, and which we therefore state as a lemma.

LEMMA α. *If $xf'(x)$ increases with x and*

$$f(x) \sim x^{a} \quad (a>0)$$

when $x\to\infty$, then $\quad f'(x) \sim ax^{a-1}$§.

(1) Suppose that the first part of Theorem 4 is known, and that the conditions of Valiron's theorem are satisfied. Writing $g(t)=t^{-1}f(t)$ in (3.1), and applying Theorem 4, we obtain

$$G(t) = \int_0^t g(u)\,du \sim Au^{\lambda}.$$

But $tG'=tg=f$ increases with t. Hence Lemma α gives $g\sim At^{\lambda-1}$, $f\sim At^{\lambda}$.

(2) Suppose that Valiron's theorem is known and that the conditions of the first part of Theorem 4 are satisfied, with $\rho=1$‖. It follows from (2.1) that

$$F(t) = \int_0^t f(u)\,du \leqslant 2t\int_0^t \frac{f(u)}{u+t}\,du \leqslant 2t\int_0^\infty \frac{f(u)}{u+t}\,du = O(t^{1-\sigma}),$$

* Valiron's theorem, as he proves it, is more general in other respects; we confine ourselves to the case of the theorem directly relevant here.

† Here and in what follows the A's are constants (functions of H, λ, ρ, or σ only) whose values are irrelevant to the explanation. The value of an A can always be determined, if required, by considering the appropriate case $f(x)=x^k$.

‡ See Landau, **4**, 218, or Hardy and Littlewood, **1**, 175.

§ This is, of course, also true when $x\to 0$.

‖ So that $\sigma<1$.

and *a fortiori* $F(t) = o(t)$, for large t. Hence we obtain, by partial integration,

$$\int_0^\infty \frac{f(t)\,dt}{t+x} = \int_0^\infty \frac{F(t)\,dt}{(t+x)^2},$$

and so

$$\int_0^\infty \frac{F(t)\,dt}{(t+x)^2} \sim \frac{H}{x^\sigma}.$$

Integrating with respect to x, and observing that

$$\int_0^x \frac{dy}{(t+y)^2} = \frac{1}{t} - \frac{1}{t+x} = \frac{x}{t(t+x)},$$

we obtain

$$\int_0^\infty \frac{F(t)\,dt}{t(t+x)} \sim Ax^{-\sigma}.$$

From this it follows, by Valiron's theorem, that $F \sim At^{1-\sigma}$.

4. It is convenient to prove three additional lemmas before we go further.

LEMMA β. *If $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for sufficiently large x, and $f(t) \sim Ht^{-\beta}$, where $\beta < 1$, when $t \to 0$, then*

$$g(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim H\Gamma(1-\beta)\,x^{-1+\beta} \tag{4.1}$$

when $x \to \infty$.

LEMMA γ. *If $e^{-xt}f(t)$ is integrable over $(0, \infty)$ for every $x > 0$, and $f(t) \sim Ht^{-\beta}$, where $\beta < 1$, when $t \to \infty$, then (4.1) holds when $x \to 0$.*

These lemmas are of a straightforward (Abelian) character, and it will be sufficient to prove the first of them. We can choose τ so that

$$(1-\epsilon)\,Ht^{-\beta} < f(t) < (1+\epsilon)\,Ht^{-\beta},$$

for $0 < t \leqslant \tau$, and then $g(x)$ lies between

$$(1 \mp \epsilon)\,H \int_0^\tau e^{-xt}t^{-\beta}\,dt \mp \int_\tau^\infty e^{-xt}\,|f(t)|\,dt.$$

The second term is not greater than

$$e^{-\frac{1}{2}x\tau} \int_\tau^\infty e^{-\frac{1}{2}xt}\,|f(t)|\,dt = O(e^{-\frac{1}{2}x\tau}),$$

when τ is fixed, and the integral in the first term is asymptotic to

$\Gamma(1-\beta)x^{-1+\beta}$. It follows that

$$(1-\epsilon)H\Gamma(1-\beta) \leqslant \overline{\lim}\, x^{\beta-1}g(x) \leqslant (1+\epsilon)H\Gamma(1-\beta),$$

which proves the lemma.

LEMMA δ. *Suppose that* $0 < s < r$, *that* $f(t)$ *is integrable in every interval* $(0, T)$, *and* $t^{-r}f(t)$ *in every interval* (T, ∞), *where* $T > 0$. *Then the relations*

$$F_1(t) = \int_t^\infty u^{-r} f(u)\, du \sim Ht^{-s} \quad (t \to 0) \tag{4.2}$$

and

$$F(t) = \int_0^t f(u)\, du \sim \frac{sH}{r-s} t^{r-s} \quad (t \to 0) \tag{4.3}$$

are equivalent.

We take $H = 1$. Suppose, first, that (4 . 2) is satisfied. Then $t^r F_1(t) \to 0$ when $t \to 0$. From this and the fact that F_1 is (except at the origin) an integral, we deduce

$$F(t) = -\int_0^t u^r F_1'(u)\, du = r\int_0^t u^{r-1}F_1(u)\, du - t^r F_1(t)$$

$$\sim r\int_0^t u^{r-s-1}\, du - t^{r-s} = \frac{s}{r-s} t^{r-s}.$$

Next suppose (4 . 3) satisfied. Then

$$t^{-r}F(t) \leqslant t^{-r}\int_0^T f(u)\, du + \int_T^t u^{-r} f(u)\, du,$$

if $0 < T < t$. The second term is less than ϵ if $T > T_0(\epsilon)$, and the first is $O(t^{-r})$ when T is fixed and $t \to \infty$. Hence $t^{-r}F(t) \to 0$ when $t \to \infty$. It follows that

$$F_1(t) = \int_t^\infty u^{-r}F'(u)\, du = r\int_t^\infty u^{-r-1}F(u)\, du - t^{-r}F(t)$$

$$\sim \frac{rs}{r-s}\int_t^\infty u^{-s-1}\, du - \frac{s}{r-s} t^{-s} = t^{-s},$$

when $t \to 0$.

5. We can now prove (1) that Theorem 1 is a corollary of Theorem 4 (when $x \to \infty$, $t \to \infty$), (2) that Theorem 3 (and therefore Theorem 2) is a corollary of Theorem 4 (when $x \to 0$, $t \to 0$), and (3) that each half of Theorem 4 is a corollary of the other. Thus the complete machinery of a Tauberian proof need not be used more than once.

(1) Suppose that the conditions of Theorem 1 are satisfied, and that $\rho > a$. Then

$$h(y) = \int_0^\infty x^{\rho-1} e^{-xy} g(x)\, dx = \int_0^\infty x^{\rho-1} e^{-xy}\, dx \int_0^\infty e^{-xt} f(t)\, dt$$

$$= \int_0^\infty f(t)\, dt \int_0^\infty x^{\rho-1} e^{-x(t+y)}\, dx = \Gamma(\rho) \int_0^\infty \frac{f(t)\, dt}{(t+y)^\rho},$$

if either repeated integral exists, a condition certainly satisfied, since $g(x)$ is continuous for $x > 0$, tends to zero when $x \to \infty$, and is less than a multiple of x^{-a} when x is small.

It follows from Lemma β and (1.1) that

$$h(y) \sim H\Gamma(\rho - a)\, y^{-\rho+a}$$

when $y \to \infty$. Hence the conditions of Theorem 4 are satisfied, and the theorem gives $F(t) \sim At^a$, where A is a constant which may be identified with the constant of (1.2).

(2) The proof that Theorem 3 is a corollary of the other half of Theorem 4 is the same in principle and may be omitted.

(3) Suppose that Theorem 4 has been proved when $x \to \infty$, $t \to \infty$, and write

$$x = \frac{1}{y}, \quad t = \frac{1}{u}, \quad u^{\rho-2} f\left(\frac{1}{u}\right) = g(u), \quad \rho = r, \quad \rho - \sigma = s.$$

Then $0 < s < r$, and the hypothesis becomes

$$\int_0^\infty \frac{g(u)}{(u+y)^r}\, du \sim \frac{A}{y^s},$$

where now $y \to 0$. The conclusion becomes

$$\int_u^\infty w^{-r} g(w)\, dw \sim Au^{-s}$$

when $u \to \infty$, and Lemma δ shows that this is equivalent to

$$\int_0^u g(w)\, dw \sim Aw^{r-s}$$

(with the appropriate values of the A's). The converse transformation goes similarly. Thus either case of Theorem 4 transforms into the other.

We should add that Titchmarsh* has given deductions of Theorem 1 (with $a < 1$) and Valiron's theorem from one another. The preceding

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* Titchmarsh, 8.

analysis shows that his deduction of the first theorem from the second may be simplified, since it is not necessary to use integrals of the Fourier type. No corresponding simplification of the converse deduction seems to be possible.

6. The preceding analysis shows that all our theorems are corollaries of either case of Theorem 4, say that in which the variables tend to infinity. It is therefore very desirable to give as direct and simple a proof of this theorem as possible. Such a proof must naturally follow the lines of our original proof of Theorem 1 or of Valiron's proof of his theorem; but Valiron's argument may be simplified considerably.

It follows from (2 . 1) that

$$(6\,.\,1)\qquad F(x) \leqslant (2x)^\rho \int_0^x \frac{f(t)\,dt}{(t+x)^\rho} \leqslant (2x)^\rho \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} = O(x^{\rho-\sigma}),$$

for large x. In particular $F(x) = o(x^\rho)$. Hence, integrating by parts, we obtain

$$(6\,.\,2)\qquad h(x) = \rho \int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+1}} = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\sigma}.$$

We prove first that (6 . 2) *may be differentiated any number of times.*

It follows from (6 . 2) that

$$\frac{d}{dx}(x^{\rho+1}h) = \rho(\rho+1)\int_0^\infty \frac{x^\rho\, tF(t)\,dt}{(t+x)^{\rho+2}} = \rho(\rho+1)\int_0^\infty \frac{uF(xu)\,du}{(u+1)^{\rho+2}}\,*,$$

an increasing function of x. From this and

$$x^{\rho+1}h \sim Ax^{\rho+1-\sigma},$$

we deduce, by Lemma α†, that

$$\frac{d}{dx}(x^{\rho+1}h) \sim Ax^{\rho-\sigma}.$$

Expanding the left-hand side, inserting the appropriate value of A, and using (6 . 2), we obtain

$$h'(x) = \rho(\rho+1)\int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+2}} \sim \frac{\sigma H}{x^{\sigma+1}}.$$

---

* The differentiations under the integral sign here and later are trivial, the derived integrals being uniformly convergent in any interval $x \geq \delta > 0$.

† We do not actually require the full force of Lemma α here; it would be enough that the result of the lemma should hold when f' increases.

This justifies one differentiation, and the argument may plainly be repeated.

We thus obtain

$$(6.3)\qquad h^{(q)}(x) = \frac{\Gamma(\rho+q+1)}{\Gamma(\rho)}\int_0^\infty \frac{F(t)\,dt}{(t+x)^{\rho+q+1}} \sim \frac{\Gamma(\sigma+q)}{\Gamma(\sigma)}\frac{H}{x^{\sigma+q}},$$

for $q = 0, 1, 2, \ldots$. Writing down this relation for $q = p, p+1, \ldots, 2p$, and observing that

$$\frac{t^p}{(t+x)^{\rho+2p+1}} = \frac{(t+x-x)^p}{(t+x)^{\rho+2p+1}} = \sum_{s=0}^{p}\binom{p}{s}\frac{(-x)^s}{(t+x)^{\rho+p+s+1}},$$

we find that

$$(6.4)\qquad I_p = \int_0^\infty \frac{t^p F(t)\,dt}{(t+x)^{\rho+2p+1}}$$

is asymptotic to a certain multiple of $x^{-\sigma-p}$. The multiple in question may obviously be determined by supposing that $F(t)$ is actually equal to the right-hand side of (2 . 2). We thus obtain

$$(6.5)\qquad I_p \sim M\int_0^\infty \frac{t^{p+\rho-\sigma}\,dt}{(t+x)^{\rho+2p+1}} = Mx^{-\sigma-p}J_p,$$

where

$$(6.61)\qquad M = \frac{H\Gamma(\rho)}{\Gamma(\sigma)\Gamma(\rho-\sigma+1)},$$

$$(6.62)\qquad J_p = \int_0^\infty \frac{u^{p+\rho-\sigma}\,du}{(u+1)^{\rho+2p+1}} = \frac{\Gamma(p+\rho-\sigma+1)\Gamma(\sigma+p)}{\Gamma(\rho+2p+1)}.$$

This is true for all positive integral p.

7. LEMMA ϵ. *If J_p is the integral in (6 . 62), and $J_{p,1}$, $J_{p,2}$, $J_{p,3}$ the corresponding integrals over the ranges $(0, 1-\zeta)$, $(1-\zeta, 1+\zeta)$, $(1+\zeta, \infty)$, where $0<\zeta<1$, then*

$$J_{p,1} = \eta_p J_p,\qquad J_{p,2} = (1-\eta_p)J_p,\qquad J_{p,3} = \eta_p J_p,$$

where $\eta_p = \eta_p(\zeta) = \eta_p(\rho, \sigma, \zeta) \to 0$ when ρ, σ, ζ are fixed and $p\to\infty$.

A straightforward application of Stirling's theorem shows that

$$(7.1)\qquad J_p \sim Ap^{-\frac{1}{2}}2^{-2p}$$

when $p\to\infty$. On the other hand, $u(1+u)^{-2}$ has a maximum $\frac{1}{4}$ when $u=1$, and its values when u is $1-\zeta$ or $1+\zeta$ are each less than a number

$\kappa = \kappa(\zeta) < \frac{1}{4}$. It follows that

(7.2) $$J_{p,1} = \int_0^{1-\zeta} \frac{u^{p+\rho-\sigma}\, du}{(u+1)^{\rho+2p+1}} < \kappa^p \int_0^{\infty} \frac{u^{\rho-\sigma}\, du}{(u+1)^{\rho+1}} = A\kappa^p.$$

The conclusion concerning $J_{p,1}$ follows from (7.1) and (7.2), and it is obvious that a similar argument disposes of $J_{p,3}$.

8. We can now prove Theorem 4. We use the following notation. We denote by η_ζ a function of ζ (and ρ and σ) which tends to zero with ζ; by η_p a function of ζ and p (and ρ and σ) which tends to zero when ζ is fixed and $p \to \infty$; and by η_x a function of ζ, p, and x (and ρ and σ) which tends to zero when ζ and p are fixed and $x \to \infty$.

By (6.4) and (6.5) we have

(8.1) $$I_p = \int_0^{\infty} \frac{t^p F(t)\, dt}{(t+x)^{\rho+2p+1}} = Mx^{-\sigma-p} J_p(1+\eta_x)^*.$$

We write

(8.2) $$I_p = \int_0^1 + \int_1^{(1-\zeta)x} + \int_{(1-\zeta)x}^{(1+\zeta)x} + \int_{(1+\zeta)x}^{\infty} = I_{p,0}+I_{p,1}+I_{p,2}+I_{p,3}.$$

If $t \geqslant 1$, then $F(t) < At^{\rho-\sigma}$, by (6.1). Hence

$$I_{p,1} < A \int_0^{(1-\zeta)x} \frac{t^{p+\rho-\sigma}\, dt}{(t+x)^{\rho+2p+1}} = Ax^{-\sigma-p} J_{p,1},$$

(8.3) $$I_{p,1} = x^{-\sigma-p} J_p \eta_p,$$

by Lemma ϵ; and similarly

(8.4) $$I_{p,3} = x^{-\sigma-p} J_p \eta_p.$$

Also

(8.5) $$I_{p,0} = O(x^{-\rho-2p-1}) = o(x^{-\sigma-p}) = x^{-\sigma-p}\eta_x.$$

From (8.1)–(8.5) we derive

(8.6) $$I_{p,2} = Mx^{-\sigma-p} J_p(1+\eta_p+\eta_x).$$

On the other hand, since $F(t)$ increases with t, we have

$$I_{p,2} \leqslant \frac{F(x+\zeta x)}{(x-\zeta x)^{\rho-\sigma}} \int_{x-\zeta x}^{x+\zeta x} \frac{t^{p+\rho-\sigma}\, dt}{(t+x)^{\rho+2p+1}} = \frac{F(x+\zeta x)}{(x+\zeta x)^{\rho-\sigma}} \left(\frac{1+\zeta}{1-\zeta}\right)^{\rho-\sigma} x^{-\sigma-p} J_{p,2}.$$

* The η_x here is independent of ζ.

Combining this with (8 . 6), we obtain

$$\text{(8 . 7)} \qquad \frac{F(x+\xi x)}{(x+\xi x)^{\rho-\sigma}} \geqslant (1+\eta_\xi)\, x^{\sigma+p} \frac{I_{p,2}}{J_{p,2}} \geqslant (1+\eta_\xi)\, x^{\sigma+p} \frac{I_{p,2}}{J_p}$$

$$\geqslant M(1+\eta_\xi+\eta_p+\eta_x).$$

Similarly

$$I_{p,2} \geqslant \frac{F(x-\xi x)}{(x-\xi x)^{\rho-\sigma}} \left(\frac{1-\xi}{1+\xi}\right)^{\rho-\sigma} x^{-\sigma-p} J_{p,2},$$

and so

$$\text{(8 . 8)} \qquad \frac{F(x-\xi x)}{(x-\xi x)^{\rho-\sigma}} \leqslant (1+\eta_\xi)\, x^{\sigma+p} \frac{I_{p,2}}{J_{p,2}} \leqslant (1+\eta_\xi+\eta_p)\, x^{\sigma+p} \frac{I_{p,2}}{J_p}$$

$$\leqslant M(1+\eta_\xi+\eta_p+\eta_x).$$

It is plain that (8 . 7) and (8 . 8) are possible only if $F(t) \sim Mt^{\rho-\sigma}$, which proves the theorem. As we have seen, Theorem 4 carries with it Theorems 1–3.

We have ignored the special cases mentioned at the end of § 2. The only one which is relevant is that in which $\sigma = \rho > 0$, $H > 0$, and the variables tend to infinity. The truth of the result in this case follows at once from (6 . 1), which shows that $F(t)$ is bounded.

9. It is known* that the condition $f(t) \geqslant 0$ in Theorem 1 may be replaced by $f(t) \geqslant -Kt^{\alpha-1}$, where K is a positive constant; the extension is immediate when $\alpha > 0$, but less obvious when $\alpha = 0$†. There are corresponding developments of Theorem 4: we confine ourselves to the case in which $x \to \infty$, $t \to \infty$.

THEOREM 5. *If $\rho > \sigma$ in Theorem 4, the condition $f(t) \geqslant 0$ may be replaced by*

$$\text{(9 . 1)} \qquad f(t) \geqslant -Kt^{\rho-\sigma-1},$$

where $K > 0$. The result remains true when $\rho = \sigma$: if $\rho > 0$ and

$$\text{(9 . 2)} \qquad f(t) \geqslant -Kt^{-1},$$

$$\text{(9 . 3)} \qquad h(x) = \int_0^\infty \frac{f(t)\,dt}{(t+x)^\rho} \sim \frac{H}{x^\rho},$$

then

$$\text{(9 . 4)} \qquad \int_0^\infty f(t)\,dt = H.$$

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* Hardy and Littlewood, **1**, 185.

† Hardy and Littlewood, **1**, 188.

When $\rho > \sigma$, Theorem 5 is an immediate corollary of Theorem 4; we need only apply Theorem 4 to

$$g(t) = f(t) + Kt^{\rho-\sigma-1}\text{*}.$$

The case $\rho = \sigma$ is more delicate, and we require Lemma η below: Lemma ζ is included for the sake of completeness and is not actually used.

LEMMA ζ. *If* (9.4) *is true, then* (9.3) *is true.*

This is the "Abelian" theorem, and follows at once from the relations

$$h(x) = \left(\int_0^T + \int_T^\infty\right) \frac{f(t)\,dt}{(t+x)^\rho} = h_1(x) + h_2(x),$$

$$|h_2(x)| \leqslant x^{-\rho}\left|\int_T^{T'} f(t)\,dt\right| \leqslant \epsilon x^{-\rho} \quad [T' > T \geqslant T_0(\epsilon)],$$

$$h_1(x) \sim x^{-\rho}\int_0^T f(t)\,dt.$$

LEMMA η. *If*

(9.5) $$G(t) = \int_0^t uf(u)\,du = o(t),$$

and, in particular, if

(9.6) $$f(t) = o(t^{-1}),$$

then (9.3) *implies* (9.4).

This is the "o-Tauberian" theorem. To prove it, we write

(9.7) $$x^\rho h(x) = \int_0^x f(t)\,dt - \int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} f(t)\,dt + x^\rho \int_x^\infty \frac{f(t)\,dt}{(t+x)^\rho}$$
$$= F(x) + l(x) + m(x),$$

say. Here

$$l(x) = -\int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} \frac{G'(t)}{t}\,dt$$
$$= -(1-2^{-\rho})\frac{G(x)}{x} + \rho x^\rho \int_0^x \frac{G(t)\,dt}{t(t+x)^{\rho+1}} - \int_0^x \left\{1 - \frac{x^\rho}{(t+x)^\rho}\right\} \frac{G(t)}{t^2}\,dt.$$

The first term is $o(1)$. The second is

$$O(x^\rho)\,O(x^{-\rho-1})\int_0^x o(1)\,dt = o(1).$$

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* Cf. Hardy and Littlewood, 1, 185.

In the third we observe that the expression in the curly bracket does not exceed

$$A\left(1-\frac{x}{t+x}\right)=\frac{At}{x},$$

so that the third term is

$$\frac{1}{x}\int_0^x o(1)\,dt = o(1).$$

Hence

$$l(x)=o(1). \tag{9.8}$$

Finally

$$m(x)=x^\rho\int_x^\infty \frac{f(t)\,dt}{(t+x)^\rho}=x^\rho\int_x^\infty\frac{G'(t)\,dt}{t(t+x)^\rho} \tag{9.9}$$

$$=-2^{-\rho}\frac{G(x)}{x}+\rho x^\rho\int_x^\infty\frac{G(t)\,dt}{t(t+x)^{\rho+1}}+x^\rho\int_x^\infty\frac{G(t)\,dt}{t^2(t+x)^\rho}$$

$$=o(1)+o\left(x^\rho\int_x^\infty\frac{dt}{t^{\rho+1}}\right)=o(1).$$

From (9 . 7), (9 . 8), (9 . 9), and (9 . 3) it follows that

$$\int_0^x f(t)\,dt\to H,$$

which proves the lemma.

10. We can now prove Theorem 5. We write

$$\phi(x)=x^{\rho+1}h(x)=\int_0^\infty\frac{x^{\rho+1}}{(t+x)^\rho}f(t)\,dt. \tag{10.1}$$

Then

$$\phi(x)\sim Hx \tag{10.2}$$

and*

$$\phi''(x)=\int_0^\infty\frac{d^2}{dx^2}\left\{\frac{x^{\rho+1}}{(t+x)^\rho}\right\}f(t)\,dt \tag{10.3}$$

$$=\rho(\rho+1)\int_0^\infty\frac{x^{\rho-1}t^2}{(t+x)^{\rho+2}}f(t)\,dt>-A\int_0^\infty\frac{x^{\rho-1}t}{(t+x)^{\rho+2}}\,dt=-\frac{A}{x}.$$

* The differentiations under the integral sign present no difficulty: compare f.n. *, p. 30.

From (10 . 2) and (10 . 3) it follows* that

(10 . 4) $$\phi'(x) \to H,$$

i.e. $$(\rho+1)\,x^{\rho}\,h(x)+x^{\rho+1}\,h'(x) \to H,$$

and so $$h'(x) = -\rho \int_0^{\infty} \frac{f(t)\,dt}{(t+x)^{\rho+1}} \sim -\rho H x^{-\rho-1}.$$

Hence $$\int_0^{\infty} \frac{tf(t)\,dt}{(t+x)^{\rho+1}} = \int_0^{\infty} \frac{f(t)\,dt}{(t+x)^{\rho}} - x\int_0^{\infty} \frac{f(t)\,dt}{(t+x)^{\rho+1}} = o(x^{-\rho}).$$

But $$g(t) = tf(t) \geqslant -K.$$

Hence we may apply the case of Theorem 5, already proved, in which ρ is replaced by $\rho+1$ and σ by $\rho < \rho+1$. It follows that

$$G(t) = \int_0^t u\,f(u)\,du = o(u),$$

and the final conclusion then follows from Lemma η.

11. It is plain that we may prove similarly

THEOREM 6. *Suppose that* $0 \leqslant \sigma \leqslant \rho$, *that*

$$a_n \geqslant -Kn^{\rho-\sigma-1},$$

that $$h(x) = \sum_1^{\infty} \frac{a_n}{(n+x)^{\rho}}$$

is convergent for $\rho > 0$, *and that*

$$h(x) \sim Hx^{-\rho},$$

when $x \to \infty$. *Then*

$$\sum_1^{\infty} a_n = H.$$

[*Added July*, 1929. Important contributions to the theory of Tauberian theorems have been made recently by Schmidt (**6**), Vijayaraghavan (**10**), and Wiener (**11**).

Schmidt proved first that the condition $na_n = O(1)$ of Littlewood's original theorem (afterwards generalized by us to $na_n > -A$) could be replaced by

$$\underline{\lim}\,(s_n - s_m) \geqslant 0,$$

* Cf. Hardy and Littlewood, **1**, 188.

where $s_n = a_0 + a_1 + \ldots + a_n$ and m and n tend to infinity in such a manner that $n/m \to 1$. This condition includes all previously known. A much simpler proof than Schmidt's was given later by Vijayaraghavan. Schmidt also proved what is substantially the corresponding generalization of Theorem 1.

We have proved the corresponding generalization of Theorem 4, in which the condition is

$$\underline{\lim} \frac{F(qx) - F(px)}{x^{\rho-\sigma}} \geqslant 0,$$

when $n \to \infty$, $q > p$, $q/p \to 1$. Our method of proof is unlike those of Schmidt or Vijayaraghavan, but we must reserve its publication until another occasion.

We may, however, take this opportunity of calling attention to Wiener's paper, in which Tauberian theorems generally are investigated from a quite novel point of view.]

References.

1. G. H. Hardy and J. E. Littlewood, "Tauberian theorems concerning power-series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* (2), 13 (1914), 174–191.
2. —— "Some theorems concerning Dirichlet's series", *Messenger of Math.*, 43 (1914), 134–147.
3. E. W. Hobson, *The theory of functions of a real variable*, 2 (ed. 2, 1926).
4. E. Landau, "Beitrage zur analytischen Zahlentheorie", *Rend. di Palermo*, 24 (1907), 81–160.
5. —— *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 1916.
6. R. Schmidt, "Uber divergente Folgen und lineare Mittelbildungen", *Math. Zeitschrift*, 22 (1925), 89–152.
7. E. C. Titchmarsh, "A theorem on infinite products", *Journal London Math. Scc.*, 1 (1926), 35–37.
8. —— "On integral functions with real negative zeros", *Proc. London Math. Soc.* (2), 26 (1927), 185–200.
9. G. Valiron, "Sur les fonctions entières d'ordre fini et d'ordre nul, et en particulier les fonctions à correspondance régulière", *Annales de Toulouse* (3), 5 (1914), 117–257.
10. T. Vijayaraghavan, "A Tauberian theorem", *Journal London Math. Soc.*, 1 (1926), 113–120.
11. N. Wiener, "A new method in Tauberian theorems", *Journal Massachusetts Inst. of Technology*, 7 (1928) 161–184.

CORRECTIONS

p. 26, *line* 7 *up*. For Au^{λ} read At^{λ}.

p. 35, *line* 3. For $=$ read $\leqslant$.

p. 37, *line* 9. For $n \to \infty$ read $p \to \infty$.

COMMENTS

Theorem 1 is the 'positive' Tauberian theorem for integrals, corresponding to the 'positive' theorem for series, 1914, 4. It may be proved by an argument similar to that given for series. Doetsch† proved it this way for $\alpha = 1$. Szasz‡ used the same argument to obtain a theorem for Stieltjes integrals, which includes both the integral and the series versions. Titchmarsh (references **7** and **8**) quoted Theorem 1 in 1926, attributing it to Hardy and Littlewood. He proved that Theorem 1, with $0 < \alpha < 1$, is equivalent to a theorem which (as Hardy and Littlewood remark) was obtained by Valiron in 1914 (reference **9**). In § 1, it is stated that the series theorem is easily deduced from the integral theorem; e.g. by taking $f(t) = a_n$ for $n \leqslant t < n+1$. The converse is also true; e.g. put

$$a_n = \int\limits_{n}^{n+1} f(t)\,dt.$$

In the present paper, Theorem 1 is deduced from Theorem 4 (i). The relations between Theorems 1–4 and Valiron's theorem are indicated by the diagram:

$$\begin{tikzcd}
 & & 4(\mathrm{i}) \arrow[rr, leftrightarrow] \arrow[d] \arrow[ddll] & & 4(\mathrm{ii}) \arrow[d] \\
 & & 1 \arrow[d] & 2 \arrow[r, leftrightarrow] & 3 \\
 \underset{\rho=1}{4(\mathrm{i})} \arrow[r, leftrightarrow] & V \arrow[r, leftrightarrow] & \underset{\alpha<1}{1} & &
\end{tikzcd}$$

In the Addendum, the new methods and results of Schmidt, Vijayaraghavan, and Wiener are mentioned. To this list the name of Karamata§ may be added. The Schmidt-type generalization of Theorem 4, mentioned in the addendum, was not published. But Hardy states in D.S. that the method used in Ch. VII, § 7.10, is that originally used to prove this generalization.

† *Math. Annalen* 82 (1921), 68–82. He makes the transformation $v = e^t$, so that the Laplace integral becomes a Mellin integral.

‡ *Sitz. d. Bayerischen Akad. d. Wiss.* 59 (1929), 325–40 (published in 1930).

§ (1) *Math. Zeit.* 32 (1930), 319–20; (2) ibid. 33 (1931), 294–9; (3) *J. für die reine u. angew. Math.* 164 (1931), 27–39.

NOTES ON THE THEORY OF SERIES (XVI): TWO TAUBERIAN THEOREMS

G. H. Hardy *and* J. E. Littlewood*.

1. Our first object in this note is to prove Theorem 3, which we required for an application in Note XV. The proof, however, depends on Theorems 1 and 2, which are in themselves more interesting. Theorem 1 is a known theorem; the most important case, in which r is integral and $\delta = 1$, was stated as long ago as 1911 by Littlewood†, and the complete theorem was proved by Andersen in his dissertation‡; but the proof which we give here is a good deal simpler than any previous proof which we have seen§. Theorem 2 is very closely connected with another theorem of Andersen, but is apparently new in the form in which we state it||.

* Received 6 August, 1931; read 12 November, 1931.

† Littlewood, **7**, 448.

‡ Andersen, **1**, 80 (Theorem 4). Andersen assumes Littlewood's result, and deduces the general theorem as in § 2, (i), below.

§ The idea of the proof is similar to that used by Landau (**6**, 12–13) in his deduction of our "positive" Tauberian theorem from Littlewood's original Tauberian theorem.

|| See § 3.

2. THEOREM 1. *If Σa_n is summable (A), and bounded (C, r), where $r > -1$, then it is summable $(C, r+\delta)$ for every positive δ.*

(i) We show first that it is enough to prove the theorem when r is integral and $\delta = 1$. Suppose that the theorem has been proved in this case, that r is non-integral, and that $s = [r]+1$. Then Σa_n is bounded (C, s) and therefore summable $(C, s+1)$, Hence, being bounded (C, r), it is summable $(C, r+\delta)$*.

(ii) If r is integral, we have

$$s_n = s_n^0 = a_0+a_1+\dots+a_n, \quad s_n^1 = s_0^0+s_1^0+\dots+s_n^0,$$

$$s_n^r = s_0^{r-1}+s_1^{r-1}+\dots+s_n^{r-1} = A_n^r a_0+A_{n-1}^r a_1+\dots+A_0^r a_n,$$

where $$A_n^r = \binom{n+r}{r} = \frac{(n+1)(n+2)\dots(n+r)}{r!}.$$

If $$v_n^{r+1} = \frac{s_n^{r+1}}{A_n^{r+1}} = \frac{s_0^r+s_1^r+\dots+s_n^r}{A_n^{r+1}},$$

we have, by an easy calculation,

$$w_n = v_n^{r+1}-v_{n-1}^{r+1} = \frac{(r+1)!}{n(n+1)\dots(n+r+1)}\{ns_n^r-(r+1)s_{n-1}^{r+1}\}$$

$$= O\left(\frac{1}{n^{r+2}}\right)O(n^{r+1}) = O\left(\frac{1}{n}\right),$$

since, by hypothesis, $$s_n^r = O(n^r).$$

Again,

$$g(x) = \sum_0^\infty w_n x^n = (r+1)!\sum_0^\infty s_n^r \frac{x^n}{(n+1)\dots(n+r+1)}$$

$$-(r+1)(r+1)!\sum_1^\infty s_{n-1}^{r+1}\frac{x^n}{n(n+1)\dots(n+r+1)},$$

$$\frac{x^{r+1}}{(r+1)!}g(x) = \sum_0^\infty s_n^r\frac{x^{n+r+1}}{(n+1)\dots(n+r+1)}-(r+1)\sum_0^\infty s_n^{r+1}\frac{x^{n+r+2}}{(n+1)\dots(n+r+2)}$$

$$= \left(\int_0^x dy\right)^{r+1}\Sigma s_n^r y^n-(r+1)\left(\int_0^x dy\right)^{r+2}\Sigma s_n^{r+1}y^n.$$

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* Andersen, 1 (Theorem 8, p. 56).

Hence, if $f(x) = \Sigma a_n x^n$, so that $(1-x)^{-r-1}f(x) = \Sigma s_n^r x^n$, we have

$$\frac{x^{r+1}}{r+1}\, g(x) = \int_0^x (x-y)^r \frac{f(y)}{(1-y)^{r+1}}\, dy - \int_0^x (x-y)^{r+1} \frac{f(y)}{(1-y)^{r+2}}\, dy$$

$$= (1-x)\int_0^x \frac{(x-y)^r}{(1-y)^{r+2}}\, f(y)\, dy.$$

If now we suppose, as we may without real loss of generality, that $f(x) \to 0$ when $x \to 1$, we have

$$\frac{x^{r+1}}{r+1}\, g(x) = (1-x)\left(\int_0^x dy\right)^{r+1} o\left\{\frac{1}{(1-y)^{r+2}}\right\} = o(1).$$

Hence $g(x) \to 0$. It follows, since $w_n = O(1/n)$, that Σw_n converges to 0, *i.e.* that $v_n^{r+1} \to 0$; and this proves the theorem.

The theorem is also true for $r = -1$, if we agree that "Σa_n is bounded $(C, -1)$" is to mean "Σa_n is bounded, and $a_n = O(1/n)$".

3. THEOREM 2. *If Σa_n is summable (A), then a necessary and sufficient condition that it should be summable (C, r), where $r > -1$, is that*

(3.1) $$t_n = a_0 + 2a_1 + \ldots + (n+1)a_n = o(n) \quad (C, r).$$

The meaning of (3.1) is that

(3.2) $$t_n^r = o(n^{r+1}),$$

where t_n^r is formed from $b_n = na_n$ as s_n^r is formed from a_n.

Theorem 2 is very nearly the same as a theorem of Andersen*. Andersen gives, instead of (3.1), the condition

(3.3) $$\frac{t_n}{n} = o(1) \quad (C, r).$$

A comparison of the two theorems shows that (3.1) and (3.3) must be equivalent when Σa_n is summable (A); and in fact it may be proved directly (though the proof demands a certain amount of calculation) that they are equivalent independently of this condition. It is, however, more convenient to prove Theorem 2 directly, especially since our proof is rather simpler than Andersen's proof of his theorem.

We require two lemmas†.

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* Andersen, 1 (Theorem 7, p. 87).

† These are given (apart from trivial differences of notation, and the limitation of Lemma 1 to integral r) by Hardy, 5, 304–305.

LEMMA 1. *Suppose that Σa_n is summable $(C, r+1)$, where $r > -1$. Then (3.2) is a necessary and sufficient condition for summability (C, r).*

We have (whether r be integral or not)

$$s_n^r = A_n^r a_0 + \ldots + A_0^r a_n, \qquad t_n^r = A_n^r b_0 + \ldots + A_0^r b_n,$$

where
$$A_n^r = \frac{\Gamma(n+r+1)}{\Gamma(r+1)\Gamma(n+1)} = \frac{(r+1)(r+2)\ldots(r+n)}{n!}.$$

From these equations it follows at once that

$$(n+r+2)s_n^r - (r+1)s_n^{r+1} = t_n^r,$$

and the lemma is a corollary of this identity.

LEMMA 2. *If r is integral and $r \geqslant -1$, then a necessary and sufficient condition that Σa_n should be summable $(C, r+1)$ is that*

$$\Sigma t_n^r \Delta^{r+1} \frac{1}{n+1} \tag{3.4}$$

should be convergent.

It is easy to verify by induction that

$$\sum_{\nu=0}^{n-r-1} t_\nu^r \Delta^{r+1} \frac{1}{\nu+1} = \frac{(r+1)!\,(n-r)!}{(n+1)!} s_{n-r-1}^{r+1}. \tag{3.5}$$

In fact (3.5) is true for $n = r+1$, and the condition that, if true for $n-1$, it should be true for n, reduces to

$$t_{n-r-1}^r = (n-r)s_{n-r-1}^{r+1} - (n+1)s_{n-r-2}^{r+1},$$

another immediate consequence of the definitions of s_n^r and t_n^r.

4. To prove Theorem 2, we observe first that the *necessity* of the condition is contained in Lemma 1, the hypothesis of summability (A) being here irrelevant.

To prove the condition *sufficient*, we suppose first that r is integral. We have then

$$f(x) = \sum_0^\infty \frac{t_n - t_{n-1}}{n+1} x^n = \sum_0^\infty t_n \Delta \frac{x^n}{n+1} = \sum_0^\infty t_n^r \Delta^{r+1} \frac{x^n}{n+1} \tag{4.1}$$

$$= \sum_0^\infty t_n^r \sum_{s=0}^{r+1} \frac{(r+1)!}{s!\,(r+1-s)!} \Delta^{r+1-s} x^n \Delta^s \frac{1}{n+r+2-s}$$

$$= \sum_{s=0}^{r+1} \frac{(r+1)!}{s!\,(r+1-s)!} f_s(x),$$

where $$f_s(x) = (1-x)^{r+1-s} \sum_0^\infty t_n^r \left(\Delta^s \frac{1}{n+r+2-s}\right) x^n.$$

If $s < r+1$, we have

$$f_s(x) = (1-x)^{r+1-s} \sum o(n^{r+1})\, O(n^{-s-1})\, x^n$$
$$= (1-x)^{r+1-s} o\{(1-x)^{-r-1+s}\} = o(1).$$

Hence the remaining term in (4 . 1), viz.

$$f_{r+1}(x) = \sum t_n^r \Delta^{r+1} \frac{1}{n} x^n,$$

tends to a limit. Since the coefficient of x^n is $o(n^{r+1})\, O(n^{-r-2}) = o(1/n)$, it follows that the series is convergent for $x = 1$ or, what is the same thing, that the series (3 . 4) is convergent. Hence, by Lemma 2, Σa_n is summable (C, r).

If r is non-integral, let $s = [r]+1$. Then $t_n = o(n)$ (C, s), and so Σa_n is summable (C, s), and *a fortiori* is summable $(C, r+1)$. Hence, by Lemma 1, it is summable (C, r). This completes the proof of the theorem.

5. THEOREM 3. *If $r > -1$, Σa_n is summable (A), and*

(5 . 1) $$t_n = O(n) \quad (C, r),$$

then Σa_n is summable $(C, r+\delta)$ for every positive δ.

If we use (5 . 1) instead of (3 . 1) in the argument of §§ 3-4, the conclusion is plainly that Σa_n is bounded (C, r). Theorem 3 (which is the proposition that we required in Note XV) then follows from Theorem 1.

It may be worth while in conclusion to state shortly an alternative proof of Theorem 1 for integral r. Suppose that r is integral and $r \geqslant -1$. We proved in **2*** that *a necessary and sufficient condition that Σa_n should be summable (C, r) is that $\Sigma a_{n,1}$, where*

$$a_{n,1} = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \dots,$$

should be summable $(C, r-1)$; the series which defines $a_{n,1}$ is itself summable $(C, r-1)$. There is plainly a corresponding theorem in which Σa_n and $\Sigma a_{n,1}$ are bounded (C, r) and $(C, r-1)$ respectively.

* Hardy and Littlewood, 2 (Theorem A, p. 69). See p. 71 for explanations of the relations of this theorem to substantially equivalent theorems of Knopp. Σa_n is summable $(C, -1)$ if it is convergent and $a_n = o(1/n)$; bounded $(C, -1)$ if it is bounded and $a_n = O(1/n)$.

When $r = -1$, Theorem 1 reduces to Littlewood's original Tauberian theorem. Let us assume that the theorem is true for r, and that Σa_n is summable (A) and bounded $(C, r+1)$. The series which defines $a_{n,1}$ is summable (A) and bounded (C, r), and so (by our assumption) summable $(C, r+1)$. It is then easy to verify* that

$$f_1(x) = \Sigma a_{n,1} x^n = \frac{1}{1-x} \int_x^1 f(t)\, dt,$$

and so that $f_1(x)$ tends to a limit. Hence $\Sigma a_{n,1}$ is summable (A) and bounded (C, r), so summable $(C, r+1)$; and therefore Σa_n is summable $(C, r+2)$. This proves the theorem by induction.

References (for **1–4**, see Note XV).

5. G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.* (2), 8 (1909), 301–320.

6. E. Landau, *Darstellung und Begrundung einiger neuerer Ergebnisse der Funktionentheorie*, 2 ed., 1929.

7. J. E. Littlewood, "The converse of Abel's theorem on power series", *Proc. London Math. Soc.* (2), 9 (1910), 434–448.

* Compare **2**, 78.

CORRECTIONS

p. 283, *line* 11 *up*. For na_n read $(n+1)a_n$.

p. 285, *line* 6. For $\Delta^{r+1}\frac{1}{n}x^n$ read $\left(\Delta^{r+1}\frac{1}{n+1}\right)x^n$.

p. 286, *References*. Add from 1931, 7 (in Vol. III):

1. A. F. Andersen, *Studier over Cesàro's Summabilitetsmetode* (Copenhagen, 1921).

2. G. H. Hardy and J. E. Littlewood, 'Solution of the Cesàro summability problem for power series and Fourier series', *Math. Zeit.* 19 (1924), 67–96.

3. —— 'The allied series of a Fourier series', *Proc. Lond. Math. Soc.* (2), 24 (1925), 211–46.

4. R. E. A. C. Paley, 'On the Cesàro summability of Fourier series and allied series', *Proc. Cambridge Phil. Soc.* 26 (1930), 173–203.

COMMENTS

Littlewood's theorem, § 1, stated in 1911, is here deduced from his O-Tauberian theorem. Anderson's proof of the result quoted in the proof of Theorem 1 (i) is reproduced by Dienes;† see also Zygmund.‡

In the proof of Theorem 1 (ii), an alternative way of manipulating $g(x)$ is to write

$$\sum w_n x^n = \sum (v_n^{r+1}-v_{n-1}^{r+1})x^n = (1-x)\sum v_n^{r+1}x^n.$$

Then $\sum v_n^{r+1}x^{n+r+1}$ is the $(r+1)$-fold integral of $(r+1)!\sum s_n^{r+1}x^n$, i.e. of $(r+1)!(1-x)^{-r-2}f(x)$. The proof is then completed as before.

The proof of Theorem 1 (ii) does not use the full force of the hypothesis $s_n^r = O(n^r)$, but only

$$w_n = v_n^{r+1}-v_{n-1}^{r+1} = O(1/n).$$

By an identity of Kogbetliantz,§

$$n(v_n^{r+1}-v_{n-1}^{r+1}) = \tau_n^{r+1},$$

where τ_n^k denotes the nth Cesàro mean of order k of the sequence na_n. It follows that the proof of Theorem 1 (ii) establishes an analogue of Theorem 3, with na_n in place of $(n+1)a_n$. The argument holds for $r > -1$, if A_n^r is defined as in Lemma 1. The analogue of Theorem 3 was obtained independently by Kogbetliantz (*Mémorial* . . ., p. 40), who also gave a one-sided version. It may be proved directly that condition (5.1) is equivalent to its analogue.

There are two versions of Lemmas 1 and 2, one involving $(n+1)a_n$, as here, and the other na_n, as in D.S., pp. 121–2. The identities in 1910, 3, § 4, where the sums run from 1 to n, correspond to those here. It is enough to redefine the s_n^r, t_n^r of 1910, 3, as s_{n-1}^r, t_{n-1}^r, and replace the a_n by a_{n-1}. On the other hand, the identities in D.S., p. 122, are the same as those given by Kogbetliantz; see *Mémorial* . . ., pp. 23 and 30. A special feature of the sequence na_n is that it may be written

$$n\nabla s_n = n(s_n-s_{n-1}) = (H^{-1}-I)s_n,$$

which is a Hausdorff transformation. It follows that

$$(n\nabla)(C,r+1)s_n = (C,r+1)(n\nabla)s_n,$$

which is Kogbetliantz's identity stated above. In 1910, 3, Lemma 1 is given for integral r. In 1913, 2, § 47, Hardy and Littlewood state Lemma 2 (2nd version) for $r > -1$.

† *The Taylor series*, pp. 428–30. Oxford University Press, 1931.

‡ *Math. Zeit.* 25 (1926), 291–6.

§ Kogbetliantz, (1), *Bull. des sci. math.* (2), 49 (1925), 234–56; (2), *Mémorial des sci. math.* 51 (1931), p. 23.

ON THE SUMMABILITY OF SERIES BY BOREL'S AND MITTAG-LEFFLER'S METHODS

G. H. HARDY*.

1. We may say that the series

$$a_0+a_1+a_2+\ldots \tag{1.1}$$

is summable (B, α), to sum s, if (i) the series

$$A_\alpha(t)=\Sigma\, a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} \tag{1.2}$$

is convergent for all t, and (ii) the integral

$$\int_0^\infty e^{-t} A_\alpha(t)\,dt=\lim_{T\to\infty}\int_0^T e^{-t} A_\alpha(t)\,dt \tag{1.3}$$

converges to s. This method of summation plays a great part in Mittag-Leffler's classical researches† in the theory of analytic continuation. When $\alpha=1$ it reduces to Borel's exponential method‡. The method becomes more "powerful", *i.e.* more capable of summing "crudely divergent" series, as α increases. Thus the series

$$1-1\,!+2\,!-3\,!+\ldots$$

is summable (B, α) when $\alpha>1$ but not when $\alpha \leqslant 1$.

Littlewood and I have often emphasized§ a general principle which it is difficult to formulate precisely, but which may be indicated roughly as follows: *the delicacy of a method of summation tends to be inversely proportional to its power.* Thus Borel's method is much more powerful than any of Cesàro's, which cannot sum a power series outside its circle of convergence. On the other hand, when we are dealing with delicately divergent series, the Cesàro methods are always more effective; and in fact a series of finite order‖, if summable (B), is necessarily summable (C)¶. There is no

* Received and read 18 January, 1934.

† Mittag-Leffler (**4**).

‡ When α is a positive integer k the method is that described by Bromwich (**1**, 299) as "Le Roy's extension of Borel's method", in which we apply Borel's method to the series obtained from (1.1) by interpolating $k-1$ zeros between each pair of terms.

§ See, for example, Hardy and Littlewood (**2**, 2).

‖ With $a_n=O(n^K)$ for some K.

¶ Hardy and Littlewood (**2**, 10).

1934, 5 Journal of the London Mathematical Society, 9, 153–7.

relation of complete inclusion between the two methods, as there is between the Cesàro and Abel or Poisson, or between the Euler and Borel*, methods, but there is a restricted relation of inclusion; if a series is summable (B), and if it is not so crudely divergent as to be outside the bounds of possible C summation, then it is certainly summable (C). My object here is to show that there is a similar relation between the Mittag-Leffler methods corresponding to different α.

2. The existence of such a relation is suggested by Mittag-Leffler's work. He determines first the *étoile de convergence* of

$$1+z+z^2+\ldots,$$

where $z=re^{i\theta}$. The boundary of the "star" is the curve

$$r=\left(\sec\frac{\theta}{\alpha}\right)^{\alpha}.$$

If $0<\alpha<2$, the curve runs to infinity in the directions $\theta=\pm\frac{1}{2}\alpha\pi$, and the star is the region to the left of the curve. If $\alpha>2$, the curve cuts itself in the point corresponding to $\theta=\pm\pi$, and the star is the interior of the loop. The star increases as α decreases, and its limit when $\alpha\to 0$ is the ordinary "Mittag-Leffler star" of the function, the plane cut from 1 to ∞. Mittag-Leffler then extends this conclusion, in a manner now classical, to any power series with a finite radius of convergence.

The theorem which follows shows that the relation indicated by Mittag-Leffler's work holds for all series without exception.

THEOREM. *If* (i) $\beta>\alpha>0$, (ii) *the series* (1.1) *is summable* (B,β), *and* (iii) *the series* (1.2) *is convergent for all* t, *then the series* (1.1) *is summable* (B,α).

As is to be expected after what I have said, the inference is from the *more* to the *less* powerful method, so that the theorem has a "Tauberian" character.

3. I give a full proof only in the special case $\alpha=1$, $\beta=2$. The proof is practically the same when α is arbitrary and $\beta=2\alpha$. In the general case it is the same in principle, but there is a good deal more formal detail.

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* Knopp (3).

Suppose that we have determined a positive solution $\phi(x)$ of the moment-equation

(3.1) $$\int_0^\infty x^n \phi(x)\,dx = \frac{\Gamma(\beta n+1)}{\Gamma(\alpha n+1)}.$$

Then

(3.2) $$A_\alpha(t) = \Sigma a_n \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} = \Sigma a_n \frac{t^{\alpha n}}{\Gamma(\beta n+1)} \int_0^\infty x^n \phi(x)\,dx$$

$$= \int_0^\infty \phi(x)\, \Sigma a_n \frac{(xt^\alpha)^n}{\Gamma(\beta n+1)}\,dx = \int_0^\infty \phi(x)\, A_\beta(x^{1/\beta} t^{\alpha/\beta})\,dx.$$

The term-by-term integration is justified by "absolute convergence".

We may now argue formally as follows. Using (3.2), and then making the substitution $x = w^\beta t^{-\alpha}$, we obtain

(3.3) $$\int_0^\infty e^{-t} A_\alpha(t)\,dt = \int_0^\infty e^{-t}\,dt \int_0^\infty \phi(x)\, A_\beta(x^{1/\beta} t^{\alpha/\beta})\,dx$$

$$= \beta \int_0^\infty e^{-t} t^{-\alpha}\,dt \int_0^\infty w^{\beta-1} \phi\left(\frac{w^\beta}{t^\alpha}\right) A_\beta(w)\,dw$$

$$= \beta \int_0^\infty w^{\beta-1} A_\beta(w)\,dw \int_0^\infty e^{-t} t^{-\alpha} \phi\left(\frac{w^\beta}{t^\alpha}\right) dt.$$

If now

(3.4) $$\int_0^\infty e^{-t} t^{-\alpha} \phi\left(\frac{w^\beta}{t^\alpha}\right) dt = \frac{1}{\beta} w^{1-\beta} e^{-w},$$

then (3.3) gives

(3.5) $$\int_0^\infty e^{-t} A_\alpha(t)\,dt = \int_0^\infty e^{-w} A_\beta(w)\,dw,$$

the result of the theorem.

4. When $\alpha = 1, \beta = 2$, we can take

(4.1) $$\phi(x) = Cx^{-\frac{1}{2}} e^{-\frac{1}{4}x},$$

where $C = 2\sqrt{\pi}$. The formula (3.4) may be verified directly. Hence, in order to complete the proof, it is only necessary to justify the inversion of the integrations in the second line of (3.3). This is (in the general as in the special case) the kernel of the proof.

It follows by "uniform convergence" that

$$\int_0^T e^{-t} t^{-\frac{1}{2}}\,dt \int_0^\infty e^{-w^2/4t} A_2(w)\,dw = \int_0^\infty A_2(w)\,dw \int_0^T \exp\left(-t - \frac{w^2}{4t}\right) t^{-\frac{1}{2}}\,dt.$$

Hence everything depends on proving that

$$\int_0^\infty A_2(w)\,dw \int_T^\infty \exp\left(-t-\frac{w^2}{4t}\right)t^{-\frac{1}{2}}\,dt \to 0 \tag{4.2}$$

as $T\to\infty$ (the assertion, of course, including that of the convergence of the integral for finite T). If we put $t=\frac{1}{2}u^2$, then (4.2) becomes

$$\int_0^\infty \psi(w)\chi(w,U)\,dw \to 0, \tag{4.3}$$

where

$$\psi(w)=e^{-w}A_2(w),\quad \chi(w,U)=\int_U^\infty \exp\left\{-\tfrac{1}{2}\left(u-\frac{w}{u}\right)^2\right\}du; \tag{4.4}$$

and if now we put

$$u-\frac{w}{u}=2x,\quad u=x+\sqrt{(x^2+w)},\quad U-\frac{w}{U}=2X,$$

then χ takes the form

$$\chi=\int_X^\infty e^{-2x^2}\,dx+\int_X^\infty \frac{xe^{-2x^2}}{\sqrt{(x^2+w)}}\,dx=\chi_1+\chi_2, \tag{4.5}$$

say.

Suppose now that $U>1$ and that w increases from 0 to ∞. Then X decreases, so that χ_1 increases, but remains less than an absolute constant C; and if we write χ_2 in the form

$$\chi_2=\chi_3-\chi_4,$$

where

$$\chi_3=\int_0^\infty \frac{xe^{-2x^2}}{\sqrt{(x^2+w)}}\,dx\quad (w<U^2),\qquad \chi_3=\int_{-X}^\infty \frac{xe^{-2x^2}}{\sqrt{(x^2+w)}}\,dx\quad (w>U^2),$$

$$\chi_4=\int_0^X \frac{xe^{-2x^2}}{\sqrt{(x^2+w)}}\,dx\quad (w<U^2),\qquad \chi_4=0\quad (w>U^2),$$

then a moment's consideration shows that χ_3 and χ_4 are also monotonic and bounded. Hence the integral (4.3) satisfies the following conditions: (i) $\psi(w)$ has a convergent integral over $(0,\infty)$; (ii) $\chi(w,U)$ is a linear combination of three functions, each positive, monotonic in w for every U, and less than an absolute constant; (iii) $\chi(w,U)$ tends to zero, when $U\to\infty$, for every w. It follows that the integral (4.3) is convergent and tends to zero.

5. It is only when $\beta=2\alpha$ that the proof can be stated so shortly. In the general case,

$$\phi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\beta s+1)}{\Gamma(\alpha s+1)}\,x^{-s-1}\,dx,$$

where $c > 0$, and this is not an elementary function but a function of hypergeometric type. It behaves, for large positive x, like a multiple of

$$e^{-Cx^{a}}x^{b},$$

where

$$a = \frac{1}{\beta-\alpha}, \quad b = \frac{1}{2(\beta-\alpha)} - 1, \quad C^{\beta-\alpha} = (\beta-\alpha)^{\beta-\alpha}\,\alpha^{\alpha}\,\beta^{-\beta}.$$

The proof runs on similar lines, but the argument of § 4 has to be replaced by one a good deal longer, which I must reserve for another occasion.

References.

1. T. J. I'A. Bromwich, *Theory of infinite series* (first ed., 1908).
2. G. H. Hardy and J. E. Littlewood, " The relations between Borel's and Cesàro's methods of summation ", *Proc. London Math. Soc.* (2), 11 (1911), 1–16.
3. K. Knopp, " Uber das Eulersche Summierungsverfahren: II ", *Math. Zeitschrift*, 18 (1923), 125–156.
4. G. Mittag-Leffler, " Sur la representation analytique d'une branche uniforme d'une fonction monogène (cinquième note) ", *Acta Math.*, 29 (1904), 101–181.

CORRECTION

p. 155, *line* 6 *up*. For $C = 2\sqrt{\pi}$ read $C = \dfrac{1}{2\sqrt{\pi}}$.

COMMENTS

The result, § 1, concerning the series $1-1!+2!-...$, stated to hold when $\alpha > 1$, has been shown by Good† to hold only when $1 < \alpha \leqslant 3$. If $\alpha > 3$, the series (1.2) converges for all $t > 0$, but the integral (1.3) is not convergent. The theorem stated in § 2 and proved for $\alpha = 1$, $\beta = 2$, has been proved in the general case by Good.‡

The method (B, α) is called (B', α) in D.S., $(B', 1)$ being Borel's integral method (B'). The method called Mittag-Leffler's method in D.S. is that defined by the limit

$$\lim_{\delta \to 0} \sum_{0}^{\infty} \frac{a_n}{\Gamma(\delta n+1)},$$

which corresponds to the case $\alpha = 0$; see Good (paper (2), § 12). The method here called Mittag-Leffler's method is called the Borel-Mittag-Leffler method by Good (paper (2)).‡

A method defined by the integral (1.3) was also given by Le Roy.§ But his definition was an extension of Borel's version of the (B^*) method, in which the series (1.2) has a finite radius of convergence, and its sum on $(0, \infty)$ is defined by analytic continuation; see the Comments on 1911, 8. In D.S., p. 226, Hardy defines a $*$-version of the (B', α) method, which also contains Le Roy's method, and states a corresponding extension of the theorem proved here.

† Good (1), *J. London Math. Soc.* 16 (1941), 180–2.

‡ Good (2), *Proc. Cambridge Phil. Soc.* 38 (1942), 144–65.

§ *Ann. de la Fac. des Sci. de l'Univ. de Toulouse* (2), 2 (1900), 317–430. See Borel (2nd edn.), pp. 146–8.

REMARKS ON SOME POINTS IN THE THEORY OF DIVERGENT SERIES

By G. H. Hardy

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1. I have collected here a number of remarks which have occurred to me when lecturing on Fourier series or on the general theory of divergent series. They contain nothing that is important, and not a great deal that is new, but may be interesting because they refer to comparatively unfamiliar topics.

A. On Fourier's own deduction of Fourier's theorem

2.1. By 'Fourier's theorem' I mean here the theorem that, if $f(x)$ belongs to an appropriate class of functions, and is 'representable' by a trigonometric series

$$\frac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)\,, \tag{2.1.1}$$

then a_n and b_n are given by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\,. \tag{2.1.2}$$

The formulae are older than Fourier, but with that I am not concerned.[1]

'Fourier's theorem' asserts, in modern language, that a trigonometrical series which represents the function must be the 'Fourier series' of the function. It is true or false according to the class of functions considered and the manner in which the function is 'represented' by the series. Thus it is true, after du Bois-Reymond and de la Vallée-Poussin, when the function is bounded or integrable and the series is convergent. If we assume only that the series is summable, by one or other of the standard methods of the theory of divergent series, then the theorem is usually false, even when the function is 0.[2] Thus the series

$$\sin x + 2\sin 2x + 3\sin 3x + \cdots \tag{2.1.3}$$

is summable to sum 0 by the Abel-Poisson method or by Cesàro means of any order greater than 1.

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[1] The fullest account of the early history of the formulae is that given by Burkhardt, 'Trigonometrische Reihen und Integrale', *Encykl. d. Math. Wiss.*, II. A. 12 (in particular §16). Burkhardt traces the first of them back to Clairaut (1757). The usual deduction by term-by-term integration is due to Euler (1777).

[2] The best results in this direction are due to Rajchmann, Zygmund, and Verblunsky. In particular Verblunsky [*Proc. London Math. Soc.* (2), 31 (1930) and 34 (1932)] has shown that a trigonometrical series cannot be summable $(C, 1)$ to 0 unless its coefficients are nul, thus confirming a conjecture of M. Riesz [*Math. Ann.* 71 (1911)].

1935, 1 Annals of Mathematics (2), 36, 167–81.

Here I am concerned primarily with the very special case which is considered by Fourier in §§207–218 of his *Théorie de la chaleur*. Suppose that $f(x)$ is odd and analytic for $|x| \leqq \pi$, so that

$$f(x) = \sum_0^\infty \frac{f^{(2h+1)}(0)}{(2h+1)!} x^{2h+1} \tag{2.1.4}$$

for $|x| \leqq \pi$, and that

$$f(x) = \sum_1^\infty b_n \sin nx \tag{2.1.5}$$

for $-\pi < x < \pi$, the series (2.1.5) being convergent in the classical sense.[3] These are in effect Fourier's assumptions, and his object is to prove that b_n is given by the second of formulae (2.1.2). It will be observed that he is attempting to prove a theorem of 'uniqueness'. He wishes to show not merely (what is trivial now) that $f(x)$ can be expanded in the form required, but also that the series so obtained is the *only* trigonometrical series for the function. The conclusion desired is true, but only on account of 'Cantor's theorem'; and a mild generalization of the interpretation of (2.1.5) would render the conclusion false.

Fourier replaces every sine in (2.1.5) by its Taylor series, and equates coefficients.[4] He thus obtains the system of equations

$$b_1 + 2^{2h+1}b_2 + 3^{2h+1}b_3 + \cdots = (-1)^h f^{(2h+1)}(0) \qquad (h = 0, 1, 2, \cdots). \tag{2.1.6}$$

It will be observed that the series here are all divergent even in the simplest cases; thus the Fourier series of $f(x) = x$ is

$$2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right),$$

and the system (2.1.6) then becomes

$$1 - 1 + 1 - \cdots = \tfrac{1}{2}, \qquad 1 - 2^{2h} + 3^{2h} - \cdots = 0 \qquad (h = 1, 2, \cdots).$$

These equations are correct when properly interpreted, but they require a theory of divergent series; and a very little use of divergent series is, as we have seen, sufficient to falsify Fourier's conclusion.

Fourier next replaces the system (2.1.6) by the corresponding system

$$\beta_1 + 2^{2h+1}\beta_2 + \cdots + k^{2h+1}\beta_k = (-1)^h f^{(2h+1)}(0) \qquad (h = 0, 1, \cdots, k-1) \tag{2.1.7}$$

of k rows and columns, solves this system for $\beta_1, \beta_2, \cdots, \beta_k$ and makes k tend to infinity. The method, as Riesz remarks, embodies an important principle,

[3] The sum of the series at the ends of the interval will naturally not be $f(x)$ but

$$\tfrac{1}{2}\{f(-\pi) + f(\pi)\} = 0.$$

[4] There is a short but very interesting discussion of Fourier's argument in F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues* (1913), pp. 2–7.

and the calculation is the more remarkable because Fourier had no determinants to help him. It would be hopeless to attempt to 'justify' the method, in this particular application, under any 'reasonable' conditions, and Fourier's calculations are now only a historical curiosity. But the *result*, viz.

$$\frac{1}{2}\pi b_n = \frac{(-1)^{n-1}}{n}\left\{f(\pi) - \frac{f''(\pi)}{n^2} + \frac{f''''(\pi)}{n^4} - \cdots\right\}, \tag{2.1.8}$$

is true fairly generally, and is a formula of considerable interest.

2.2. The formulae (2.1.7) and (2.1.8) suggest several different questions. The first is whether (2.1.8) is in fact a valid representation of the Fourier coefficient of $f(x)$.

In the first place, if $f(x)$ is analytic for $-\pi \leqq x \leqq \pi$,[5] we have

$$\frac{1}{2}\pi b_n = \int_0^\pi f(x)\sin nx\,dx = (-1)^{n-1}\left\{\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3} + \cdots + (-1)^h\frac{f^{(2h)}(\pi)}{n^{2h+1}} + \frac{(-1)^h}{n^{2h+1}}\int_0^\pi f^{(2h+1)}(x)\cos nx\,dx\right\},$$

by repeated partial integration. Since the last term is $O(n^{-2h-2})$ for large n, we conclude that *the series* (2.1.8) *is an asymptotic series for* b_n, *for any analytic* $f(x)$.

The series will be convergent, for sufficiently large n, if

$$|f^{(2h)}(\pi)| < K^{2h}$$

for some K. For this, it is necessary and sufficient that $f(x)$ be an integral function of order 1 and finite type, *i.e.* that

$$|f(x)| < Ae^{B|x|} \tag{2.2.1}$$

for some A and B. If $B < 1$, in which case also $K < 1$, then the series is convergent for $n \geqq 1$. All this is familiar.[6]

We are led to ask whether the series

$$\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3} + \frac{f''''(\pi)}{n^5} - \cdots \tag{2.2.2}$$

or

$$\frac{f(\pi)}{n} + 0 - \frac{f''(\pi)}{n^3} + 0 + \frac{f''''(\pi)}{n^5} + \cdots \tag{2.2.3}$$

are summable by recognised methods: it is the second series which presents itself most naturally. If $f(x)$ is an integral function, we have

$$\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3}\frac{t^2}{2!} + \frac{f''''(\pi)}{n^5}\frac{t^4}{4!} - \cdots = \frac{1}{2n}\left\{f\left(\pi + \frac{it}{n}\right) + f\left(\pi - \frac{it}{n}\right)\right\}, \tag{2.2.4}$$

[5] That is to say, in a region of the complex plane containing the portion of the real axis from $-\pi$ to π. This is naturally a more general hypothesis than Fourier's.

[6] See in particular G. Pólya, *Math. Zeitschrift*, 29 (1929), 549–640 (especially pp. 578–580).

and so

$$(2.2.5)\qquad \int_0^\infty e^{-t}\left\{\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3}\frac{t^2}{2!} + \cdots\right\}dt$$

$$= \frac{1}{2n}\int_0^\infty e^{-t}\left\{f\left(\pi + \frac{it}{n}\right) + f\left(\pi - \frac{it}{n}\right)\right\}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-nt}\,\{f(\pi + it) + f(\pi - it)\}\,dt,$$

if the integral is convergent. In these circumstances the series (2.2.3) is summable (B), *i.e.* by Borel's exponential integral. The series (2.2.2) is summable in the sense which I have called $(B,\ 2)$.[7] It is summable $(B,\ 1)$, *i.e.* (B), whenever

$$\frac{f(\pi)}{n} - \frac{f''(\pi)}{n^3}\frac{t}{1!} + \frac{f''''(\pi)}{n^5}\frac{t^2}{2!} - \cdots$$

is convergent for all t; for example, when $f(x)$ is of finite type, in which case the series is convergent for large n and summable (B) for all positive n.

Suppose now that $f(x) = f(\xi + i\eta)$ is (integral and) of finite type in the half-strip R defined by

$$-\pi \leqq \xi \leqq \pi,\quad \eta \geqq 0;$$

that is to say that (2.2.1) is satisfied in R. Then, applying Cauchy's theorem to

$$\int e^{nix} f(x)\,dx$$

and R, we obtain

$$\pi b_n = \frac{1}{i}\int_{-\pi}^{\pi} e^{nix} f(x)\,dx$$

$$(2.2.6)\qquad = \int_0^\infty e^{-n\pi i - n\eta} f(-\pi + i\eta)\,d\eta - \int_0^\infty e^{n\pi i - n\eta} f(\pi + i\eta)\,d\eta$$

$$= (-1)^{n-1}\int_0^\infty e^{-n\eta}\,\{f(\pi + i\eta) + f(\pi - i\eta)\}\,d\eta\,.$$

Comparing (2.2.5) and (2.2.6), we see that the series (2.2.3) is summable (B) to sum $(-1)^{n-1}\frac{1}{2}\pi b_n$. That is to say, when $f(x)$ satisfies the conditions stated, *Fourier's formula is correct for sufficiently large n if the infinite series is interpreted as a Borel sum. It is correct in this sense for all n if the B of* (2.2.1) *is less than* 1. For example, these conditions are satisfied when

$$f(x) = x\cos ax^2 \qquad (2a\pi < 1)\,.$$

[7] See G. H. Hardy, *Journal London Math. Soc.*, 8 (1934), 153–157. A series is summable $(B,\ k)$ when the series obtained by interpolating $k - 1$ zeros between each pair of successive terms is summable (B).

It is to be observed that the series may be summable, or even converge, without representing b_n; thus it vanishes identically whenever $f(x)$ has the period 2π. In this case $f(x)$, if of finite type, is a trigonometrical polynomial, and $b_n = 0$ for sufficiently large n.

There is another generalization of Borel's method which is fairly familiar and which enables us to extend the range of validity of Fourier's formula further. The generalization is that often used in summing the series

$$\sum a_n x^n = 1 - 1!\, x + 2!\, x^2 - \cdots .$$

In this case the series

$$\sum a_n \frac{(xt)^n}{n!} = 1 - xt + x^2t^2 - \cdots$$

is convergent only when $t \mid x \mid < 1$, but is summable (B) when $\Re(xt) \geqq -1$, *i.e.* when xt lies within the Borel polygon of summability. This condition will be satisfied, for all positive t, if and only if $\Re(x) > 0$; and then we may write

$$1 - 1!\, x + 2!\, x^2 - \cdots = \int_0^\infty \frac{e^{-t}}{1 + xt}\, dt ,$$

and say that the series is summable (B^2), *i.e.* by a repeated application of Borel's method.

We assumed that $f(x)$ is an integral function only in writing down the sum of the series in (2.2.4). This equation will be true, in the Borel sense, whenever all the arguments of $f(x)$ in question lie within the Borel polygon of the series; and this will be true, for every t, if *all the singularities of* $f(x)$ *are on the real axis and at a distance greater than* π *from the origin.* If this condition is fulfilled, and $f(x)$ satisfies (2.2.1) in R, then our conclusions about Fourier's formula remain valid with the substitution of (B^2) for (B).

As an example, suppose

$$f(x) = \frac{1}{a - x} - \frac{1}{a + x} \qquad (a > \pi) .$$

2.3. There remain Fourier's formulae (2.1.6). Fourier's series are divergent in the most trivial cases, but the simplest definitions are sufficient to sum them and to re-establish the formulae. Riesz,[8] for example, remarks that the formulae are correct if the series are interpreted as Abel-Poisson sums. Much more indeed is true, since Young[9] has shown that the k^{th} derived series of the Fourier series of $f(x)$ is summable (C, k), at any internal point, whenever $f^{(k)}(x)$ is continuous. However, the Abel-Poisson method is the simplest and most natural which will sum *all* the series, and it may be worth while to give a very simple proof that it will do so.

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[8] l. c., p. 5.

[9] *Proc. London Math. Soc.* (2) 17 (1918), 195–236.

Since $f(x)$ is regular for $-\pi \leqq x \leqq \pi$, we have

$$b_n = \frac{1}{\pi i}\int_{-\pi}^{\pi} f(x)\, e^{nix}\, dx = \frac{1}{\pi i}\int_{C_1} f(x)\, e^{nix}\, dx\,,$$

where C_1 is a curve from $-\pi$ to π a little above the real axis. Hence, if $\delta > 0$,

$$\sum b_n e^{-\delta n} = \frac{1}{\pi i}\int_{C_1} f(x) \sum e^{nix-\delta n}\, dx = \frac{1}{\pi i}\int_{C_1} f(x)\, \frac{e^{ix-\delta}}{1-e^{ix-\delta}}\, dx\,.$$

Differentiating $2h+1$ times with respect to δ, and then replacing the derivative under the integral sign by the corresponding derivative with respect to x, we obtain

$$\sum n^{2h+1} b_n e^{-\delta n} = \frac{(-1)^{h-1}}{\pi}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{e^{ix-\delta}}{1-e^{ix-\delta}}\, dx\,.$$

When $\delta \to 0$, the right hand side tends to

$$\frac{(-1)^{h-1}}{\pi}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{e^{ix}}{1-e^{ix}}\, dx = \frac{(-1)^{h}}{2\pi i}\int_{C_1} f(x)\left(\frac{d}{dx}\right)^{2h+1} \cot\frac{1}{2}x\, dx$$

$$= \frac{(-1)^{h}}{2\pi i}\int_{C} f(x)\left(\frac{d}{dx}\right)^{2h+1} \frac{1}{2}\cot\frac{1}{2}x\, dx\,,$$

where C is a complete contour round the origin. This is

$$\frac{(-1)^{h-1}}{2\pi i}\int_{C} f^{(2h+1)}(x)\, \frac{1}{2}\cot\frac{1}{2}x\, dx = (-1)^h f^{(2h+1)}(0)\,;$$

and this is accordingly the Abel-Poisson sum of the series $\Sigma n^{2h+1} b_n$.

A slightly modified form of the argument will prove that the k^{th} derived series of the Fourier series of $f(x)$ is summable (A) to $f^{(k)}(x)$ at any internal point of $(-\pi, \pi)$.[10] Suppose that

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{nix}\,,\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-nix} f(x)\, dx\,,$$

and that $-\pi < \xi < \pi$. Then we can express the sums

$$\sum_{n \geqq 0} c_n e^{ni\xi-\delta n}\,,\quad \sum_{n<0} c_n e^{ni\xi+\delta n} \qquad (\delta > 0)$$

by integrals along paths C_2, C_1 from $-\pi$ to π, C_1 above and C_2 below the real axis. The proof then follows the same lines.

It is also easy to show that the series are summable (B); we may prove this directly from the definitions, or use the theorem that a power series is summable (B) at any regular point on the circle of convergence.

[10] This was pointed out to me by Professor Pólya.

B. The series $\sum a^r \phi^{(r)}(x)$

3.1. The series of §2.2 were of the type

$$\sum_{r=0}^{\infty} a^r \phi^{(r)}(x),$$

and we expressed these sums by integrals involving $\phi(x)$ and exponential functions. A more entertaining form can be given to the results by considering series infinite in both directions. We write

$$\phi^{(-r)}(x) = \phi_r(x) = \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} \phi(t)\, dt \qquad (r > 0),$$

so that a derivative of negative order $-r$ is a 'Riemann-Liouville' integral of order r and with origin 0; and we consider the series

$$y(x) = \sum_{-\infty}^{\infty} a^r \phi^{(r)}(x) = \sum_{-\infty}^{\infty} b^{-r} \phi^{(r)}(x) \qquad \left(b = \frac{1}{a}\right). \tag{3.1.1}$$

I suppose, in the first instance, that x and a are positive.

If the series (3.1.1) is uniformly convergent in any interval of x, then

$$y' = by,\ y = Ce^{bx}.$$

Hence we may expect the sum of (3.1.1) to be an exponential function, whatever the form of $\phi(x)$.[11]

We write

$$y = \sum b^{-r} \phi^{(r)}(x) = \sum_{r<0} + \sum_{r\geqq 0} = y_1 + y_2. \tag{3.1.2}$$

The first series may be summed at once, since

$$y_1 = \sum_{r>0} \frac{b^r}{(r-1)!} \int_0^x (x-t)^{r-1} \phi(t) dt = be^{bx} \int_0^x e^{-bt} \phi(t)\, dt. \tag{3.1.3}$$

For this no more is necessary than that $\phi(t)$ should be integrable. If we can prove that (with some interpretation of the series)

$$y_2 = \sum_{r\geqq 0} b^{-r} \phi^{(r)}(x) = b \int_0^{\infty} e^{-bu} \phi(x+u)\, du = be^{bx} \int_x^{\infty} e^{-bt} \phi(t)\, dt, \tag{3.1.4}$$

then (3.1.3) and (3.1.4) will give

$$y = Ce^{bx},\ C = b \int_0^{\infty} e^{-bt} \phi(t)\, dt. \tag{3.1.5}$$

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[11] See Pólya and Szegö, *Aufgabe aus der Analysis*, I (problems **165, 251,** pp. 30, 133, 185, 314). The formal remark is probably to be found in the work of Heaviside.

These formulae are certainly true, for example, if $\phi(x)$ is integral and of type less than b; in this case the series are convergent.

The work of §2.2 suggests that (3.1.4) should be true under considerably more general conditions when the series is interpreted as a Borel sum. The ordinary Borel sum is

$$\int_0^\infty e^{-t} \sum_0^\infty \frac{(at)^r}{r!} \phi^{(r)}(x)\, dx = \int_0^\infty e^{-t}\phi(x + at)\, dt = b \int_0^\infty e^{-bu}\phi(x + u)\, du$$

if (i) $\phi(x)$ is an integral function and (ii) the integrals are convergent. The conditions that the series should be summable (B^2) are more general. Suppose, for example, that $a > 0$, $x > 0$, and that $\phi(\xi + i\eta)$ is regular for $\xi > 0$. Then the series

$$\sum \frac{(at)^r}{r!} \phi^{(r)}(x) \tag{3.1.6}$$

has a positive radius of convergence, and $x + at$ lies, for every positive t, inside its Borel polygon of summability, so that the series is summable (B) to $\phi(x + at)$. Hence (3.1.4) *is true, in the* (B^2) *sense, whenever* (i) $\phi(\xi + i\eta)$ *is regular for* $\xi > 0$ *and* (ii) *the integral is convergent.*

For example, we may take

$$\phi(x) = \frac{x^c}{\Gamma(c)},$$

where $c > -1$: if $c = 0$, $\phi(x) = 0$. We thus obtain the formula

$$\sum_{-\infty}^{\infty} a^r \frac{x^{c-r}}{\Gamma(c - r)} = a^c\, e^{x/a}. \tag{3.1.7}$$

Here it is understood that terms for which $c - r$ is 0 or a negative integer are interpreted as 0.

The formula (3.1.7) is proved, in the first instance, for $x > 0$, $a > 0$, $c > -1$. These conditions may be relaxed. In the first place, replacing c by $c + 1, c + 2, \cdots$, we may remove the restriction on c. We may also suppose c complex. Finally we may extend the formula to complex values of x and a. For here $\phi(\xi + i\eta)$ is regular except at the origin, and not merely in the right half-plane. Hence, if

$$|\arg x - \arg a| < \tfrac{1}{2}\pi,$$

or $\Re(x/a) > 0$, $x + at$ lies, for all positive t, within the Borel polygon of (3.1.6). Hence (3.1.7) is true whenever $\Re(x/a) > 0$.

When c is an integer, the series reduces to the ordinary exponential series. The formula itself seems to be due substantially to Heaviside.[12]

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[12] See H. Jeffreys, *Operational methods in mathematical physics*, Camb. Math. Tracts, No. 23, p. 91.

Ingham and Jeffreys[13] have shown that Heaviside's formula is true asymptotically, that is to say that

$$\sum_R^\infty \frac{x^{c-r}}{\Gamma(c-r)} = e^x + O(x^{c-R-1})$$

for large positive x. It is natural to ask whether (3.1.4) and (3.1.5) are true in the same sense. We take $a = 1$.

If x is positive, and $\phi(\xi + i\eta)$ regular for $\xi > 0$, then

$$\sum_0^R t^r \frac{\phi^{(r)}(x)}{r!} = \frac{1}{2\pi i} \int_C \frac{\phi(u)}{u - x} \sum_0^R \left(\frac{t}{u - x}\right)^r du,$$

where C is a circle, of radius less than x, round $u = x$. We may replace C by the circle γ whose centre is $x + t$ and whose radius is $\frac{1}{2}x + t$. Then

$$\sum_0^R t^r \frac{\phi^{(r)}(x)}{r!} = \frac{1}{2\pi i} \int_\gamma \frac{\phi(u)}{u - x - t} \left\{1 - \left(\frac{t}{u - x}\right)^{R+1}\right\} du$$

$$= \phi(x + t) - \frac{1}{2\pi i} \int_\gamma \frac{\phi(u)}{u - x - t} \left(\frac{t}{u - x}\right)^{R+1} du = \phi(x + t) - J,$$

say.

Suppose now that $\phi(u) = O(|u|^K)$, where $K > 0$, in the right hand half plane. Then, on γ, $|\phi(u)| < A(x^K + t^K)$, $|u - x - t| > Ax$, $|u - x| > Ax$, where the A are independent of u, x, t, R. Hence

$$|J| < A(x^K + t^K)\left(\frac{t}{x}\right)^{R+1} < A(x^{K-R-1} t^{R+1} + x^{-R-1} t^{K+R+1}),$$

$$\left|\int_0^\infty e^{-t} J\, dt\right| < A\{\Gamma(R + 2)\, x^{K-R-1} + \Gamma(K + R + 2)\, x^{-R-1}\}$$

$$< A\, \Gamma(K + R + 2)\, x^{K-R-1},$$

$$\sum_0^R \phi^{(r)}(x) = \int_0^\infty e^{-t} \sum_0^R t^r \frac{\phi^{(r)}(x)}{r!}\, dt = \int_0^\infty e^{-t} \phi(x + t)\, dt + O(x^{K-R-1}),$$

for large x. In this sense the series is asymptotic. When, as in Heaviside's formula, $\phi(x)$ is a power of x, the series is asymptotic in the ordinary sense, the error after the term of order x^{c-R} being of order x^{c-R-1}.

C. The W, V, and B methods of summation of Fourier series

4.1. We may say that $\sum a_n$ is summable (A, α), where $\alpha > 0$, to sum s, if $\sum a_n e^{-\delta n^\alpha}$ is convergent for all positive δ and tends to s when $\delta \to 0$. The

[13] Jeffreys, *l. c.*

particular case which is relevant here is the case $\alpha = 2$. In this case it is convenient to replace δ by $(\delta\pi)^2$, so that

(4.1.1) $$\sum a_n e^{-(n\delta\pi)^2} \to s .$$

In these circumstances I shall say, for reasons which will appear in a moment, that the series is summable (W).

A series may be summable (W) without being summable (A); thus

$$1 - x + x^2 - x^3 + \cdots$$

is summable (W), to sum $(1 + x)^{-1}$, for some $x > 1$. It is however easy to prove that *if* $\sum a_n$ *is summable* (W), *and* $\sum a_n e^{-ny}$ *is convergent for all positive* y, *then* $\sum a_n$ *is summable* (A).[14] In a word, the (W) method is more crudely powerful, but, for delicately divergent series, the ordinary Abel-Poisson method is always the more effective. In particular a Fourier series, if summable (W), is certainly summable (A).

It is not quite so easy to give an example of a Fourier series which is summable (A) and not summable (W). I have however shown[15] that the Dirichlet series

$$\sum n^{-s} e^{Ai(\log n)^2} \qquad (s > 0, A > 0)$$

is summable (A) if $s > 1 - A\pi$ and summable (W) if $s > 1 - \frac{1}{2}A\pi$, these inequalities being the best of their kind. Hence the series

$$\sum n^{-1+\delta} e^{Ai(\log n)^2} \cos nx ,$$

with

$$0 < \delta < \tfrac{1}{2},\ \tfrac{1}{2}A\pi < \delta < A\pi$$

is a Fourier series summable (A), but not summable (W), for $x = 0$.

If

$$\tfrac{1}{2}A_0 + \sum A_n(t) = \tfrac{1}{2}a_0 + \sum (a_n \cos nt + b_n \sin nt)$$

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[14] From

$$e^{-ny} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-n^2x^2y^2 - 1/x^2} \frac{dx}{x^2}$$

we deduce

$$\sum a_n e^{-ny} = \frac{2}{\sqrt{\pi}} \int_0^\infty f(xy) e^{-1/x^2} \frac{dx}{x^2},$$

where $f(u) = \Sigma a_n e^{-(nu)^2}$, and the conclusion follows from the continuity of $f(u)$ at the origin. Much more general theorems concerning the relations of the methods (A, α) and (A, β) have been proved by M. L. Cartwright, *Proc. London Math. Soc.* (2), 31 (1930), 81–96.

[15] G. H. Hardy, 'The application of Abel's method of summation to Dirichlet's series', *Quarterly Journal*, 47 (1916), 176–192 (192).

is the Fourier series of $f(t)$, then

$$A_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt,$$

$$\frac{1}{2}A_0 + \sum A_n(x)\,e^{-(n\delta\pi)^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,K(t-x)\,dt,$$

where

$$K(u) = \frac{1}{2} + \sum_1^{\infty} \cos nu\; e^{-(n\delta\pi)^2} = \frac{1}{2\delta\sqrt{\pi}}\sum_{-\infty}^{\infty} e^{-(2n\pi-u)^2/4\pi^2\delta^2},$$

by a familiar formula of elliptic functions. It is plain that, when we substitute the second series for $K(t-x)$ under the sign of integration, all the terms are trivial except that for which $n = 0$. Hence, *in order that the series be summable* (W) *for* $t = x$, *it is necessary and sufficient that*

$$\text{(4.1.2)} \qquad \frac{1}{2\delta\sqrt{\pi}}\int_{-\pi}^{\pi} f(t)\,e^{-(t-x)^2/4\pi^2\delta^2}\,dt \to s,$$

i.e. that Weierstrass's singular integral for $f(t)$ *should converge to* s.

The (W) method, like the (A) method, sums the series at any point of 'mean continuity'. Mean continuity is defined as follows. We write, as usual,

$$\phi(t) = \phi(t, x) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\},$$

with arbitrary s, and

$$\phi_1(t) = \int_0^t \phi(u)\,du,\quad \phi_2(t) = \int_0^t \phi_1(u)\,du,\ \cdots;$$

and we say that $f(t)$ has mean continuity at x, or tends to s in mean at x, if

$$\text{(4.1.3)} \qquad \phi_k(t) = o(t^k)$$

for some k.

The series will be summable (W) if

$$\text{(4.1.4)} \qquad \frac{1}{\delta}\int \phi(t)\,e^{-t^2/4\pi^2\delta^2}\,dt = o(1),$$

where the integration is over a small fixed interval including the origin; or, what is the same thing, if

$$\frac{1}{\delta}\int \phi_k(t)\left(\frac{d}{dt}\right)^k (e^{-t^2/4\pi^2\delta^2})\,dt = o(1).$$

Now

$$\left(\frac{d}{dt}\right)^k e^{-t^2/4\pi^2\delta^2} = \delta^{-k} Q_k\!\left(\frac{t}{\delta}\right) e^{-t^2/4\pi^2\delta^2},$$

where $Q_k(u)$ is a polynomial of degree k in u; and

$$\frac{1}{\delta}\int\left(\frac{t}{\delta}\right)^k Q_k\left(\frac{t}{\delta}\right)e^{-t^2/4\pi^2\delta^2}\,dt$$

is bounded in δ for every k. From this it follows in the ordinary way that (4.1.3) implies (4.1.4).

4.2. Another method of summation which has the property just proved for the (W) method is that of de la Vallée-Poussin. We say that $\sum a_n$ is summable (V) to s if

$$a_0+\frac{n}{n+1}a_1+\frac{n(n-1)}{(n+1)(n+2)}a_2+\cdots+\frac{n(n-1)\cdots 1}{(n+1)(n+2)\cdots 2n}a_n\to s\,.$$

Since

$$\frac{1}{2}+\frac{n}{n+1}\cos t+\frac{n(n-1)}{(n+1)(n+2)}\cos 2t+\cdots=2^{2n-1}\frac{(n!)^2}{2n!}\left(\cos\frac{1}{2}t\right)^{2n}$$

and

$$\frac{2^{2n-1}(n\,!)^2}{(2n)\,!}\sim\tfrac{1}{2}\sqrt{n\pi}\,,$$

a necessary and sufficient condition for the summability of the Fourier series to s is that

$$\frac{1}{2}\sqrt{\frac{n}{\pi}}\int_{-\pi}^{\pi}f(t)\left\{\cos\frac{1}{2}(t-x)\right\}^{2n}dt\to s\,,$$

or, what is the same thing, that

$$\text{(4.2.1)}\qquad \sqrt{n}\int\phi(t)\left(\cos\frac{1}{2}t\right)^{2n}dt=o(1)\,,$$

the integration being again over a small fixed interval including the origin.

The relations between the (W) and (V) methods are much closer than those between either of them and the Abel-Poisson method. In particular, *for Fourier series, the (W) and (V) methods are equivalent.*[16]

To prove this we have to show that the hypotheses (4.1.4) and (4.2.1) are equivalent. If we write $1/\nu$ for $\pi^2\delta^2$ in (4.1.4), it becomes

$$\text{(4.2.2)}\qquad \sqrt{\nu}\int\phi(t)\,e^{-\frac{1}{4}\nu t^2}\,dt=o(1)\,.$$

Here however ν tends to infinity continuously, whereas n is an integer in (4.2.1).

We therefore prove first that if

$$N<\nu=N+f<N+1\,,\qquad \pi^2\delta^2=\frac{1}{\nu}\,,\qquad \pi^2\delta_0^2=\frac{1}{N}\,,$$

[16] Mr. J. M. Hyslop has shown that the two methods are equivalent for all series in which $a_n=O(n^k)$ for some k.

and $a_n = o(1)$, then

(4.2.3) $$\sum a_n \{e^{-(n\delta\pi)^2} - e^{-(n\delta_0\pi)^2}\} \to 0$$

when $N \to \infty$, uniformly in f. This will show that, when $a_n = o(1)$, (4.1.1), if true for the special sequence (δ_0), is true generally, so that we may suppose ν an integer in (4.2.2). For this, it is enough to prove

$$\sum \{e^{-n^2/(N+1)} - e^{-n^2/N}\}$$

bounded; and this sum is less than

$$\sum \{e^{-n/(N+1)} - e^{-n/N}\} = \frac{1}{1 - e^{-1/(N+1)}} - \frac{1}{1 - e^{-1/N}}$$
$$= N + 1 + O(1) - N + O(1) = O(1).$$

We may therefore replace ν, in (4.2.2), by n.

We choose a value of α between $\frac{3}{8}$ and $\frac{1}{2}$. If $t > n^{-\alpha}$ then

$$e^{-\frac{1}{4}n t^2} < e^{-\frac{1}{4}n^{1-2\alpha}}$$

and[17]

$$\left(\cos\frac{1}{2}t\right)^{2n} = e^{2n \log \cos \frac{1}{2}t} < e^{4n \log\left(1 - \frac{1}{16}t^2\right)} < e^{-\frac{1}{4}n t^2} < e^{-\frac{1}{4}n^{1-2\alpha}}$$

Hence the parts of the integrals in which $t > n^{-\alpha}$ are negligible, and it is sufficient to prove that

$$J = \int_0^{n^{-\alpha}} |\phi(t)| \left\{\left(\cos\frac{1}{2}t\right)^{2n} - e^{-\frac{1}{4}n t^2}\right\} dt = o\left(n^{-\frac{1}{2}}\right).$$

Here

$$\left(\cos\frac{1}{2}t\right)^{2n} = e^{-\frac{1}{4}n t^2 + O(n t^4)},$$

$$\left(\cos\frac{1}{2}t\right)^{2n} - e^{-\frac{1}{4}n t^2} = O(nt^4),$$

and so

$$J = O\left\{n \int_0^{n^{-\alpha}} t^4 \,|\, \phi(t) \,|\, dt\right\} = O(n^{1-4\alpha}) = o\left(n^{-\frac{1}{2}}\right),$$

which proves the equivalence of the two methods.

4.3. I conclude by a few remarks concerning the summability of Fourier series by Borel's method. This method is much less useful than those which

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[17] $\cos t < 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 < \left(1 - \frac{1}{4}t^2\right)^2.$

we have been considering since §4.1, as has been shown by C. N. Moore[18] (and as will appear incidentally in the sequel) it is not always effective at a point of continuity. Littlewood and I have however shown[19] that Borel's and similar methods can be of service in the theory of convergence of Fourier series.

In what follows I suppose that $\phi(t)$ tends to 0 with t. Since a_n and b_n are $o(1)$, the various forms of Borel's definition are equivalent. I take the definition

$$s = \lim_{\xi\to\infty} e^{-\xi} \sum s_n \frac{\xi^n}{n!}.$$

Since

$$\sum \frac{\xi^n}{n!} \sin\left(n + \frac{1}{2}\right) t = e^{\xi \cos t} \sin\left(\xi \sin t + \frac{1}{2} t\right),$$

the series will be summable to s if and only if

$$\int \phi(t)\, e^{-\xi(1-\cos t)} \frac{\sin\left(\xi \sin t + \frac{1}{2} t\right)}{\sin \frac{1}{2} t}\, dt \to 0. \tag{4.3.1}$$

Here we may, on the usual grounds, omit the $\frac{1}{2}t$ in the argument of

$$\sin\left(\xi \sin t + \frac{1}{2} t\right),$$

and replace $\sin \frac{1}{2}t$ in the denominator by $\frac{1}{2}t$. The condition thus takes the form

$$\int \phi(t)\, e^{-\xi(1-\cos t)} \frac{\sin (\xi \sin t)}{t}\, dt \to 0. \tag{4.3.2}$$

The condition (4.3.2) may be simplified further. Since $1 - \cos t > \frac{1}{3}t^2$ for small t, $\phi(t) = o(1)$, and

$$\int_{\xi^{-\frac{1}{2}}}^{\infty} \frac{e^{-\frac{1}{3}\xi t^2}}{t}\, dt = \int_1^{\infty} \frac{e^{-\frac{1}{3}u^2}}{u}\, du$$

is bounded, we may ignore the part of the integral for which $t > \xi^{-\frac{1}{2}}$. I $t < \xi^{-\frac{1}{2}}$ then

$$e^{-\xi(1-\cos t)} \sin (\xi \sin t) - \sin \xi t = O(\xi t^2)$$

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[18] *Proc. Nat. Acad. of Sci.*, 11 (1925), 284–287. See also E. Hille and J. D. Tamarkin, *Math. Annalen*, 108 (1933), p. 557.

[19] 'Some new convergence criteria for Fourier series', *Annali d. R. Sc. di Pisa* (2), 3 (1934), 43–62.

and

$$\int_0^{\xi^{-\frac{1}{2}}} |\, \phi(t) \,|\, O(\xi t)\, dt = o\left(\xi \int_0^{\xi^{-\frac{1}{2}}} t\, dt\right) = o(1).$$

Hence *a necessary and sufficient condition for summability*[20] *is that*

$$(4.3.3) \qquad \int_0^{\xi^{-\frac{1}{2}}} \phi(t) \frac{\sin \xi t}{t}\, dt \to 0.$$

Suppose now that

$$(4.3.4) \qquad \phi(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\}.$$

Then

$$J(\xi) = \int_0^{\xi^{-\frac{1}{2}}} \phi(t) \frac{\sin \xi t}{t}\, dt = \int_0^{\xi^{-1}} + \int_{\xi^{-1}}^{\xi^{-\frac{1}{2}}} = J_1(\xi) + J_2(\xi).$$

Here

$$J_1(\xi) = o\left(\int_0^{\xi^{-1}} \xi\, dt\right) = o(1),$$

and

$$J_2(\xi) = o\left\{\int_{\xi^{-1}}^{\xi^{-\frac{1}{2}}} \frac{dt}{t \log(1/t)}\right\} = o(\log 2) = o(1).$$

Hence *the series is summable* (*B*), *to s, whenever* $\phi(t)$ *satisfies* (4.3.4). It is this result, in a different and generalised form, which Littlewood and I use in the paper just referred to. Combined with Tauberian theorems, it yields interesting criteria for ordinary convergence.

It is easy to deduce from (4.3.3) that Borel summation is not always effective at a point of continuity. The proofs that the Fourier series of a continuous function is not necessarily convergent depend, at bottom, on the fact that

$$\int_0^a \frac{\sin^2 \xi t}{t}\, dt$$

is not bounded. Since this is equally true of

$$\int_0^{\xi^{-\frac{1}{2}}} \frac{\sin^2 \xi t}{t}\, dt,$$

the same is true of Borel summability.

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[20] When $\phi(t) = o(1)$; in particular, when $f(t)$ is continuous at x.

CORRECTIONS

p. 171, *line* 4 *up*. For (C, k) read $(C, k+1)$.
p. 174, *lines* 17–20. Replace $\Gamma(c)$ by $\Gamma(c+1)$. Omit 'if $c = 0$, $\phi(x) = 0$'. Replace $\Gamma(c-r)$ by $\Gamma(c-r+1)$. Omit '0 or'.
p. 175, *line* 3. Replace $\sum\limits_{R}^{\infty}$ by $\sum\limits_{-\infty}^{R}$, and $\Gamma(c-r)$ by $\Gamma(c-r+1)$. Cf. 1945, 3, § 1.

COMMENTS

Section A

A further discussion of Fourier's proof of 'Fourier's theorem' is given in D.S., Ch. II. An English translation of Fourier's *Théorie de la chaleur*, by Freeman (1875), is reprinted by Dover Publications (1955).

The methods (B) and $(B, 2)$ are called (B') and $(B', 2)$ in D.S., p. 83. The (B^2) method is given in Borel (1st edn., p. 99, 2nd edn., p. 129); see also D.S., p. 346.

Section B

The series (3.1.7) is given here as an example of the series (3.1.1). In 1945, 3, Hardy observes that it is also a case of Riemann's form of Taylor's theorem:†

$$f(k+x) = \sum_{s=-\infty}^{\infty} \frac{x^{c+s} D^{c+s} f(k)}{\Gamma(c+s+1)}.$$

Put $k = 0$, $s = -r$, $f(x) = e^{x/a}$ and‡ $D^{\nu} e^{bx} = b^{\nu} e^{bx}$.

In 1945, 3, Hardy points out that the asymptotic formula, attributed to Ingham and Jefferies in § 3, had been given by Barnes§ in 1906, for complex x in an angle.

† For reference see 1945, 3 and the Comments on that paper.
‡ Cf. Liouville, *J. de l'École polytech.* 13, cah, 21 (1832), 1–69.
§ See the Addendum to 1945, 3.

NOTES ON THE THEORY OF SERIES (XX): ON LAMBERT SERIES

By G. H. Hardy *and* J. E. Littlewood.

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1. A series Σa_n may be said to be "Lambert summable", or "summable (L)", to sum l, if

$$g(y) = \sum_1^\infty \frac{nye^{-ny}}{1-e^{-ny}} a_n \to l \tag{1.1}$$

when $y \to 0$ through positive values. This method of summation is one of the most interesting of the less familiar methods because of its curious connections with the analytic theory of numbers†.

There are known theorems which state simple relations between Lambert summability and summability by more usual methods.

Theorem 1‡. *If Σa_n is summable (C, k), for any k, then it is summable (L) to the same sum.*

Theorem 2§. *If Σa_n is summable (L), then it is summable (A) to the same sum.*

In fact $(C) \prec (L) \prec (A)$; the Lambert method includes the Cesàro methods and is included in the Abel-Poisson method. The proof of Theorem 1 is straightforward‖; but that of Theorem 2 demands the "prime number theorem" or an equivalent. Both theorems are "Abelian" in character; they state relations of complete inclusion, without supplementary conditions.

† See Hardy and Littlewood (4); Wiener (10, 39; 11, 112).

‡ Hardy (2), Theorem 2.

§ Hardy and Littlewood (4).

‖ It is a corollary of general theorems of Bohr, Bromwich, and Hardy, concerning series $\Sigma a_n \chi_n(y)$.

2. It is natural to ask for "Tauberian" theorems corresponding to Theorem 2: in what circumstances does summability (A) imply summability (L)? The supplementary or "Tauberian" conditions may be expected to bear, not directly on the coefficients a_n, but upon the function

$$f(y) = \sum_1^\infty a_n e^{-ny}. \tag{2.1}$$

One such theorem has been proved by Ananda Rau†.

THEOREM 3. *If*

$$f(y) \to l \tag{2.2}$$

and

$$f'(y) = O\big(\phi(y)\big), \tag{2.3}$$

where $\phi(y)$ *is a positive decreasing function such that*

$$\int_0 \phi(y)\,dy < \infty, \tag{2.4}$$

then

$$g(y) \to l. \tag{2.5}$$

That is to say, summability (A) implies summability (L) under the supplementary condition (2.3). We shall give a new proof of this theorem in §3, and see later (§8) that it is a "best possible" theorem of its kind. There are, however, other theorems (neither including Theorem 3 nor included by it) in which the supplementary condition has a simpler form.

3. THEOREM 4. *If* $f(y) \to l$ *and*

$$f''(y) > -\frac{C}{y^2}, \tag{3.1}$$

then $g(y) \to l$‡.

We may suppose $l = 0$. Then, since

$$g(y) = \sum_{m=1}^\infty \frac{mye^{-my}}{1-e^{-my}} a_m = y \sum_{m=1}^\infty \sum_{n=1}^\infty ma_m e^{-mny}$$

$$= y \sum_{n=1}^\infty \sum_{m=1}^\infty ma_m e^{-mny} = -y \sum_{n=1}^\infty f'(ny),$$

† Ananda Rau (1), Theorem 2.2.

‡ Here, and elsewhere, C is a positive number independent of y and (when n occurs) of n.

(2.5) is equivalent to

$$(3.2)\qquad h(y) = -\frac{g(y)}{y} = \sum_1^\infty f'(ny) = o\left(\frac{1}{y}\right).$$

Now $$f(y) = -\int_y^\infty f'(t)\,dt = -\sum_{n=1}^\infty \int_{ny}^{(n+1)y} f'(t)\,dt.$$

Hence

$$(3.3)\qquad f(y)-g(y) = f(y)+yh(y) = \sum_{n=1}^\infty u_n(y) = S(y),$$

where

$$(3.4)\qquad u_n(y) = \int_{ny}^{(n+1)y} \Big(f'(ny)-f'(t)\Big)\,dt.$$

Similarly,

$$(3.5)\quad f(y)-yf'(y)-g(y) = f(y)-yf'(y)+yh(y) = \sum_{n=1}^\infty v_n(y) = T(y),$$

where

$$(3.6)\qquad v_n(y) = \int_{ny}^{(n+1)y} \Big(f'\big((n+1)y\big)-f'(t)\Big)\,dt.$$

We use both (3.3) and (3.5) in proving Theorem 4.

The conditions of Theorem 4 imply that†

$$(3.7)\qquad f'(y) = o\left(\frac{1}{y}\right),$$

so that each term of $S(y)$ or $T(y)$ tends to zero. If the conditions of Theorem 3 are satisfied, then $S(y)$ is majorized by

$$y\sum\phi(ny) < y\int_0^\infty \phi(ty)\,dt < C,$$

so that $S(y)\to 0$ and therefore $g(y)\to 0$. We thus obtain an immediate proof of Ananda Rau's theorem.

Returning to Theorem 4, we write

$$S(y) = \sum_1^\infty u_n(y) = \sum_1^N u_n(y) + \sum_{N+1}^\infty u_n(y) = S_1(y)+S_2(y).$$

In $S_2(y)$ we have

$$f'(ny)-f'(t) = -(t-ny)f''(\tau) \quad (ny<\tau<t),$$

and so $$u_n(y) \leqslant y\,.\,y\,.\,\frac{C}{n^2y^2} = \frac{C}{n^2}.$$

† Hardy and Littlewood (3), or Landau (5, 58).

Hence
$$S_2(y) \leqslant C \sum_N^\infty \frac{1}{n^2} < \epsilon$$

for $N = N(\epsilon)$. Also $S_1(y) \to 0$ when N is fixed. It follows that

$$\overline{\lim}\, S(y) \leqslant 0,$$

and therefore, from (3.3), that

$$\overline{\lim}\, yh(y) \leqslant 0. \tag{3.8}$$

Similarly we prove that

$$\underline{\lim}\, T(y) \geqslant 0,$$

and therefore, from (3.5) and (3.7), that

$$\underline{\lim}\, yh(y) \geqslant 0. \tag{3.9}$$

Finally, (3.8) and (3.9) show that $yh(y) \to 0$.

4. There is another theorem of some interest which neither includes nor is included by Theorem 4.

We write

$$f^\alpha(y) = f_{-\alpha}(y) = \Sigma\, n^\alpha a_n e^{-ny}, \tag{4.1}$$

for all real α. When α is a positive integer, $f^\alpha(y)$ is, apart from sign, the α-th derivative of $f(y)$. Also

$$f_\beta(y) = \frac{1}{\Gamma(\beta)} \int_y^\infty (t-y)^{\beta-1} f(t)\, dt \tag{4.2}$$

when $\beta > 0$, and

$$f^{\alpha-\gamma}(y) = \frac{1}{\Gamma(\gamma)} \int_y^\infty (t-y)^{\gamma-1} f^\alpha(t)\, dt \tag{4.3}$$

when $\gamma > 0$. Our formulae embody the natural definitions of the derivatives of arbitrary order.

THEOREM 5. *If $f(y) \to l$ and*

$$f^{1+\delta}(y) = O(y^{-1-\delta}) \tag{4.4}$$

for some positive δ, then $g(y) \to l$.

It will be observed that (4.4) is not a "one-sided" condition, as (3.1) is; we return to this point in a moment.

We use the identity (3.3). The hypotheses again involve (3.7)†, and $u_n(y) \to 0$ for every n.

Next, by (4.3),

$$-f'(t) = f^1(t) = \frac{1}{\Gamma(\delta)} \int_t^\infty (u-t)^{\delta-1} f^{1+\delta}(u)\, du,$$

and so

$$f'(ny) - f'(t) = -\frac{1}{\Gamma(\delta)} \int_{ny}^t (u-ny)^{\delta-1} f^{1+\delta}(u)\, du$$

$$+\frac{1}{\Gamma(\delta)} \int_t^\infty \left((u-t)^{\delta-1} - (u-ny)^{\delta-1} \right) f^{1+\delta}(u)\, du = J_1 + J_2,$$

say. Here

$$J_1 = O\left((ny)^{-1-\delta} \int_{ny}^{(n+1)y} (u-ny)^{\delta-1} du \right) = O\left((ny)^{-1-\delta} y^\delta \right) = O(n^{-1-\delta} y^{-1}),$$

uniformly for $ny < t < (n+1)y$ and all n; and

$$J_2 = O\left((ny)^{-1-\delta} \int_t^\infty \left((u-t)^{\delta-1} - (u-ny)^{\delta-1} \right) du \right)$$

$$= O\left((ny)^{-1-\delta} (t-ny)^\delta \right) = O(n^{-1-\delta} y^{-1}).$$

Hence $$|u_n(y)| < Cn^{-1-\delta},$$

and $S(y)$ is dominated by $C \Sigma n^{-1-\delta}$. It follows that $S(y) \to 0$.

5. It might be thought that there should be a more general theorem with

(5.1) $$f^{1+\delta}(y) > -Cy^{-1-\delta}$$

in place of (3.1) or (4.4); such a theorem would include both Theorem 4 and Theorem 5. It seems that this is not so; that (5.1) is not a sufficient condition unless $\delta \geqslant 1$; and that each of Theorems 4 and 5 is in a sense "best possible". We do not prove this formally, but the remarks which follow will probably persuade the reader.

There are well-known relations between the orders of magnitude of the derivatives of $f(y)$. Let us suppose for the moment that $f(y)$ is a general

† See M. Riesz (7). Riesz works with a different definition of the fractional derivatives, but the difference is of no importance here.

function of the real variable y†. Then

(5.2) $$f(y) \to l$$

and

(5.3) $$f^{\alpha}(y) = o(y^{-\alpha})$$

imply

(5.4) $$f^{\beta}(y) = o(y^{-\beta})$$

for $0 < \beta < \alpha$‡. If α and β are integers, (5.3) may be replaced by

(5.5) $$f^{\alpha}(y) > -Cy^{-\alpha}§,$$

but this is not true without reservation for general α and β; (5.2) and (5.5) imply (5.4) when $0 < \beta \leqslant \alpha - 1$, but not necessarily when

$$\alpha - 1 < \beta < \alpha.$$

Suppose, for example, that $\alpha = 2$ and $C = 0$, so that (5.5) is

(5.6) $$f''(y) > 0.$$

Then (5.4) is true for $0 < \beta < 1$, but $f^{\beta}(y)$ may have infinities, when $\beta > 1$, in every neighbourhood of the origin; in this case $f''(y)$ will also have infinities, but this is consistent with (5.6). Thus (5.2) and (5.6) do not imply

$$f^{\beta}(y) = O\big(\phi(y)\big)$$

for any monotonic ϕ, however rapidly ϕ may tend to infinity, when $\beta > 1$.

Here we are concerned with power series (so that infinities are impossible), but the limitations on theorems of this kind are effectively the same. Thus

$$f(y) = \Sigma a_n e^{-ny} \to l, \quad f''(y) = \Sigma n^2 a_n e^{-ny} > 0$$

† In this case our definitions of the derivatives must be framed differently. We must define $f_{\beta}(y)$, for $\beta > 0$, by (4.2), and

$$f^{k-\beta}(y) = f_{\beta-k}(y) \quad (k = 1, 2, \ldots)$$

by

$$f_{\beta-k}(y) = (-1)^k \left(\frac{d}{dy}\right)^k f_{\beta}(y).$$

The functions are no longer power-series, and the question of "consistency" requires attention, but the general situation is much the same.

‡ See Hardy and Littlewood (3), M. Riesz (7). Theorems of this type are by now familiar.

§ See Hardy and Littlewood (3).

do not imply

$$f^{\beta}(y) = \Sigma n^{\beta} a_n e^{-ny} = o(y^{-\beta})$$

when $\beta > 1$. Hence Theorem 4 is not (as we might have expected it to be) a corollary of Theorem 5. And (5.1), for some positive δ, does not, even when $C = 0$, imply (4.4) for all smaller positive δ, so that we cannot replace (4.4) by (5.1), in Theorem 5, in the way which is naturally suggested†.

Negative results.

6. We shall not attempt to prove formally that Theorems 3, 4, 5 have all a best possible character (though this is the natural lesson of our analysis). But it is desirable to show by definite examples that one particular condition, viz.

$$f(y) = o\left(\frac{1}{y}\right), \tag{6.1}$$

is not a sufficient substitute for (2.3), (3.1), or (4.4); and this will lead us to the study of some special series which have an independent interest.

The proposition to be negatived is

$$f(y) \to l \,.\, f'(y) = o\left(\frac{1}{y}\right) . \twoheadrightarrow g(y) \to l‡. \tag{A}$$

With this we may associate two other propositions, viz.

$$f(y) = O(1) \,.\, f'(y) = O\left(\frac{1}{y}\right) . \twoheadrightarrow g(y) = O(1); \tag{A_1}$$

and

$$|f(y)| < C \,.\, |f'(y)| < \frac{D}{y} . \twoheadrightarrow |g(y)| < E, \tag{A_2}$$

where $E = E(C, D)$. We may call (A_1) the "O form" and (A_2) the "inequality form" of (A). The three propositions stand or fall together; if we can prove (A_2) false, for example, we can deduce the falsity of (A)

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† It may be shown that (5.2) and (5.5) imply

$$\underline{\lim}\, y^{\beta} f^{\beta}(y) \geqslant 0$$

when $0 < \alpha - 1 < \beta < \alpha$.

The significance of the difference 1 between α and β becomes clearer if we suppose that $f^{\alpha}(y) > 0$ and observe that

$$f^{\beta}(y) = \frac{1}{\Gamma(\alpha-\beta)} \int_y^{\infty} (t-y)^{\alpha-\beta-1} f^{\alpha}(t)\, dt$$

increases (when y decreases) if $\alpha - \beta \geqslant 1$, but not necessarily if $\alpha - \beta < 1$.

‡ $\twoheadrightarrow$ is Hilbert's symbol for implication.

by a familiar, though possibly tiresome, routine†. In what follows we shall consider (A_1) or (A_2), and leave the transition to (A) to the reader's imagination.

We have two ways of proving (A_1) or (A_2) false. The first depends, at bottom, on the facts that

$$h(y) = \Sigma f'(ny) = y^{-s}\zeta(s)$$

when

$$f(y) = \frac{y^{1-s}}{1-s}$$

and that $\zeta(s)$ is not bounded on the line $s = 1+it$. This is the most natural line of proof, but there are complications because (i) y^{-s} is not a power series in e^{-y}, and (ii) Σn^{-s} is not convergent when $s = 1+it$.

We take

$$(6.2) \qquad f(y) = -\frac{(1-e^{-y})^{-ia}}{ia}, \qquad f'(y) = e^{-y}(1-e^{-y})^{-1-ia},$$

where

$$a = [Y^{-\frac{1}{2}}]$$

and Y is small, so that $a > 1$. This $f(y)$ obviously satisfies the hypotheses of (A_2), with $C = D = 1$. We consider its behaviour for $y = Y$.

We have

$$(6.3) \qquad -g(Y) = Yh(Y) = Y\Sigma f'(nY) = Y\Sigma e^{-nY}(1-e^{-nY})^{-1-ia}$$
$$= Y\sum_{n\leqslant a} + Y\sum_{a<n\leqslant 1/Y} + Y\sum_{n>1/Y} = S_1+S_2+S_3,$$

say.

In S_1, $nY = \eta$ is small, and

$$e^{-\eta} = 1+O(\eta), \quad 1-e^{-\eta} = \eta+O(\eta^2), \quad (1-e^{-\eta})^{-1-ia} = \eta^{-1-ia}+O(a).$$

Hence

$$(6.4) \qquad S_1 = Y\sum_{n\leqslant a}\Big((nY)^{-1-ia}+O(a)\Big) = Y^{-ia}\sum_{n\leqslant a} n^{-1-ia}+O(Ya^2)$$
$$= Y^{-ia}\Big(\zeta(1+ia)+O(1)\Big)+O(1)\ddagger = Y^{-ia}\zeta(1+ia)+O(1).$$

† Compare, for example, the three false propositions:

(a) if $f(x) \to l$ when $x \to 0$, then

$$J_n = \frac{2}{\pi}\int_0^\delta f(x)\,\frac{\sin nx}{x}\,dx \to l;$$

(a_1) if $f(x)$ is bounded, then J_n is bounded;

(a_2) if $|f(x)| < C$, then $|J_n| < E(C)$.

‡ It is familiar that $\zeta(1+it) = \sum_{n\leqslant t} n^{-1-it}+O(1)$. See, for example, Titchmarsh (8), 13.

Next

$$(6.5)\quad S_2 = \Sigma\, Y e^{-nY}(1-e^{-Y})^{-1-ia} = \Sigma\, u_n(Y) + \Sigma\, v_n(Y) = S_2' + S_2'',$$

where
$$u_n(Y) = \int_{nY}^{(n+1)Y} e^{-t}(1-e^{-t})^{-1-ia}\, dt,$$

$$v_n(Y) = \int_{nY}^{(n+1)Y} \left(e^{-nY}(1-e^{-nY})^{-1-ia} - e^{-t}(1-e^{-t})^{-1-ia}\right) dt,$$

and the summations are over $a < n \leqslant 1/Y$. Here

$$(6.6)\quad S_2' = \int_\tau^T e^{-t}(1-e^{-t})^{-1-ia}\, dt = \left[-\frac{1}{ia}(1-e^{-t})^{-ia}\right]_\tau^T = O\left(\frac{1}{a}\right) = O(1)$$

(the particular values of τ and T being irrelevant). The integrand in $v_n(Y)$ is

$$-\int_{nY}^{t} \frac{d}{du}\left(e^{-u}(1-e^{-u})^{-1-ia}\right) du = O\left(Y\,.\,\frac{1}{nY}\right) + O\left(Y\,.\,\frac{a}{n^2 Y^2}\right) = O\left(\frac{a}{n^2 Y}\right),$$

uniformly in n, and so

$$(6.7)\quad S_2'' = O\left(aY \sum_{n>a} \frac{1}{n^2 Y}\right) = O\left(a \sum_{n>a} \frac{1}{n^2}\right) = O(1).$$

Finally, it is obvious that

$$(6.8)\quad S_3 = O(1);$$

and (6.3)–(6.8) give

$$(6.9)\quad -g(Y) = Y^{-ia}\zeta(1+ia) + O(1).$$

Since $\zeta(1+ia)$ is not bounded for large a, or small Y, this contradicts (A_2).

7. There is an alternative method which leads to more interesting analysis. We take as our goal the refutation of (A_1). Suppose that

$$(7.1)\quad f'(y) = \operatorname{cosech} y \cos(a \operatorname{cosech} y + \beta) = \frac{2e^{-y}}{1-e^{-2y}} \cos\left(\frac{2ae^{-y}}{1-e^{-2y}} + \beta\right).$$

Then $f(y)$ tends to a limit† and $yf'(y)$ is bounded. We wish to show that $g(y)$ is not bounded, and it is plainly enough to prove that

$$(7.2)\quad S = \sum_{n<1/y} f'(ny) \neq O\left(\frac{1}{y}\right).$$

† Since
$$\int_0 f'(y)\, dy = \lim_{\epsilon\to 0} \int_\epsilon f'(y)\, dy$$
exists.

If we replace the first factor cosech ny in $f'(ny)$ by $1/ny$, the error introduced into a term of S is $O(ny)$, and that in S is

$$O\Big(y \sum_{n<1/y} n\Big) = O\Big(\frac{1}{y}\Big).$$

Also

$$\cos\Big(\frac{a}{\sinh ny}+\beta\Big)-\cos\Big(\frac{a}{ny}+\beta\Big) = O(1)\sin\big(O(ny)\big) = O(ny),$$

and so

$$\sum_{n<1/y}\frac{1}{ny}\Big(\cos\Big(\frac{a}{\sinh ny}+\beta\Big)-\cos\Big(\frac{a}{ny}+\beta\Big)\Big) = \sum_{n<1/y} O(1) = O\Big(\frac{1}{y}\Big).$$

Hence (7.2) is equivalent to

$$T = \sum_{n<1/y}\frac{1}{n}\cos\Big(\frac{a}{ny}+\beta\Big) \neq O(1), \tag{7.3}$$

and it is sufficient for our purpose to prove that *the functions*

$$P(x) = \sum_{n<x}\frac{1}{n}\cos\frac{x}{n}, \qquad Q(x) = \sum_{n<x}\frac{1}{n}\sin\frac{x}{n} \tag{7.4}$$

are not bounded for large x†.

8. THEOREM 6. $\quad P(x) = \Omega(\log\log x)$‡.

If c is fixed and

$$c\sqrt{x} \leqslant \mu < \nu \leqslant x,$$

then

$$\sum_{\mu}^{\nu}\frac{1}{n}\cos\frac{x}{n} - \int_{\mu}^{\nu+1}\frac{1}{t}\cos\frac{x}{t}\,dt$$

$$= \sum_{\mu}^{\nu}\int_{n}^{n+1}\Big(\frac{1}{n}\cos\frac{x}{n}-\frac{1}{t}\cos\frac{x}{t}\Big)\,dt$$

$$= \sum_{\mu}^{\nu}\Big(O\Big(\frac{1}{n^2}\Big)+O\Big(\frac{x}{n^3}\Big)\Big) = O(1)+O\Big(x\sum_{n>c\sqrt{x}}\frac{1}{n^3}\Big) = O(1).$$

† Either function would suffice; but there are interesting differences between them, and we discuss both. Since

$$\sum_{n\geqslant x}\frac{1}{n}\sin\frac{x}{n} = O\Big(x\sum_{n\geqslant x}\frac{1}{n^2}\Big) = O(1),$$

the second series may equally well be extended to infinity.

‡ $|P| > C\log\log x$ for appropriate large x.

Hence

$$P(x) = \sum_{n<\mu} \frac{1}{n} \cos \frac{x}{n} + O(1) = P^*(x) + O(1), \tag{8.1}$$

the O being uniform in μ for $c\sqrt{x} \leqslant \mu \leqslant x$.

We suppose k to be a large positive integer, and take

$$x = 2\pi k! = 2\pi K, \quad x_j = jx \quad (j = 1, 2, \ldots, K), \quad \mu = K.$$

Then μ lies between $\sqrt{(x_j/2\pi)}$ and x_j for all j, and (8.1) is true when $\mu = K$ and $x = x_j$, uniformly in j.

Now

$$P^*(x_j) = \sum_{n=1}^{k} \frac{1}{n} + \sum_{n=k+1}^{K} \frac{1}{n} \cos \frac{x_j}{n} > \log k + \sum_{n=k+1}^{K} \frac{1}{n} \cos \frac{x_j}{n},$$

and so

$$\frac{1}{K} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) P^*(x_j) > \frac{K-1}{2K} \log k + \frac{1}{K} \sum_{n=k+1}^{K} \frac{1}{n} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) \cos \frac{jx_1}{n}.$$

But

$$\tfrac{1}{2} + \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) \cos j\theta$$

is the first Cesàro mean of the series $\frac{1}{2} + \cos\theta + \cos 2\theta + \ldots$, and is therefore non-negative for all θ; and so

$$\frac{1}{K} \sum_{j=1}^{K} \left(1 - \frac{j}{K}\right) P^*(x_j) > \frac{K-1}{2K} \log k - \frac{1}{2K} \sum_{1}^{K} \frac{1}{n} > (\tfrac{1}{2} - \epsilon) \log k$$

for every ϵ and sufficiently large k. It follows that

$$P^*(x_j) > (1 - \epsilon) \log k$$

for every ϵ, sufficiently large k, and some j. The same is true of $P(x_j)$, by (8.1). Finally,

$$x_j < 2\pi K^2 < 2\pi (k!)^2, \quad \log x_j < Ck \log k, \quad \log\log x_j < C \log k,$$

and the theorem follows.

It is plain that the same argument would prove that

$$\sum_{n<x} \frac{1}{n \log n} \cos \frac{x}{n} = \Omega(\log\log\log x),$$

and that it could be developed so as to prove Ananda Rau's theorem the best possible of its kind†.

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† As indeed, no doubt, could tha 6.

9. THEOREM 7. $Q(x) = \Omega\left((\log\log x)^{\frac{1}{2}}\right)$.

We denote by q a number of the form Πp^a, where the p are the primes $4m+1$, and take

$$K = \prod_{q \leqslant 4k+1} q,$$

$$x = \tfrac{1}{2}\pi K, \quad x_j = (4j+1)x \quad (j = 1, 2, \ldots, K).$$

As before, it is sufficient to consider

$$Q^*(x) = \sum_{n=1}^{K} \frac{1}{n} \sin\frac{x}{n}$$

for the values x_j.

If $n \mid K$, then K/n is an integer $4m+1$, and $\sin(x_j/n) = 1$. Hence

$$Q^*(x_j) = \sum_{n \mid K} \frac{1}{n} + R(x_j) = \lambda(k) + R(x_j),$$

say, where
$$R(x) = \Sigma' \frac{1}{n} \sin\frac{x}{n},$$

Σ' denoting a sum over n which do not exceed and do not divide K; and†

$$\frac{1}{K}\sum_{j=1}^{K} Q^*(x_j) = \lambda(k) + \frac{1}{K}\Sigma'_n \frac{1}{n}\sum_{j=1}^{K} \sin\frac{(4j+1)x}{n}. \tag{9.1}$$

In the inner sum on the right of (9.1), $2x/n$ differs from the nearest multiple of π by at least C/n, so that

$$\sum_{j=1}^{K} \sin\frac{(4j+1)x}{n} = O\left(\left|\operatorname{cosec}\frac{2x}{n}\right|\right) = O(n),$$

and the repeated sum is

$$O\left(\frac{1}{K}\sum_{n=1}^{K}\frac{1}{n}\,.\,n\right) = O(1).$$

Hence
$$\frac{1}{K}\sum_{j=1}^{K} Q^*(x_j) > (1-\epsilon)\lambda(k)$$

and
$$Q^*(x_j) > (1-\epsilon)\lambda(k),$$

for some j, in the sense of § 8.

† There is no advantage this time in taking a more elaborate mean.

Finally, it is plain that

$$\lambda(k) \geqslant \sum_{q \leqslant 4k+1} \frac{1}{q}.$$

If $N(n)$ is the number of q which do not exceed n, then

$$\sum_{q \leqslant 4k+1} \frac{1}{q} = \sum_{n \leqslant 4k+1} \frac{N(n)-N(n-1)}{n} \geqslant \sum_{n \leqslant 4k+1} \frac{N(n)}{n(n+1)}.$$

But†

$$N(n) > \frac{Cn}{(\log n)^{\frac{1}{2}}},$$

and so

$$Q^*(x_j) = \Omega\left((\log k)^{\frac{1}{2}}\right) = \Omega\left((\log \log x_j)^{\frac{1}{2}}\right).$$

We cannot prove that $Q(x)$ is ever as large as $\log \log x$, though this is true both of $P(x)$ and of

$$Q_1(x) = \sum \frac{(-1)^n}{2n+1} \sin \frac{x}{2n+1}.$$

The "O-problems" for all these sums are much like the corresponding problems for $\zeta(1+it)$.

The series are also connected by identities with series of a quite different type. Thus

$$f(z) = \sum_1^\infty \frac{1}{n}(1-e^{-z/n}) = 2\log z + 2\gamma$$
$$-2\sum_1^\infty \left\{K_0\left(\sqrt{(2n\pi iz)}\right) + K_0\left(\sqrt{(-2n\pi iz)}\right)\right\},$$

if $\mathfrak{R}(z) > 0$; here γ is Euler's constant and K_0 is "Basset's function"‡. Such identities lead to interesting translations of our theorems. Suppose, for example, that

$$f(s) = \sum_1^\infty n^{-\frac{1}{2}} e^{-s\sqrt{n}}$$

if $s = \sigma + it$, $\sigma > 0$, and $f(it)$ is the limit of $f(s)$ when $\sigma \to 0$. Then

$$f(it) = \Omega(|t|^{\frac{1}{2}} \log \log |t|).$$

That $f(it) = O(|t|^{\frac{1}{2}+\epsilon})$, for every positive ϵ, has been proved by Dr. Eric Phillips, but the proof has not been published.

† Landau (6).
‡ Watson (9, 77).

References.

1. K. Ananda Rau, " On Lambert's series ", *Proc. London Math. Soc.* (2), 19 (1921), 1–20.
2. G. H. Hardy, " Note on Lambert's series ", *ibid.*, 13 (1914), 192–198.
3. G. H. Hardy and J. E. Littlewood, " Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive ", *ibid.*, 13 (1914), 174–191.
4. ———, " On a Tauberian theorem for Lambert's series, and some fundamental theorems in the analytic theory of numbers ", *ibid.*, 19 (1921), 21–29.
5. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2 Auflage (Berlin, 1929).
6. ———, " Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate ", *Archiv. der Math.* (3), 13 (1908), 305–312.
7. M. Riesz, " Sur un théorème de la moyenne et ses applications ", *Acta Univ. Hungaricae Szeged,* 1 (1923), 114–126.
8. E. C. Titchmarsh, *The zeta-function of Riemann*, Cambridge Math. Tracts, 26 (1930).
9. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922).
10. N. Wiener, *The Fourier integral and certain of its applications* (Cambridge, 1933).
11. ———, " Tauberian theorems ", *Annals of Math.*, 33 (1932), 1–100.

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COMMENTS

Theorems 1 and 2 are proved in 1914, 5 and 1921, 6 (in vol. II) respectively; see also D.S., Appendix IV.

The hypotheses in Theorem 3 imply that $\phi(y) = o(1/y)$ as $y \to 0$, and hence that (3.7) holds. They also imply, since $f'(y)$ is continuous for $y > 0$ and $O(e^{-y})$ as $y \to \infty$, that $f(y) \in BV(0, \infty)$, i.e. that $\sum a_n$ is *absolutely* summable (A).

The definition (4.1), for the fractional derivatives and integrals of a sum of negative powers of e^y, and also the formula (4.2), are due to Liouville;† (4.2) is also used in 1914, 4, § 6.

In the proof that the hypotheses of Theorem 5 imply (3.7), the functions f and $-f'$ may be expressed as fractional integrals of $f^{1+\delta}$, by (4.2). Then it is enough to prove the two similar propositions (where $y \to 0+$): (i) $F_{1+\delta} \to L$ and $F_\delta = O(y^{-1})$ imply $F_1 = o(y^{-\delta})$, (ii) $F_1 = o(y^{-\delta})$ and $F = O(y^{-1-\delta})$ imply $F_\delta = o(y^{-1})$. Since the integrands are exponentially small at infinity, Riesz's analysis for integrals with a finite base may be adapted.

† *J. de l'École polytech.* 13, cah. 21 (1832), 1–67.

NOTE ON A DIVERGENT SERIES

By G. H. HARDY

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1. The series
$$\sum_0^\infty n!\,z^n \tag{1·1}$$
is the simplest and most familiar power series whose radius of convergence is zero. It is natural to regard it as a development of the function $G(z)$ defined, when
$$z = x+iy = re^{i\theta} \quad (0<\theta<2\pi), \tag{1·2}$$
by
$$G(z) = \int_0^\infty \frac{e^{-t}}{1-zt}\,dt. \tag{1·3}$$
For
$$G(z) = \int_0^\infty e^{-t}\{1+zt+\ldots+(zt)^n\}\,dt + \int_0^\infty e^{-t}\frac{(zt)^{n+1}}{1-zt}\,dt$$
$$= 1+1!\,z+\ldots+n!\,z^n+R_n(z),$$
say; and
$$|R_n(z)| \leqslant r^{n+1}\int_0^\infty e^{-t}t^{n+1}\frac{dt}{|1-zt|} \leqslant \frac{(n+1)!}{|\sin\theta|}\,r^{n+1}$$
or
$$|R_n(z)| \leqslant (n+1)!\,r^{n+1},$$
according as x is positive or negative*. Thus the series (1·1) is an asymptotic series for $G(z)$ in the sense of Poincaré.

It is easy to verify by standard methods that the origin is the only finite singularity of $G(z)$, and that
$$G(z) = \frac{1}{z}e^{-1/z}\log\left(-\frac{1}{z}\right)+H(z),$$
where $H(z)$ is regular at infinity. Thus $G(z)$ has an infinity of branches differing by multiples of
$$\frac{2\pi i}{z}e^{-1/z}.$$
There is one branch which is positive on the negative real axis and tends to 1 when $z\to 0$ along the axis. We call this branch, which is that represented asymptotically by the series, the principal branch; and, by $G(z)$, we shall mean the principal branch of $G(z)$.

* $|1-zt|^2 = 1-2rt\cos\theta+r^2t^2$ has the minimum $\sin^2\theta$ or 1, according as $\cos\theta$ is positive or negative.

2. We denote by P the region $0 < \theta < 2\pi$ (the plane cut along the positive real axis), and by $P(R, \eta)$ the closed region

$$0 \leqslant r \leqslant R, \quad 0 < \eta \leqslant \theta \leqslant 2\pi - \eta < 2\pi, \tag{2·1}$$

where $\eta < \pi$. The function $G(z)$ is regular in P, and it is natural to suppose that there should be some method of summation which sums the series (1·1), to the sum $G(z)$, throughout P and uniformly in any $P(R, \eta)$. But the more obvious methods of summation are not so effective as this. Thus we might try to sum the series by a double application of Borel's method. If we apply Borel's method, in its usual form, to (1·1), we obtain

$$\int_0^\infty e^{-t}\, \Sigma (zt)^n\, dt. \tag{2·2}$$

The series is convergent only if $rt < 1$, but we may sum it in its turn by Borel's method whenever

$$\Re(zt) = rt \cos\theta < 1, \tag{2·3}$$

thus obtaining the sum $G(z)$. But (2·3) is satisfied, for all positive t, only when $r\cos\theta \leqslant 0$, so that the method is effective only in the half-plane $x \leqslant 0$.

The method of summation expressed by the formulae

$$c_n = \int_0^\infty \phi(w)\, w^n\, dw,$$

$$\Sigma a_n z^n = \int_0^\infty \phi(w)\, \Sigma \frac{a_n}{c_n} (wz)^n\, dw,$$

is a generalization of Borel's method, to which it reduces when $\phi(w) = e^{-w}$, $c_n = n!$. It is natural to try to sum (1·1) by taking $c_n = (n!)^2$, in which case

$$\phi(w) = 2K_0(2w^{\frac{1}{2}})$$

(in the notation of Watson's *Bessel functions*). It will be found, however, that this method also succeeds only when $x \leqslant 0$.

3. It is well known that the "Lindelöf" method of summation, defined by

$$\Sigma a_n z^n = \lim_{\delta \to 0} \Sigma e^{-\delta n \log n}\, a_n z^n,$$

sums the geometric series Σz^n throughout its Mittag-Leffler star, and this suggests another method of summation which will be found to sum (1·1) throughout P.

We write*

$$\lambda_0 = \lambda_1 = \lambda_2 = 0, \quad \lambda_n = n \log n \log\log n \quad (n \geqslant 3), \tag{3·1}$$

* The point is to choose λ_n so that

$$\Sigma e^{-\delta\lambda_n}\, n!\, z^n$$

is an integral function, for every positive δ, but with "little to spare".

and say that Σc_n is *summable* (A), to sum s, if

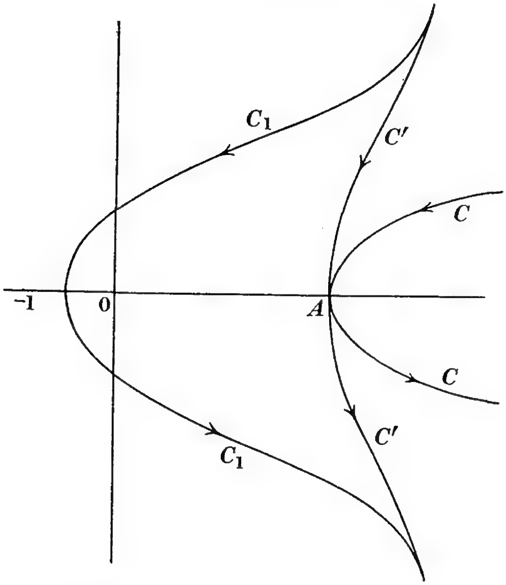
$$\lim_{\delta \to 0} \Sigma e^{-\delta \lambda_n} c_n = s. \tag{3·2}$$

THEOREM 1. *The series* (1·1) *is summable* (A), *to sum* $G(z)$, *throughout* P (*i.e. for all* z *except positive* z), *and uniformly summable in any* $P(R, \eta)$.

We suppose that z lies in $P(R, \eta)$; that

$$0 < 2\pi\delta < \eta; \tag{3·3}$$

and that $A = 3{\cdot}5$ (so that $\log\log A > 0$). We define two contours C and C', in the plane of $u = \rho e^{i\phi}$, as follows. C is a loop passing through $u = A$ and enclosing the



part of the real axis to the right of this point. C' has the general shape shown in the figure. On its distant parts

$$u = \rho e^{i\phi} \quad (-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi), \quad \cos\phi = \frac{\frac{1}{2}(\pi + \eta)}{\log \rho}, \tag{3·4}$$

and

$$\cos\phi \geqslant \frac{\frac{1}{2}(\pi + \eta)}{\log \rho} \tag{3·5}$$

to the right of C'.*

Then, if we write

$$\lambda(u) = u \log u \log\log u = u\, lu\, l_2 u, \tag{3·6}$$

we have

$$G_4(z, \delta) = \sum_4^{\infty} e^{-\delta\lambda_n} n!\, z^n = \int_C \frac{e^{-\delta\lambda(u)}\,\Gamma(u+1)}{e^{2\pi i u} - 1} z^u\, du. \tag{3·7}$$

* That is to say, of the distant part of C'. The inequality is not satisfied near $u = A$.

4. We begin by showing that we can replace C by C' in (3·7). For this, we require majorants of the various factors in the integrand. Simple calculations give

$$|e^{-\delta\lambda(u)}| = \exp\{-\delta\rho\, l\rho\, l_2\rho\cos\phi + \delta\rho\, l_2\rho\,\phi\sin\phi + \delta\rho\,\phi\sin\phi + o(\rho)\}, \tag{4·1}$$

$$|\Gamma(u+1)| = \exp\{\rho\, l\rho\cos\phi - \rho\phi\sin\phi - \rho\cos\phi + o(\rho)\}, \tag{4·2}$$

$$\left|\frac{1}{e^{2\pi iu}-1}\right| = O(1), \tag{4·3}$$

$$|z^u| = \exp(\rho\cos\phi\, lr - \rho\sin\phi\,\theta). \tag{4·4}$$

It then follows from (3·5) that the product of the four functions is majorized by

$$\exp\{-\delta\rho\, l\rho\, l_2\rho\cos\phi + \delta\rho\, l_2\rho\,\phi\sin\phi + O(\rho)\}$$
$$\leqslant \exp\{-\tfrac{1}{2}(\pi+\eta)\,\delta\rho\, l_2\rho + \tfrac{1}{2}\pi\delta\rho\, l_2\rho + O(\rho)\} = \exp\{-\tfrac{1}{2}\eta\delta\rho\, l_2(\rho) + O(\rho)\}$$

throughout the distant parts of the region between C and C'; and so that

$$G_4(z,\delta) = \int_{C'} \frac{e^{-\delta\lambda(u)}\,\Gamma(u+1)}{e^{2\pi iu}-1} z^u\,du. \tag{4·5}$$

5. We prove next that the integral in (4·5) is uniformly convergent for $\delta \geqslant 0$ and z in $P(R,\eta)$. We need only consider the distant parts of C', on which

$$\cos\phi = \frac{\tfrac{1}{2}(\pi+\eta)}{l\rho}, \quad |\phi| = \tfrac{1}{2}\pi + O\left(\frac{1}{l\rho}\right), \quad |\sin\phi| = 1 + O\left(\frac{1}{l\rho}\right). \tag{5·1}$$

Taking first the upper half of C', the first two factors are now majorized by

$$\exp\{-\tfrac{1}{2}(\pi+\eta)\,\delta\rho\, l_2\rho + \tfrac{1}{2}\pi\delta\rho\, l_2\rho + \tfrac{1}{2}\pi\delta\rho + o(\rho)\}$$
$$= \exp\{-\tfrac{1}{2}\eta\delta\rho\, l_2\rho + \tfrac{1}{2}\pi\delta\rho + o(\rho)\} \tag{5·2}$$

and

$$\exp\{\tfrac{1}{2}(\pi+\eta)\,\rho - \tfrac{1}{2}\pi\rho + o(\rho)\} = \exp\{\tfrac{1}{2}\eta\rho + o(\rho)\}. \tag{5·3}$$

The third is bounded; and these three estimates are all independent of z. Finally, the fourth factor is majorized by

$$\exp(\rho\cos\phi\, lr - \rho\sin\phi\,\theta) < \exp\left\{\tfrac{1}{2}(\pi+\eta)\,\rho\frac{lR}{l\rho} - \eta\rho + o(\rho)\right\}$$
$$= \exp\{-\eta\rho + o(\rho)\}, \tag{5·4}$$

where the $o(\rho)$ is uniform for z in $P(R,\eta)$. Hence, after (3·3), the integrand is majorized by

$$\exp\{\tfrac{1}{2}\pi\delta\rho - \tfrac{1}{2}\eta\rho + o(\rho)\} < \exp\{-\tfrac{1}{4}\eta\rho + o(\rho)\}, \tag{5·5}$$

and the integrand over the upper part of C' is uniformly convergent.

On the lower half of C', ϕ is negative, and (5·4) must be replaced by

$$\exp(\rho\cos\phi\, lr - \rho\sin\phi\,\theta) < \exp\left\{\tfrac{1}{2}(\pi+\eta)\,\rho\frac{lR}{l\rho} + (2\pi-\eta)\,\rho + o(\rho)\right\}$$
$$= \exp\{(2\pi-\eta)\,\rho + o(\rho)\}.$$

Here there is an additional factor $e^{2\pi\rho}$. But now

$$\left|\frac{1}{e^{2\pi iu}-1}\right| = O\,|e^{-2\pi iu}| = O(e^{2\pi\rho\sin\phi}) = O\{e^{-2\pi\rho+o(\rho)}\};$$

and so this part of the integral is also uniformly convergent.

It now follows that
$$G_4(z,\delta) \to \int_{C'} \frac{\Gamma(u+1)}{e^{2\pi i u}-1} z^u\, du \tag{5·6}$$
when $\delta \to 0$, uniformly for z in $P(R,\eta)$. Incidentally we have proved that the integral on the right-hand side is uniformly convergent in $P(R,\eta)$, and so represents a function regular in P.

6. It is plain that
$$\sum_0^3 e^{-\delta\lambda_n} n!\, z^n \to \sum_0^3 n!\, z^n = \int_{C''} \frac{\Gamma(u+1)}{e^{2\pi i u}-1} z^u\, du,$$
where C'' is a contour round the points $u = 0, 1, 2, 3$ (and no other poles of the integrand). Hence
$$G(z,\delta) = \sum_0^\infty e^{-\delta\lambda_n} n!\, z^n \to \int_{C_1} \frac{\Gamma(u+1)}{e^{2\pi i u}-1} z^u\, du, \tag{6·1}$$
where C_1 is C' diverted, as in the figure, so as to cross the real axis between -1 and 0. We may suppose that
$$\Re u \geqslant -1+\mu > -1 \tag{6·2}$$
everywhere on C_1.

We have to identify the last integral with $G(z)$. For this, we may suppose that
$$z = -r \quad (0<r<1), \quad \theta = \pi. \tag{6·3}$$
Then
$$\int_{C_1} \frac{\Gamma(u+1)}{e^{2\pi i u}-1} z^u\, du = \int_{C_1} \frac{z^u}{e^{2\pi i u}-1}\, du \int_0^\infty e^{-t} t^u\, dt = \int_0^\infty e^{-t} \chi(zt)\, dt, \tag{6·4}$$
where
$$\chi(w) = \int_{C_1} \frac{w^u}{e^{2\pi i u}-1}\, du, \tag{6·5}$$
provided that the inversion of the order of integration is legitimate.

The part of the repeated integral for which $|u|$ is not large is majorized, after (6·2), by
$$\int_0^\infty e^{-t} t^{-1+\mu}\, dt \int \left| \frac{z^u}{e^{2\pi i u}-1} \right| |du|,$$
which is plainly convergent.

On the "distant" upper part of C_1 (which is the same as that of C'),
$$|(zt)^u| = \exp\{\rho\cos\phi \log(rt) - \pi\rho\sin\phi\};$$
and the corresponding part of the repeated integral is majorized by
$$\int_{\rho_0}^\infty e^{-\pi\rho\sin\phi}\, r^{\rho\cos\phi}\, d\rho \int_0^\infty e^{-t} t^{\rho\cos\phi}\, dt = \int_{\rho_0}^\infty e^{-\pi\rho\sin\phi}\, r^{\rho\cos\phi}\, \Gamma(\rho\cos\phi+1)\, d\rho,$$
for a certain ρ_0. Here $\quad \cos\phi = \dfrac{\frac{1}{2}(\pi+\eta)}{\log\rho}, \quad r^{\rho\cos\phi} = e^{o(\rho)},$
$$\Gamma(\rho\cos\phi+1) < \exp\left\{\frac{\frac{1}{2}(\pi+\eta)}{\log\rho}\,\rho\log\rho\right\} = e^{\frac{1}{2}(\pi+\eta)\rho},$$
$$e^{-\pi\rho\sin\phi} = e^{-\pi\rho+o(\rho)}.$$

Hence the repeated integral is majorized by

$$\int_{\rho_0}^{\infty} e^{-\frac{1}{2}(\pi-\eta)\rho+o(\rho)}\,d\rho<\infty.$$

On the distant lower part of C_1, we have, as at the end of § 5, $\phi<0$ and additional factors $e^{2\pi\rho}$ and $e^{2\pi\rho\sin\phi}$; and the argument goes similarly. It follows that the repeated integral is absolutely convergent and the inversion in (6·4) and (6·5) justifiable. Thus

$$G(z,\delta)\to\int_0^{\infty} e^{-t}\chi(zt)\,dt, \tag{6·6}$$

where $\chi(w)$ is defined by (6·5).

Finally, it is familiar that

$$\chi(w)=\int_{C_1}\frac{w^u\,du}{e^{2\pi iu}-1}=\frac{1}{1-w} \tag{6·7}$$

for all w in P. Hence $\displaystyle\int_0^{\infty} e^{-t}\chi(zt)\,dt=\int_0^{\infty}\frac{e^{-t}}{1-zt}\,dt,$

and this completes the proof of Theorem 1.

7. We can generalize Theorem 1 as follows.

THEOREM 2. *If*

$$f(z)=\sum_0^{\infty} a_n z^n \tag{7·1}$$

is regular at the origin and in P, and

$$f(z)=O(|z|^k) \tag{7·2}$$

for large $|z|$ and some k, uniformly in any $P(\infty,\eta)$, then the series*

$$\sum_0^{\infty} n!\,a_n z^n \tag{7·3}$$

is summable (A), to sum

$$g(z)=\int_0^{\infty} e^{-t}f(zt)\,dt, \tag{7·4}$$

in P, and uniformly in any $P(R,\eta)$.

I prove summability in P, leaving the gloss about uniformity to the reader.

Suppose that

$$z=re^{i\theta}\quad(0<\theta<2\pi)$$

and that C is the contour in the u-plane formed by (i) the arc of the circle $\rho=\rho_0$ on which $\theta<\theta+\zeta\leqslant\phi\leqslant 2\pi+\theta-\zeta$, and (ii) the two radii from the ends of this arc to infinity. When u describes C,

$$w=\frac{z}{u}=\frac{r}{\rho}e^{i(\theta-\phi)}=Re^{i\Theta}$$

describes a contour Γ consisting of (i) the arc of the circle $R=r/\rho_0$ on which $-2\pi+\zeta\leqslant\Theta\leqslant-\zeta$, and (ii) the two radii from the ends of this arc to the origin. This contour is the boundary of $P(r/\rho_0,\zeta)$, and so $G(w,\delta)\to G(w)$ uniformly on Γ.

* $P(\infty,\eta)$ is the region which is the limit of $P(R,\eta)$ when $R\to\infty$.

We write $$f_{k+1}(z) = f(z) - a_0 - a_1 z - \ldots - a_k z^k,$$
and similarly for other power series. Then
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(u)}{u^{n+1}}\,du$$
for $n > k$,* and so
$$g_{k+1}(z,\delta) = \sum_{k+1}^{\infty} e^{-\delta\lambda_n} n!\, a_n z^n = \frac{1}{2\pi i}\int_C \frac{f(u)}{u}\sum_{k+1}^{\infty} e^{-\delta\lambda_n} n!\left(\frac{z}{u}\right)^n du$$
$$= \frac{1}{2\pi i}\int_C \frac{f(u)}{u} G_{k+1}\left(\frac{z}{u},\delta\right) du. \tag{7·5}$$

But $$G_{k+1}\left(\frac{z}{u},\delta\right) \to G_{k+1}\left(\frac{z}{u}\right)$$
uniformly on C, and is $O(|u|^{-k-1})$ at infinity; and so
$$g_{k+1}(z,\delta) \to \frac{1}{2\pi i}\int_C \frac{f(u)}{u} G_{k+1}\left(\frac{z}{u}\right) du. \tag{7·6}$$

8. Now $$g_{k+1}(z) = g(z) - a_0 - 1!\,a_1 z - \ldots - k!\,a_k z^k \dagger$$
$$= \int_0^\infty e^{-t} f_{k+1}(zt)\,dt, \tag{8·1}$$
and $$\frac{f_{k+1}(zt)}{(zt)^{k+1}} = \frac{1}{2\pi i}\int_C \frac{f(u)}{u^{k+1}(u-zt)}\,du. \tag{8·2}$$
Hence $$g_{k+1}(z) = \int_0^\infty e^{-t}\,dt\,\frac{1}{2\pi i}\int_C \left(\frac{zt}{u}\right)^{k+1}\frac{f(u)}{u-zt}\,du$$
$$= \frac{1}{2\pi i}\int_C \frac{f(u)}{u}\,du \int_0^\infty e^{-t}\left(\frac{zt}{u}\right)^{k+1}\frac{dt}{1-(zt/u)}, \tag{8·3}$$
provided once more that we may invert the order of integration. If we take this for granted for the moment, then
$$g_{k+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(u)}{u} G_{k+1}\left(\frac{z}{u}\right) du;$$
and it follows from (7·6) that
$$g_{k+1}(z,\delta) \to g_{k+1}(z)$$
and so $$g(z,\delta) \to g(z).$$

* We express a_n first by an integral round a contour lying inside the circle of convergence of the series (7·1). We may deform this contour into C because $f(z)$ is regular in P and satisfies (7·2).

† This is the definition of $g_{k+1}(z)$.

9. It remains only to verify that the inversion in (8·3) is legitimate. This is obvious so far as the circular part of C is concerned. If u is on one of the rectilinear parts of C, say on the radius $\phi = \theta + \zeta$, then

$$|u - zt| = |\rho e^{i\zeta} - rt| = \sqrt{(\rho^2 - 2\rho rt\cos\zeta + r^2t^2)},$$

and the repeated integral is majorized by a numerical multiple of

$$\int_0^\infty e^{-t}dt \int_{\rho_0}^\infty \left(\frac{rt}{\rho}\right)^{k+1} \frac{\rho^k d\rho}{\sqrt{(\rho^2 - 2\rho rt\cos\zeta + r^2t^2)}} = r^k \int_0^\infty e^{-t}t^k\,dt \int_{\rho_0/rt}^\infty \frac{d\sigma}{\sigma\sqrt{(\sigma^2 - 2\sigma\cos\zeta + 1)}}.$$

The inner integral here is less than

$$K(\zeta)\,\mathrm{Max}\left(\log\frac{\rho_0}{rt},\ 1\right),$$

where $K(\zeta)$ depends only on ζ, so that the repeated integral is absolutely convergent. The other rectilinear part of C may be disposed of similarly.

10. Suppose for example that

$$a_n = \binom{n+\alpha}{n} m_n,$$

where m_n is a "moment constant"

$$m_n = \int_0^1 w^n\,d\chi(w)$$

and $\chi(w)$ has bounded variation. Then

$$f(z) = \int_0^1 \frac{d\chi(w)}{(1-zw)^{1+\alpha}}$$

is regular in P. If $\alpha \geqslant -1$, then $|1 - zw| \geqslant |\sin\theta|$ if $x > 0$ and $|1 - zw| \geqslant 1$ if $x < 0$, so that

$$|f(z)| \leqslant |\sin\eta|^{-1-\alpha} \int_0^1 |d\chi|$$

in $P(\infty, \eta)$. If $\alpha < -1$, then $|1 - zw| \leqslant |z - 1|$ for large z, and

$$|f(z)| = O(|z|^{-1-\alpha})$$

for large z in $P(\infty, \eta)$. In any case the conditions of Theorem 2 are satisfied, and

$$\Sigma n!\binom{n+\alpha}{n} m_n z^n$$

is summable (A) in P. Similarly

$$\Sigma n!\, n^\alpha m_n z^n$$

is summable (A) in P.

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COMMENTS

The series $x-1!x^2+2!x^3-\ldots$ was regarded by Euler as a solution of the differential equation

$$x^2y'+y = x.$$

This has a solution

$$y = xG(-x) \quad \text{for} \quad x > 0.$$

See D.S., pp. 26–7, where the calculations 'easy to verify', § 1, are given.

Le Roy† showed that the series (1.1) is summable (B^*) to $G(z)$ throughout the complex plane, cut along the positive real axis; see also D.S., p. 192. This trivial solution to Hardy's problem may be excluded, by interpreting 'some method of summation', § 2, as some method *not depending on analytic continuation.*

The 'double application' of Borel's method (§ 2) is the (B^2) method; see 1935, 1, § 2.2.

The property, § 3, that $\sum z^n$ is summable by Lindelöf's method $(A, n\log n)$ throughout P was given by Lindelöf;‡ see D.S., Theorem 32.

† *Ann. de la Fac. des Sci. de l'Univ. de Toulouse* (2), 2 (1900), 317–430.

‡ *J. de math. pures et appl.* (5), 9 (1903), 213–21.

NOTES ON THE THEORY OF SERIES (XXII)*: ON THE TAUBERIAN THEOREM FOR BOREL SUMMABILITY

G. H. HARDY and J. E. LITTLEWOOD†.

1. If $A_n = a_0 + a_1 + \ldots + a_n$ and

$$e^{-x} A(x) = e^{-x} \sum_0^\infty A_n \frac{x^n}{n!} \to A \tag{1.1}$$

when $x \to \infty$, then we say that Σa_n is summable (B) to sum A, and write

$$\Sigma a_n = A \quad (B). \tag{1.2}$$

This is Borel's "exponential" definition. The distinction between his exponential and integral definitions is irrelevant here, since they are equivalent whenever $a_n \to 0$, and we are concerned only with series which satisfy this condition. For the same reason it is immaterial whether $x \to \infty$ continuously or through integral values m.

We are concerned with the group of theorems which assert that (1.1), together with some supplementary condition, implies the convergence of Σa_n. We proved in 1912 (1) that

$$a_n = o(n^{-\frac{1}{2}}) \tag{1.3}$$

is a sufficient supplementary condition. In 1917 (2) we proved, by a much more sophisticated argument, that

$$a_n = O(n^{-\frac{1}{2}}) \tag{1.4}$$

is sufficient; and later writers have generalized the condition further, its most general form being that of R. Schmidt, Valiron and Vijayaraghavan, that

$$\varliminf (A_n - A_m) \geqslant 0 \tag{1.5}$$

subject to

$$n > m, \quad m \to \infty, \quad m^{-\frac{1}{2}}(n-m) \to 0. \tag{1.6}$$

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* The last two notes of this series, published in *Proc. London Math. Soc.* (2), 41 (1936), 257–270, and *Quarterly Journal* (Oxford), 31 (1937), 161–172, were both numbered XX.

† Received 1 December, 1943; read 16 December, 1943.

There are two quite different lines of proof for these theorems. The first, due in its simplest form to Vijayaraghavan (5), depends on a development of the ideas of 2; the second, of which the simplest version is that of Pitt (4), on the theory of Fourier transforms and the ideas of Wiener (6). Whichever line is followed, a good deal of ingenuity is required, but the most essential difficulties remain those involved in the passage from (1.3) to (1.4).

Our object here is to give a proof of the theorem, with the hypothesis (1.4), which is a good deal simpler than any published so far. The essential idea is that of using "Vitali's theorem"* at the critical moment. We have nothing to add concerning the more general hypothesis (1.5): we can pass from (1.4) to (1.5) by following Vijayaraghavan. We present the proof so as to be complete in itself, suppressing only details of approximation by Stirling's theorem.

2. We use two definitions of the sum of a divergent series besides Borel's.

(i) If $c > 0$ and

$$\sqrt{\left(\frac{c}{\pi n}\right)} \Sigma e^{-ch^2/n} A_{n+h} \to A, \tag{2.1}$$

where the summation runs from $h = -\infty$ to $h = \infty$, and $A_{n+h} = 0$ if $n+h < 0$, then we write

$$\Sigma a_n = A \quad (e, c). \tag{2.2}$$

(ii) If $\Sigma a_n x^n$ converges to $f(x)$ for $|x| < 1$, $0 < k < 1$, and

$$\sum_0^\infty b_n = \sum_0^\infty \frac{k^n}{n!} f^{(n)}(1-k) = A, \tag{2.3}$$

that is to say if the Taylor's series for $f(x) = f(1-k+ky)$ converges to A for $y = 1$, then we write

$$\Sigma a_n = A \quad (\gamma, k). \tag{2.4}$$

We call this method of summation the "circle" method.

* For Vitali's theorem see Titchmarsh, *Theory of functions*. 168.

We need an explicit formula for $B_m = b_0+b_1+\ldots+b_m$ in terms of A_n. If $[\phi(z)]_{z^m}$ is the coefficient of z^m in the expansion of $\phi(z)$, then

$$(2.5)\quad B_m = \left[\frac{f(1-k+ky)}{1-y}\right]_{y^m} = \left[\frac{kf(x)}{1-x}\right]_{y^m} = \left[k\sum_0^{\infty} A_n(1-k+ky)^n\right]_{y^m}$$

$$= k^{m+1}\left\{A_m+(m+1)(1-k)A_{m+1}+\frac{(m+1)(m+2)}{2!}(1-k)^2A_{m+2}+\ldots\right\}$$

$$= k^{m+1}\sum_{n=m}^{\infty}\binom{n}{m}(1-k)^{n-m}A_n.$$

We shall be concerned with three classes of series Σa_n, viz. those which satisfy the conditions

$$(2.6a)\quad a_n = o(1), \qquad (2.6b)\quad A_n = o(n^{\frac{1}{2}}), \qquad (2.6c)\quad a_n = O(n^{-\frac{1}{2}}).$$

We call these classes A, B, C. A includes C, while B does not; but we shall find that all series of A (and *a fortiori* of C), summable by any of our methods, belong to B. Our main theorem concerns C, but most of our preliminary lemmas are true for all series of A.

3. We begin with two elementary lemmas, whose content is familiar, concerning the exponential and binomial series. The proofs demand only straightforward applications of Stirling's theorem, and need not be set out in detail.

LEMMA 1. *Suppose that m is a positive integer and*

$$(3.1)\qquad u_n = u_n(m) = e^{-m}\frac{m^n}{n!}$$

(*so that* $\Sigma u_n = 1$). *Then the largest* u_n *are* u_{m-1} *and* u_m (*which are equal*). *Further, if* $\frac{1}{2} < \eta < \frac{2}{3}$ *and* $n = m+h$, *then*

$$(3.2)\qquad \sum_{|h|\geqslant m^{\eta}} u_n = O(e^{-m^{\zeta}}),$$

where $\zeta = \zeta(\eta) > 0$; *and*

$$(3.3)\qquad u_n = \sqrt{\left(\frac{c}{\pi m}\right)}\, e^{-ch^2/m}\left\{1+O\left(\frac{|h|+1}{m}\right)+O\left(\frac{|h|^3}{m^2}\right)\right\},$$

where $c=\frac{1}{2}$, *when* $|h| < m$. *The conclusion* (3.2) *is not affected if* u_n *is multiplied by any power of n.**

* There is a slip (without material consequences) on p. 37 of 2 (3.3) being asserted without the limitation that $|h|^3/m^2$ is small.

LEMMA 2. *Suppose that m is a positive integer, $0 < k < 1$, and*

$$(3.4) \quad u_n(m) = 0 \quad (n < m), \qquad u_n(m) = \binom{n}{m} k^{m+1} (1-k)^{n-m} \quad (n \geq m),$$

so that

$$\Sigma u_n = k^{m+1} \left\{ 1 + (m+1)(1-k) + \frac{(m+1)(m+2)}{2!} (1-k)^2 + \ldots \right\} = 1.$$

Then the largest u_n is u_N, where

$$N = [m/k]$$

(two terms being equal if m/k is an integer). If $\frac{1}{2} < \eta < \frac{2}{3}$ and $n = N+h$, then (3.2) holds (with the same gloss as in Lemma 1), and

$$(3.5) \qquad u_n = \sqrt{\left(\frac{c}{\pi N}\right)} e^{-ch^2/N} \left\{ 1 + O\left(\frac{|h|+1}{m}\right) + O\left(\frac{|h^3|}{m^2}\right) \right\},$$

with

$$2c = k/(1-k).$$

4. Our next pair of lemmas establishes the equivalences

$$(4.1) \quad (B) \equiv (e, \tfrac{1}{2}), \qquad (4.2) \quad (\gamma, k) \equiv \left(e, \frac{k}{2(1-k)}\right)$$

for series of A or B. The notation implies that any series of either of these classes, summable by one of the methods referred to, is summable to the same sum by the other.

LEMMA 3. *If $a_n = o(1)$, and Σa_n is summable by any one of the methods (B), (e, c), (γ, k), then $A_n = o(n^{\frac{1}{2}})$.*

If $\Sigma a_n = A(B)$, and m is an integer, then

$$(4.3) \qquad A_m = e^{-m} \Sigma \frac{m^n}{n!} (A_m - A_n) + A + o(1);$$

and we may replace the sum here, after Lemma 1, by

$$\sum_{|h|<m^\eta} u_{m+h}(m)(A_m - A_{m+h}) = O\left\{ \sum_{|h|<m^\eta} \frac{e^{-h^2/2m}}{\sqrt{(2\pi m)}} |A_{m+h} - A_m| \right\}.$$

But $A_{m+h} - A_m = o(|h|)$, since $a_n = o(1)$; and so the last sum is

$$o(m^{-\frac{1}{2}} \Sigma e^{-h^2/2m} |h|) = o\left(m^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-t^2/2m} |t| \, dt\right) = o(m^{\frac{1}{2}}).$$

If $\Sigma a_n = A(\gamma, k)$, then the conclusion follows similarly from Lemma 2. If $\Sigma a_n = A(e, c)$, the proof is simpler, no preliminary approximations being required.

Lemma 4. *The equivalences* (4.1) *and* (4.2) *hold for all series of* A *or* B.

We take (4.1). It is plainly sufficient, after Lemmas 1 and 3, to prove that

$$\sum_{|h|<m^{\eta}} e^{-m}\frac{m^{m+h}}{(m+h)!}A_{m+h}-\frac{1}{\sqrt{(2\pi m)}}\sum_{|h|<m^{\eta}} e^{-h^2/2m}A_{m+h}=o(1)$$

whenever $A_n=o(n^{\frac{1}{2}})$. But the difference is

$$m^{-\frac{1}{2}}\sum_{|h|<m^{\eta}} e^{-h^2/2m}\left\{O\left(\frac{|h|+1}{m}\right)+O\left(\frac{|h|^3}{m^2}\right)\right\}o(m^{\frac{1}{2}})$$

$$=o\left(\frac{1}{m}\int_{-\infty}^{\infty}e^{-t^2/2m}|t|\,dt\right)+o\left(\frac{1}{m^2}\int_{-\infty}^{\infty}e^{-t^2/2m}|t|^3\,dt\right)=o(1).$$

This proves (4.1), and the proof of (4.2) requires only the use of Lemma 2 instead of Lemma 1. The proofs apply to all series of B, whether they belong to A or not.

5. Our next group of lemmas establishes the implication

(5.1) $$(e, c)\to(e, d)\quad(0<d<c)$$

for series of A. We use the "circle" method as a convenient intermediary.

Lemma 5. *The* (γ, k) *method is regular.*

For, if we write (2.5) as $B_m=\sum c_{m,n}A_n$, then

$$c_{m,n}\geqslant 0,\quad \sum_n|c_{m,n}|=\sum_n c_{m,n}=k^{m+1}\{1-(1-k)\}^{-m-1}=1,$$

and $c_{m,n}\to 0$, for fixed n, when $m\to\infty$. Thus the standard conditions of Toeplitz and Schur are satisfied.

Lemma 6. *If* $\sum a_n=A(\gamma, k)$ *and* $0<l<k$, *then* $\sum a_n=A\ (\gamma, l)$.

This is a corollary of Lemma 5. For if

$$x=1-k+ky=1-l+lz,\quad f(x)=\sum a_n x^n=\sum b_n y^n=\sum c_n z^n,$$

then $$y=1-\frac{l}{k}+\frac{lz}{k}.$$

If $\sum a_n$ is summable (γ, k), then $\sum b_n$ is convergent; and therefore, by Lemma 5, summable $(\gamma, l/k)$. But this means that $\sum c_n$ is convergent, *i.e.* that $\sum a_n$ is summable (γ, l).

Lemma 7. *If* $a_n=o(1)$, $\sum a_n=A(e, c)$, *and* $0<d<c$, *then*

$$\sum a_n=A(e, d).$$

This is plainly a corollary of Lemmas 4 and 6.

We need one further lemma, concerning series of the narrower class C.

LEMMA 8. *If* $a_n = O(n^{-\frac{1}{2}})$ *and* $\Sigma a_n = A(B)$, *then* $A_n = O(1)$.

For $\Sigma a_n = A(e, \frac{1}{2})$, by Lemma 4, so that

(5.2) $$\sqrt{\left(\frac{c}{\pi m}\right)}\, \Sigma e^{-ch^2/m} A_{m+h} = A + o(1)$$

with $c = \frac{1}{2}$. Also

$$\sqrt{\left(\frac{c}{\pi m}\right)}\, \Sigma e^{-ch^2/m} = 1 + O\left\{\sqrt{\left(\frac{c}{\pi m}\right)}\right\}$$

and so

(5.3) $$A_m\{1+o(1)\} = \sqrt{\left(\frac{c}{\pi m}\right)}\, \Sigma e^{-ch^2/m} A_m$$
$$= \sqrt{\left(\frac{c}{\pi m}\right)}\, \Sigma e^{-ch^2/m}(A_m - A_{m+h}) + A + o(1).$$

We may restrict the summation to $|h| < m^\eta$, when $A_{m+h} - A_m = O(m^{-\frac{1}{2}}|h|)$; and then the sum is

$$O\left(m^{-1} \sum_{|h|<m^\eta} e^{-ch^2/m}|h|\right) = O\left(m^{-1}\int_{-\infty}^{\infty} e^{-ct^2/m}|t|\,dt\right) = O(1).$$

Hence $A_m\{1+o(1)\} = O(1)$, *i.e.* $A_m = O(1)$.

6. *Proof of the theorem.* We now suppose $c = \gamma + i\delta$ complex. We choose δ_0, γ_0 and γ_1 so that $\delta_0 > 0$, $0 < \gamma_0 < \frac{1}{2} < \gamma_1$; denote by R the rectangle $\gamma_0 \leqslant \gamma \leqslant \gamma_1$, $|\delta| \leqslant \delta_0$ in the plane of c; and write

$$\phi_m(c) = \sqrt{\left(\frac{c}{\pi m}\right)}\, \Sigma e^{-ch^2/m} A_{m+h}.$$

We may also suppose that $|a_n| < n^{-\frac{1}{2}}$ for large n.

If $\Sigma a_n = A(B)$ and $a_n = O(n^{-\frac{1}{2}})$, then $A_n = O(1)$, by Lemma 8. Hence, first, $\phi_m(c)$ is, for each m, an analytic function of c regular in R. Next,

$$\phi_m(c) = O\left\{\sqrt{\left(\frac{|c|}{m}\right)}\, \Sigma e^{-\gamma h^2/m}\right\} = O\left\{\sqrt{\left(\frac{\gamma}{m}\right)} \int_{-\infty}^{\infty} e^{-\gamma t^2/m}\,dt\right\} = O(1)$$

uniformly for c in R and all m. Finally, by Lemmas 4 and 7, $\Sigma a_n = A(e, c)$ for $0 < c \leqslant \frac{1}{2}$, so that $\phi_m(c) \to A$ for every c on the stretch $(\gamma_0, \frac{1}{2})$ of the real axis. It follows from Vitali's theorem that $\phi_m(c) \to A$ for all c of R. In particular, since γ_0 and γ_1 are arbitrary, $\Sigma a_n = A(e, c)$ for all positive c.

Thus (5.3) is true for all positive c. We may restrict the summation to $|h| < m^\eta$, and then $|A_{m+h} - A_m| < 3m^{-\frac{1}{2}}|h|$ for large m. Hence, if S

is the sum on the right of (5.3), thus restricted,

$$|S| \leqslant \frac{3}{m}\sqrt{\left(\frac{c}{\pi}\right)}\,\Sigma\, e^{-ch^2/m}|h|$$

$$\leqslant \frac{3}{m}\sqrt{\left(\frac{c}{\pi}\right)}\left\{\int_{-\infty}^{\infty} e^{-ct^2/m}|t|\,dt+\sqrt{\left(\frac{2m}{ec}\right)}\right\} = \frac{3}{\sqrt{(c\pi)}}+o(1).$$

It now follows from (5.3) that

$$\overline{\lim}\,|A_m - A| \leqslant 3(c\pi)^{-\frac{1}{2}},$$

and therefore, since c is arbitrary, that $A_m \to A$.

7. In the course of the argument we have proved a number of subsidiary theorems, such as those expressed by (4.1), (4.2), and (5.1), valid for larger classes of series than the class C relevant to the main theorem. It has not been necessary to consider how large these classes may be made, and what we have proved falls far short of the ultimate truth. Thus Hyslop (3) has proved that $(B) \equiv (e, \frac{1}{2})$ whenever $a_n = O(n^K)$ for some K, and the scope of (5.1) could be extended similarly. It could probably be extended still further by the use of different weapons.

Thus it may be proved that, if

$$f_a(x) = \sqrt{\left(\frac{a}{\pi x}\right)}\int_{-\infty}^{\infty} e^{-at^2/x} f(t+x)\,dt$$

and $0 < b < a$, then

$$f_b(x) = \sqrt{\left(\frac{ab}{\pi}\right)}\int_0^{ax/b} \frac{(a-b)\,x}{(ax-bt)^{\frac{3}{2}}}\exp\left\{-\frac{ab(t-x)^2}{ax-bt}\right\} f_a(t)\,dt$$

under very general conditions; and this formula seems the most appropriate starting point for a general discussion of (5.1). But the indirect method which we have followed here is much simpler, and sufficient for our actual purpose.

References.

1. G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1912), 1–16.
2. G. H. Hardy and J. E. Littlewood, *Rendiconti Palermo*, 41 (1916), 36–53.
3. J. M. Hyslop, *Proc. London Math. Soc.* (2), 41 (1936), 243–256.
4. H. R. Pitt, *Proc. London Math. Soc.* (2), 44 (1938), 243–288.
5. T. Vijayaraghavan, *Proc. London Math. Soc.* (2), 27 (1927), 316–326.
6. N. Wiener, *Annals of Math.* (2), 33 (1932), 1–100 (67–72).

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COMMENTS

The O-Tauberian theorem for Borel summability was first proved by Hardy and Littlewood in 1916, 8. In the present proof, Vitali's theorem plays the role of 'repeated differentiation'. Vitali's theorem may be stated in the form: *if $\phi_n(z)$ is a sequence of analytic functions, regular and uniformly bounded in an open rectangle R, and convergent at an infinite set of points with a limit point in R, then $\phi_n(z)$ converges in R, uniformly on any interior closed set, to a function which is regular in R.*

In (3.3) of Lemma 1 and (3.5) of Lemma 2, h should be restricted so that $|h| < m^\eta$, $\frac{1}{2} < \eta < \frac{2}{3}$, as in Lemma 4 (cf. D.S., p. 200). This is implied by the footnote to Lemma 1, where a similar omission is pointed out in (2.125) of 1916, 8, which is (3.3) of the present paper.

Hardy says in D.S., p. 220, that the formula stated in § 7, expressing $f_b(x)$ in terms of $f_a(t)$, $0 < b < a$, holds *whenever the integral defining $f_b(x)$ is convergent.* The formula depends on the identity

$$\sqrt{\left(\frac{b}{\pi x}\right)}\exp\left\{-\frac{b(w-x)^2}{x}\right\}$$

$$=\sqrt{\left(\frac{ab}{\pi}\right)}\int\limits_0^{ax/b}\frac{(a-b)x}{(ax-bt)^{\frac{3}{2}}}\exp\left\{-\frac{ab(t-x)^2}{(ax-bt)}\right\}\cdot\sqrt{\left(\frac{a}{\pi t}\right)}\exp\left\{-\frac{a(w-t)^2}{t}\right\}dt.$$

If we substitute this into the convergent integral defining $f_b(x)$:

$$f_b(x)=\sqrt{\left(\frac{b}{\pi x}\right)}\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{b(w-x)^2}{x}\right\}f(w)\,dw,$$

and invert the repeated integral, the formula is obtained.

To verify the identity, write it as:

$$1=\sqrt{\left(\frac{ab}{\pi}\right)}\int\limits_0^{ax/b}\frac{(a-b)x}{(ax-bt)^{\frac{3}{2}}}\exp\left\{-\frac{ab(t-x)^2}{(ax-bt)}-\frac{a(w-t)^2}{t}+\frac{b(w-x)^2}{x}\right\}\sqrt{\left(\frac{ax}{bt}\right)}\,dt,$$

and make the substitutions $bt = axv$, $v = y/(1+y)$, $(a-b)xy = bwY^2$. Then the right-hand side becomes $J\{(a-b)w\}$, where

$$J(c)=2\sqrt{\left(\frac{c}{\pi}\right)}\int\limits_0^{\infty}\exp\{-c(Y-Y^{-1})^2\}\,dY=\sqrt{\left(\frac{c}{\pi}\right)}\int\limits_0^{\infty}\exp\{-c(Y-Y^{-1})^2\}(1+Y^{-2})\,dY$$

$$=\sqrt{\left(\frac{c}{\pi}\right)}\int\limits_{-\infty}^{\infty}e^{-cW^2}\,dW=1.$$

The justification of the inversion is straightforward.

NOTE ON THE MULTIPLICATION OF SERIES BY CAUCHY'S RULE

By G. H. HARDY

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1. I proved in 1908 (1) that *if* $A = \Sigma a_m$ *and* $B = \Sigma b_n$ *are convergent, and*

$$a_m = O(m^{-1}), \quad b_n = O(n^{-1}) \tag{1}$$

for large m and n, then

$$C = \Sigma c_p = \Sigma(a_0 b_p + a_1 b_{p-1} + \dots + a_p b_0)$$

is convergent (necessarily to AB); and this theorem has been extended in a number of directions both by other writers and by myself. Thus we may replace (1), when a_m and b_n are real, by

$$ma_m > -1, \quad nb_n > -1; \tag{2}$$

we may use conditions unsymmetrical in a_m and b_n; we may put the same problem for the product of any number of series; and we may consider modes of 'Dirichlet multiplication' based on sequences λ_m and μ_n, reducing to Cauchy's when $\lambda_m = m$ and $\mu_n = n$. The appropriate references will be found in (2).

I confine myself here to Cauchy multiplication of two convergent series. Then Neder (3) proved a theorem which exhausts the problem so long as our conditions on a_m and b_n are symmetrical, viz. that

$$\sum_{\frac{1}{2}x}^{x} |a_m| = O(1), \quad \sum_{\frac{1}{2}x}^{x} |b_n| = O(1) \tag{3}$$

is a sufficient pair of conditions for the convergence of C. It is plain that these conditions include (1); and it is easy to see that, when A and B are convergent, they also include (2). But it seems that there is still something to be said about unsymmetrical conditions.

Thus Rosenblatt and I proved that, if $\phi(m)$ is any function of the type

$$(\log m)^{\alpha} (\log\log m)^{\beta} \dots,$$

then

$$a_m = O\left\{\frac{\phi(m)}{m}\right\}, \quad b_n = O\left\{\frac{1}{n\phi(n)}\right\} \tag{4}$$

is a sufficient pair of conditions; but no one seems to have noticed that such a pair is quite unnecessarily strong. I shall show here, for example, that it is sufficient that

$$a_m = O(m^{-\delta}), \quad b_n = O\{(n \log n)^{-1}\} \tag{5}$$

for any positive δ.

I need a slight generalization of Neder's theorem: *if A and B are convergent*; $x = y + z$, *where* $y = y(x)$ *and* $z = z(x)$ *increase steadily to infinity with* x; *and*

$$\sum_{y}^{x} |a_m| = O(1), \quad \sum_{z}^{x} |b_n| = O(1); \tag{6}$$

then C converges to AB.

We have $$C(x) = \sum_{p \leq x} c_p = \sum_{m+n \leq x} a_m b_n = \sum_T a_m b_n,$$

 1944, 2 *Proceedings of the Cambridge Philosophical Society*, 40, 251–2.

where T is the triangle $m \geqq 0$, $n \geqq 0$, $m+n \leqq x$. We suppose that $0<\eta<y$, $0<\zeta<z$, and that neither η nor ζ is integral; and divide T into the five regions

$$(T_1) \quad 0 \leqq m < y, \quad 0 \leqq n < z;$$

$$(T_2) \quad 0 \leqq n < \zeta, \quad y \leqq m \leqq x-n; \qquad (T_3) \quad \zeta < n \leqq z, \quad y \leqq m \leqq x-n;$$

$$(T_4) \quad 0 \leqq m < \eta, \quad z \leqq n \leqq x-m; \qquad (T_5) \quad \eta < m \leqq y, \quad z \leqq n \leqq x-m.$$

Here T_2+T_3 and T_4+T_5 are the triangles left when the rectangle T_1 is removed from T, and these triangles are subdivided by the lines $n = \zeta$ and $m = \eta$. We denote the five sums into which $C(x)$ is divided by $S_1, S_2, \ldots, S_5$. Plainly $S_1 \to AB$, and it is sufficient to prove that $S_2+S_3 \to 0$ and $S_4+S_5 \to 0$.

We choose ζ so that
$$\left|\sum_{n_1}^{n_2} b_n\right| < \epsilon \quad (n_2 > n_1 > \zeta).$$

Then
$$|S_3| = \left|\sum_y^{x-\zeta} a_m \sum_\zeta^{x-m} b_n\right| \leqq \sum_y^{x-\zeta} |a_m| \left|\sum_\zeta^{x-m} b_n\right| \leqq \epsilon \sum_y^x |a_m| < H\epsilon$$
for a constant H, and
$$|S_2| = \left|\sum_0^\zeta b_n \sum_y^{x-n} a_m\right| \leqq \sum_0^\zeta |b_n| \left|\sum_y^{x-n} a_m\right| \to 0$$
when ζ is fixed and x, y, z tend to infinity. Hence $|S_2+S_3| < 2H\epsilon$ for large x. Thus $S_2+S_3 \to 0$, and the proof that $S_4+S_5 \to 0$ is similar.

Let us suppose, for example (taking a pair of one-sided conditions), that a_m and b_n are real, $0<\delta<1$, and
$$a_m > -m^{-\delta}, \quad b_n > -(n \log n)^{-1} \tag{7}$$
for large m, n. We write a_m^+ and a_m^- for the positive and negative a_m, so that $a_m = a_m^+ + a_m^-$ and $|a_m| = a_m^+ - a_m^-$, and similarly for b_n; and take $y = x - x^\delta$, $z = x^\delta$. Then
$$\sum_y^x |a_m| = \sum_y^x a_m - 2\sum_y^x a_m^- < o(1) + O\{y^{-\delta}(x-y)\} = O(1),$$
$$\sum_z^x |b_n| = \sum_z^x b_n - 2\sum_z^x b_n^- < o(1) + O\left(\sum_{x^\delta}^x \frac{1}{n \log n}\right) = O(1).$$
Thus the conditions (7) are sufficient, and (5) are sufficient *a fortiori*.

It is plain that, if ψ_m is any function which decreases to 0, then we can find a χ_n such that $\Sigma\chi_n = \infty$ and
$$a_m = O(\psi_m), \quad b_n = O(\chi_n)$$
form a sufficient pair of conditions, and thus obtain a chain of theorems ranging from that first quoted to that of Mertens, in which B is absolutely convergent and no condition on a_m is needed. For example, we can find a χ_n when $\psi_m = (\log m)^{-1}$; but we shall require that
$$\sum_{\log x}^x \chi_n = O(1),$$
and the divergence of $\Sigma\chi_n$ will be so slow as to elude the logarithmic scale.

REFERENCES

(1) Hardy, G. H. *Proc. London Math. Soc.* (2), 6 (1908), 410.
(2) Hardy, G. H. *Journal London Math. Soc.*, 2 (1927), 169.
(3) Neder, L. *Proc. London Math. Soc.* (2), 23 (1924), 176.

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COMMENTS

Earlier results on the multiplication of series are given in 1908, 2; 1912, 2; 1913, 2, §§ 44–50; 1914, 11; 1927, 10. See also D.S., Chapter X.

The theorem under conditions (4) was given by Hardy in 1908, 2, with o in place of O, and by Rosenblatt† with O.

See also D.S., §§ 10.4–5, where Hardy's extension of Neder's theorem is proved with a diagram, and some applications are given. Neder's theorem was originally given for 'Dirichlet multiplication'.

† *Bull. de l'Acad. Polonaise* (*A*), 1913, 603–31.

RIEMANN'S FORM OF TAYLOR'S SERIES

G. H. Hardy†.

1. In his fragment "Versuch einer allgemeinen Auffassung der Integration und Differentiation" [*Werke* (1876), 331–344], Riemann gives the formula

$$f(x+h) = \Sigma \frac{h^{m+r}}{\Gamma(m+r+1)} D^{m+r} f(x). \tag{1.1}$$

† Received 9 March, 1945; read 17 May, 1945.

Here r is fixed, and in general non-integral; m (as always when there is no indication to the contrary) runs through all, positive or negative, integers; and D^{m+r} is an operation of generalized differentiation. If r is an integer then the terms for which $m < -r$ disappear, and we obtain the usual form of Taylor's series. The most noteworthy special cases are those in which $f(x) = e^{cx}$ and $f(x) = x^n$. If we suppose that c, x, and h are positive, and that the operator D is so defined in the two cases that

$$(1.2) \qquad D^p e^{cx} = c^p e^{cx}, \quad D^p x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-p)} x^{n-p},$$

then (1.1) reduces in the first case to

$$(1.3) \qquad e^z = \Sigma \frac{z^{m+r}}{\Gamma(m+r+1)} \dagger,$$

and in the second to

$$(1.4) \qquad (x+h)^n = \Sigma \frac{\Gamma(n+1)}{\Gamma(m+r+1)\Gamma(n-m-r+1)} x^{n-m-r} h^{m+r},$$

generalizations of the exponential and binomial series respectively.

Riemann writes down (1.4), but not (1.3), explicitly. Both series appear, much later, in the work of Heaviside‡, and (1.3) is usually referred to as "Heaviside's exponential series". Neither Riemann nor Heaviside attempts any discussion of the "validity" of the formulae. Riemann's point of view is avowedly heuristic, his object being to define D^p in such a way as to secure formal agreement with (1.1); and Heaviside, as always in his work, deliberately avoids definitions of any kind§.

The "exponential" series (1.3) has attracted some attention from mathematicians. Thus Ingham and Jeffreys have shown that it is valid asymptotically||, *i.e.*, that

$$\sum_{-M}^{\infty} \frac{z^{m+r}}{\Gamma(m+r+1)} = e^z + O(z^{r-M-1})$$

for large positive z, and I have shown that the series is summable to e^z

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† Here we have rejected a factor e^{cx} and written z for ch.

‡ *Electromagnetic theory* (1899), vol. 2, ch. 8.

§ "I have avoided defining the meaning of equivalence. The definitions will make themselves in time" (*Electromagnetic theory*, 447). They had already done so before Heaviside's volume appeared (though no doubt not when this passage was written), Borel's early writings on divergent series having been published during 1895–1899.

|| See Jeffreys, *Operational methods in mathematical physics* (Cambridge, 1929), 91–92.

by a method of the Borel type†. But nothing seems to have been written about the validity of the general formula (1.1), which I shall consider here. I shall not aim particularly at generality, which would hardly be appropriate; my object is merely to show that the formula is valid, with suitable definitions, in a few of the most obviously interesting cases.

2. We may suppose without real loss of generality that $0 < r < 1$. I shall also suppose h positive, to avoid any difficulties about the interpretation of h^r; it will appear later (§ 6) that this restriction is quite natural. We use the definitions

$$(2.1) \quad D^p f(x) = D^{-q} f(x) = I^q f(x) = \frac{1}{\Gamma(q)} \int_a^x (x-t)^{q-1} f(t)\, dt \quad (q = -p > 0),$$

$$(2.2) \quad D^{m+p} f(x) = \left(\frac{d}{dx}\right)^m D^p f(x) \quad (p < 0,\ m = 1, 2, 3, \ldots),$$

and we confine ourselves to two cases, suggested by the special cases (1.3) and (1.4). We shall be interested primarily in real x, and the many-valued functions which occur will have their real values; but it will be convenient to make assumptions about the behaviour of $f(x)$ as an analytic function of the complex variable‡.

(A) $f(x)$ is an integral function, and

$$(2.3) \quad f(x) = O(e^{c\mathfrak{R}x}) \quad (c > 0)$$

for $\mathfrak{R}x < 0$. In this case we take $a = -\infty$. Then $D^p f(x)$ is an integral function for every p,

$$(2.4) \quad D^p D^{p'} f(x) = D^{p'} D^p f(x) = D^{p+p'} f(x)$$

for all p and p', and $D^p f(x) = f^{(p)}(x)$ for positive integral p. Also, if $F(x) = D^p f(x)$, then

$$(2.5) \quad F(x+h) = \sum_0^\infty \frac{h^m}{m!} F^{(m)}(x),$$

the series converging uniformly in any finite interval of x. The typical case is $f(x) = e^{cx}$.

† *Annals of Math.*, 36 (1935), 167–181.
‡ In order to ensure the truth of (2.5).

(B) $f(x)$ is regular for $\Re x > 0$, $f(x) = O(|x|^n)$, where $n > 0$, for x small and $\Re x > 0$, and $a = 0$. Then $D^p f(x)$ is also regular for $\Re x > 0$, and (2.5) is true for $x > 0$, $|h| < x$, and in particular for $0 < h < x$. In this case we cannot assert (2.4) without reservations. But if $F(x) = D^r f(x)$, where $0 < r < 1$, then

$$(2.6) \quad F(x) = \frac{d}{dx} D^{r-1} f(x) = \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_0^x (x-t)^{-r} f(t)\, dt$$

$$= \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_0^x \frac{(x-t)^{1-r}}{1-r} f'(t)\, dt\dagger = \frac{1}{\Gamma(1-r)} \int_0^x (x-t)^{-r} f'(t)\, dt,$$

i.e. $D^r f(x) = D^{r-1} f'(x)$. The typical case is $f(x) = x^n$, with $n > 0$.

We write

$$(2.7) \qquad S = \sum_{-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)} D^{m+r} f(x) = \sum_{-\infty}^{-1} + \sum_{0}^{\infty} = S_1 + S_2,$$

and consider S_1 and S_2 separately. We shall use the formulae

$$(2.8) \qquad \begin{cases} \displaystyle\int_t^h (h-u)^{r-1} (u-t)^{-r}\, du = \frac{\pi}{\sin r\pi}, \\ \displaystyle\int_0^h (h-u)^{r-1} (x+u-t)^{-r-1}\, du = \frac{h^r}{r(x+h-t)} (x-t)^{-r}, \end{cases}$$

where $x > t$, $0 < r < 1$.

3. The series S_2 is convergent. We write

$$(3.1) \qquad F(x) = D^r f(x) = D^{r-1} f'(x) = \frac{1}{\Gamma(1-r)} \int_a^x (x-t)^{-r} f'(t)\, dt,$$

where a is $-\infty$ or 0. Then

$$h^{-r}\Gamma(r)\, S_2 = \sum_0^\infty \frac{\Gamma(r)\Gamma(m+1)}{\Gamma(m+r+1)} \frac{h^m}{m!} D^m F(x) = \sum_0^\infty \frac{h^m}{m!} D^m F(x) \int_0^1 (1-t)^{r-1} t^m\, dt$$

$$= \int_0^1 (1-t)^{r-1} \sum_0^\infty \frac{(ht)^m}{m!} D^m F(x)\, dt = \int_0^1 (1-t)^{r-1} F(x+ht)\, dt$$

$$= h^{-r} \int_0^h (h-u)^{r-1} F(x+u)\, du,$$

† Integrating by parts and observing that $f(t) \to 0$ when $t \to 0$.

since the Taylor series for $F(x+ht)$ converges uniformly, in either case, for $0 \leqslant t \leqslant 1$. Hence

$$S_2 = \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \int_a^{x+u} (x+u-t)^{-r} f'(t)\, dt$$

$$= \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \left(\int_a^x + \int_x^{x+u} \right) (x+u-t)^{-r} f'(t)\, dt = S_3 + S_4,$$

say. Here (writing $x+w$ for t)

$$S_4 = \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \int_0^u (u-w)^{-r} f'(x+w)\, du$$

$$= \frac{\sin r\pi}{\pi} \int_0^h f'(x+w)\, dw \int_w^h (h-u)^{r-1}(u-w)^{-r}\, du$$

$$= \int_0^h f'(x+w)\, dw = f(x+h) - f(x),$$

$$S_3 = \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \int_a^x (x+u-t)^{-r} f'(t)\, dt$$

$$= \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \left\{ u^{-r} f(x) - r \int_a^x (x+u-t)^{-r-1} f(t)\, dt \right\}$$

$$= f(x) - Q(x, h),$$

where

$$Q(x, h) = r\, \frac{\sin r\pi}{\pi} \int_0^h (h-u)^{r-1}\, du \int_a^x (x+u-t)^{-r-1} f(t)\, dt$$

$$= r\, \frac{\sin r\pi}{\pi} \int_a^x f(t)\, dt \int_0^h (h-u)^{r-1}(x+u-t)^{-r-1}\, du = h^r\, \frac{\sin r\pi}{\pi} \int_a^x \frac{(x-t)^{-r}}{x+h-t} f(t)\, dt,$$

by (2.8). Combining these results, we obtain

$$(3.2) \quad S_2 = f(x+h) - \frac{\sin r\pi}{\pi}\, h^r \int_a^x \frac{(x-t)^{-r}}{x+h-t} f(t)\, dt = f(x+h) - \phi(x, h),$$

say, for $x > 0$, $h > 0$ and for $0 < h < x$ in the two cases.

4. The series

$$(4.1) \qquad S_1 = \sum_{-\infty}^{-1} \frac{h^{m+r}}{\Gamma(m+r+1)} D^{m+r} f = \sum_1^{\infty} \frac{h^{r-\mu}}{\Gamma(r+1-\mu)} I^{\mu-r} f$$

is usually divergent. In case (A), however, it gives an asymptotic formula for $f(x+h)$ for large h. For

$$\sum_1^M \frac{h^{r-\mu}}{\Gamma(r+1-\mu)} I^{\mu-r} f = h^r \frac{\sin r\pi}{\pi} \sum_1^M (-1)^{\mu-1} h^{-\mu} \int_{-\infty}^x (x-t)^{\mu-r-1} f(t)\, dt$$

$$= h^r \frac{\sin r\pi}{\pi} \int_{-\infty}^x (x-t)^{-r-1} f(t) \frac{\xi+(-1)^{M+1}\xi^{M+1}}{1+\xi}\, dt = J_1+J_2,$$

where

$$\xi = \frac{x-t}{h}, \quad J_1 = h^r \frac{\sin r\pi}{\pi} \int_{-\infty}^x \frac{(x-t)^{-r}}{x+h-t} f(t)\, dt = \phi(x, h),$$

$$J_2 = (-1)^{M+1} \frac{\sin r\pi}{\pi} h^{r-M-1} \int_{-\infty}^x (x-t)^{M-r} \frac{h}{x+h-t} f(t)\, dt,$$

so that J_2 is plainly $O(h^{r-M-1})$. It follows from (3.2) that

$$\sum_{-M}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)} D^{m+r} f(x) = f(x+h) + O(h^{r-M-1}),$$

and that Riemann's series is an asymptotic series for $f(x+h)$.

5. This analysis does not apply to case (B), when h must be less than x to secure the convergence of S_2. In any case we are not concerned particularly with *large* h; and it is probably more interesting, even in case (A), to prove the series summable than to prove it asymptotic. Since S_1 usually diverges rapidly, a rather drastic method of summation will be required.

If the series
$$\sum_0^\infty a_\mu \frac{w^\mu}{\mu!}$$

is convergent for small w, the function $a(w)$ represented by it is regular for $w > 0$, and

$$\int_0^\infty e^{-w} a(w)\, dw = s,$$

then we say that Σa_μ is summable (B^*) to s. When $a(w)$ is an integral function, the method reduces to Borel's. We shall prove that the series (4.1) is summable (B^*) to sum $\phi(x, h)$, so that Riemann's series is in this sense summable to $f(x+h)$.

Here (taking $a_0 = 0$, $m = -\mu$), we have

$$a(w) = \sum_1^\infty \frac{h^{r-\mu}}{\Gamma(r-\mu+1)} \frac{w^\mu}{\mu!} I^{\mu-r} f(x)$$

$$= h^r \frac{\sin r\pi}{\pi} \sum_1^\infty (-1)^{\mu-1} \frac{(w/h)^\mu}{\mu!} \int_a^x (x-t)^{\mu-r-1} f(t)\, dt.$$

This series is majorized by a multiple of

$$\Sigma \frac{(w/h)^\mu}{\mu!} \int_a^x (x-t)^{\mu-r-1} |f(t)|\, dt = \int_a^x \frac{|f(t)|}{(x-t)^{r+1}} \{e^{(x-t)w/h}-1\}\, dt,$$

which is convergent for small w. Thus the series is convergent for small w, and

$$a(w) = h^r \frac{\sin r\pi}{\pi} \int_a^x \frac{f(t)}{(x-t)^{r+1}} \sum_1^\infty \frac{(-1)^{\mu-1}}{\mu!} \left\{\frac{w(x-t)}{h}\right\}^\mu dt$$

$$= h^r \frac{\sin r\pi}{\pi} \int_a^x \frac{f(t)}{(x-t)^{r+1}} \{1-e^{-(x-t)w/h}\}\, dt,$$

the term-by-term integration being justified by our majorization. It is plain that $a(w)$ is an integral function in case (B), while in case (A) it is regular for all positive w, and indeed for $\Re w > -ch$. Finally

$$\int_0^\infty e^{-w} a(w)\, dw = h^r \frac{\sin r\pi}{\pi} \int_0^\infty e^{-w}\, dw \int_a^x \frac{f(t)}{(x-t)^{r+1}} \{1-e^{-(x-t)w/h}\}\, dt$$

$$= h^r \frac{\sin r\pi}{\pi} \int_a^x \frac{f(t)}{(x-t)^{r+1}}\, dt \int_0^\infty e^{-w}\{1-e^{-(x-t)w/h}\}\, dw = h^r \frac{\sin r\pi}{\pi} \int_a^x \frac{(x-t)^{-r}}{x+h-t} f(t)\, dt,$$

if we can justify the change in the order of integration. Since the last integral is $\phi(x, h)$, this will complete the proof.

We may certainly invert the integrations when w is limited to a finite interval $(0, W)$. It is therefore sufficient to prove that

$$I(W) = \int_a^x \frac{f(t)}{(x-t)^{r+1}}\, dt \int_W^\infty e^{-w}\{1-e^{-(x-t)w/h}\}\, dw$$

$$= e^{-W} \int_a^x \frac{f(t)}{(x-t)^{r+1}} \left\{1-\frac{he^{-(x-t)W/h}}{x+h-t}\right\} f(t)\, dt$$

is convergent for $W > 0$ and tends to 0 when $W \to \infty$. We write

$$I(W) = e^{-W}\left(\int_a^{x-h/W} + \int_{x-h/W}^x\right) = I_1(W) + I_2(W).$$

In $I_1(W)$,

$$x-t \geqslant \frac{h}{W}, \quad (x-t)^{-r-1} \leqslant \left(\frac{W}{h}\right)^{r+1},$$

so that
$$|I_1(W)| \leqslant e^{-W} \left(\frac{W}{h}\right)^{r+1} \int_a^x |f(t)|\, dt \to 0.$$

In $I_2(W)$,

$$1-\frac{h}{x+h-t}\,e^{-(x-t)W/h}=1-\left(1-\frac{x-t}{x+h-t}\right)e^{-(x-t)W/h}$$

$$\leqslant 1-e^{-(x-t)W/h}+\frac{x-t}{h}\leqslant (W+1)\,\frac{x-t}{h}\leqslant 2W\,\frac{x-t}{h}$$

for $W>1$, so that

$$|\,I_2(W)\,|\leqslant\frac{2We^{-W}}{h}\int_a^x\frac{|\,f(t)\,|}{(x-t)^r}\,dt\to 0.$$

Thus $I(W)\to 0$.

6. If $f(x)=e^{cx}$ $(c>0)$, we obtain (1.3); if $f(x)=x^n$ $(n>0)$, we obtain (1.4). The first formula is proved for $z>0$, the second for $0<h<x$. In each case the right-hand half of the series is convergent, the left-hand half summable (B^*). In the second case the $a(w)$ of § 5 is an integral function, so that the (B^*) may be replaced by (B). In the first, $a(w)$ is regular for $\mathfrak{R}w>0$, so that the positive axis is included in its polygon of summability. Thus the series for $a(w)$ is summable (B) for $w>0$, and we may replace (B^*) by (B^2), a repeated application of Borel's method.

I add a few miscellaneous remarks. It is natural to ask whether (1.3) is not true, with a proper choice of z^r, for negative or complex z. It is not difficult to show that the formula is valid for $\mathfrak{R}z>0$ if z^r is defined as $|z|^r e^{ri\arg z}$. But difficulties arise if, for example, $z<0$. If then we interpret z^r as either $|z|^r e^{ri\pi}$ or $|z|^r e^{-ri\pi}$, the left-hand half of the series is

$$|z|^r e^{\pm ri\pi}\sum_{-\infty}^{-1}\frac{(-1)^m}{\Gamma(m+r+1)}\,|z|^m,$$

and $\Gamma(m+r+1)$ has the sign $(-1)^{m-1}$. Thus the series is a divergent series of negative terms, and so certainly not summable. It is therefore not surprising that Heaviside (p. 466) should find himself in difficulties when he tries to adapt his series to negative z. There is the same difficulty with (1.4), so that the limitations $z>0$, $x>0$, $h>0$ are more natural than they might seem at first.

If $h>x$ in (1.4), then the left-hand half of the series is convergent, the right-hand half summable. The formula has been proved only when $n>0$, this restriction being required to make $f(t)$ integrable in the definition (2.1) and $f'(t)$ in (2.6). It is true generally, and may be proved directly in various ways, but some restriction is inevital if it is to be regarded

as a case of (1.1), since we have defined $D^{\nu} f(x)$ only for integrable $f(x)$. It is not trivial even for $n = 0, 1, 2, \ldots$: thus when $n = 0$ it becomes

$$\Sigma \frac{\sin(m+r)\pi}{m+r} y^{m+r} = \pi,$$

where $y = h/x$. It is familiar that this is true for $y = 1$ (when the series converges at both ends).

The formula may be regarded as one in the theory of hypergeometric series, viz.,

$$\frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n+1)} y^{-r}(1+y)^n$$
$$= F(1, -n+r, r+1, -y) + \frac{r}{(n-r+1)y} F\left(1, -r+1, n-r+2, -\frac{1}{y}\right).$$

7. There is an integral analogue of (1.1), viz.,

$$(7.1) \qquad f(x+h) = \int_{-\infty}^{\infty} \frac{h^y}{\Gamma(y+1)} D^y f(x)\,dy:$$

the parameter r now disappears. The special cases corresponding to (1.3) and (1.4) are

$$e^z = \int_{-\infty}^{\infty} \frac{z^y}{\Gamma(y+1)}\,dy, \quad (x+h)^n = \int_{-\infty}^{\infty} \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} x^{n-y} h^y\,dy.$$

These formulae are true when interpreted appropriately, but for the moment I merely record them. We can deduce (7.1) from (1.1) heuristically by the transformations

$$\int_{-\infty}^{\infty} \frac{h^y}{\Gamma(y+1)} D^y f(x)\,dy = \sum_{-\infty}^{\infty} \int_m^{m+1} \frac{h^y}{\Gamma(y+1)} D^y f(x)\,dy$$
$$= \sum_{-\infty}^{\infty} \int_0^1 \frac{h^{m+y}}{\Gamma(m+y+1)} D^{m+y} f(x)\,dy = \int_0^1 f(x+h)\,dy = f(x+h).$$

I am indebted to Dr. L. S. Bosanquet for a number of valuable suggestions.

[*Added October* 20, 1945. Prof. G. N. Watson points out to me that the theorem concerning the asymptotic character of Heaviside's series (1.2), attributed to Ingham and Jeffreys on p. 49, is included, in a more complete form, in those proved by Barnes, "On functions defined by

simple types of hypergeometric series", *Trans. Cambridge Phil. Soc.*, 20 (1906), 253–79. Barnes proves there (pp. 268–9) that

$$e^x - \sum_{n=0}^{\infty} \frac{x^{n+\rho}}{\Gamma(n+\rho+1)} \sim \sum_{n=1}^{\infty} \frac{x^{\rho-n}}{\Gamma(\rho-n+1)},$$

the series on the right being asymptotic for $|\arg x| < \pi$.]

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CORRECTIONS

p. 52, *line* 6. For the second du read dw.
p. 54, *line* 6 *up*. Read W/h in the exponent.

COMMENTS

Riemann's fragment, dated 1847, was written when he was a student, and published posthumously in his *Gesammelte Werke*.†

The formula $(2.8)_{ii}$ may be evaluated, for $x-t < h$, by expanding the factor $(x-u-t)^{-r-1}$ in powers of $u/(x-t)$.

† 1st edn. 1876; 2nd edn. 1892; Dover edn., 1953, pp. 353–66 (Paper XIX).

ARRANGEMENT OF THE VOLUMES

VOLUME VI

VI. Theory of series

VOLUME VII

VII. 1 Integral equations and integral transforms

VII. 2 Miscellaneous papers

VII. 3 Questions from the *Educational Times*

VII. 4 Obituary notices by G. H. Hardy

VII. 5 List of other writings

LIST OF PAPERS BY G. H. HARDY

Abbreviations

N.I.C. Notes on some points in the integral calculus
D.A. Some problems of Diophantine approximation
P.N. Some problems of 'Partitio Numerorum'
N.S. Notes on the theory of series

1899

1. Question 13848, *Educational Times*, 70, 43.
2. Question 13917, *Educational Times*, 70, 78–79.
3. Question 14124, *Educational Times*, 71, 100–101.
4. Question 14005, *Educational Times*, 71, 111–112.

VII. 3

1900

1. On a class of definite integrals containing hyperbolic functions, *Messenger of Mathematics*, 29, 25–42. V
2. Question 14243, *Educational Times*, 72, 80–81.
3. Question 14271, *Educational Times*, 73, 36–37.
4. Question 14179, *Educational Times*, 73, 53–54.
5. Question 14317, *Educational Times*, 73, 61–63.

VII. 3

1901

1. On differentiation and integration under the integral sign, *Quarterly Journal of Mathematics*, 32, 66–140. (Corrected in 1915, 2.) V
2. General theorems in contour integration, with some applications, *Quarterly Journal of Mathematics*, 32, 369–384. V
3. N.I.C. I: On the formula for integration by parts, *Messenger of Mathematics*, 30, 185–187. V
4. N.I.C. II: Two general convergence theorems, *Messenger of Mathematics*, 30, 187–190. V
5. Question 14496, *Educational Times*, 74, 37–38.
6. Question 14447, *Educational Times*, 74, 98–100.
7. Question 14467, *Educational Times*, 74, 111–112.
8. Question 14028, *Educational Times*, 74, 122–123.
9. Question 14369, *Educational Times*, 75, 135–136.

VII. 3

1902

1. The elementary theory of Cauchy's principal values, *Proceedings of the London Mathematical Society*, (1) 34, 16–40. V
2. The theory of Cauchy's principal values (Second Paper: the use of principal values in some of the double limit problems of the integral calculus), *Proceedings of the London Mathematical Society*, (1) 34, 55–91. V

1902 (*cont.*)

3. On the Frullanian integral

$$\int_0^\infty \frac{\phi(ax^m)-\psi(bx^n)}{x}(\log x)^p\,dx,$$

Quarterly Journal of Mathematics, 33, 113–144. V

4. N.I.C. III: On the logarithmic criteria for the absolute convergence of an integral whose upper limit is ∞, *Messenger of Mathematics*, 31, 1–6. V
5. N.I.C. IV: On the integral $\int^\infty \sin x\psi(x)\,dx$, *Messenger of Mathematics*, 31, 6–8. V
6. A new proof of Kummer's series for log $\Gamma(a)$, *Messenger of Mathematics*, 31, 31–33. IV. 1 (*d*)
7. N.I.C. V: On absolutely convergent integrals of functions which are infinitely often infinite, *Messenger of Mathematics*, 31, 73–76. V
8. N.I.C. VI: Absolute convergence of infinite multiple integrals, *Messenger of Mathematics*, 31, 125–128. V
9. N.I.C. VII: On differentiation under the integral sign, *Messenger of Mathematics*, 31, 132–134. V
10. N.I.C. VIII: Absolutely convergent integrals of irregular types, *Messenger of Mathematics*, 31, 177–183. V
11. On the zeroes of the integral function

$$x-\sin x = \sum_1^\infty (-)^{n-1}\frac{x^{2n+1}}{2n+1!},$$

Messenger of Mathematics, 31, 161–165. IV. 1 (*a*)

12. Questions 1423, 2316, 3941, 4794, *Educational Times*, (2) 1, 25.
13. Question 14851, *Educational Times*, (2) 1, 58–59.
14. Question 14055, *Educational Times*, (2) 2, 41–42.

(12–14) VII. 3

1903

1. The theory of Cauchy's principal values (Third Paper: differentiation and integration of principal values), *Proceedings of the London Mathematical Society*, (1) 35, 81–107. V
2. On the continuity and discontinuity of definite integrals which contain a continuous parameter, *Quarterly Journal of Mathematics*, 34, 28–53. V
3. Note on the limiting values of the elliptic modular functions, *Quarterly Journal of Mathematics*, 34, 76–86. IV. 1 (*d*)
4. N.I.C. IX: On the integral $\int^\infty \{A-\phi(\sin^2 x)\}\psi(x)\,dx$, *Messenger of Mathematics*, 32, 1–3. V
5. On the zeroes of certain integral functions, *Messenger of Mathematics*, 32, 36–45. IV. 1 (*a*)
6. On the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\log(ax^2+2bx+c)^2}{\alpha x^2+2\beta x+\gamma}\,dx$, *Messenger of Mathematics*, 32, 45–50. V
7. N.I.C. X: On conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 32, 92–97. V
8. N.I.C. XI: Some conditionally convergent infinite double integrals, *Messenger of Mathematics*, 32, 159–165. V
9. N.I.C. XII: On the operation which is the inverse of double integration, *Messenger of Mathematics*, 32, 187–192. V
10. Question 14988, *Educational Times*, (2) 3, 94–95.
11. Question 14989, *Educational Times*, (2) 4, 69–70.
12. Question 15019, *Educational Times*, (2) 4, 75.
13. Question 15265, *Educational Times*, (2) 4, 109–110.

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1904

1. On the convergence of certain multiple series, *Proceedings of the London Mathematical Society*, (2) 1, 124–128. VI
2. A general theorem concerning absolutely convergent series, *Proceedings of the London Mathematical Society*, (2) 1, 285–90. VII. 2
3. On differentiation and integration of divergent series, *Transactions of the Cambridge Philosophical Society*, 19, 297–321. VI
4. Researches in the theory of divergent series and divergent integrals, *Quarterly Journal of Mathematics*, 35, 22–66. VI
5. A theorem concerning the infinite cardinal numbers, *Quarterly Journal of Mathematics*, 35, 87–94. VII. 2
6. Note on the function $\int_x^\infty e^{\frac{1}{2}(x^2-t^2)}dt$, *Quarterly Journal of Mathematics*, 35, 193–207. VII. 1
7. The asymptotic solution of certain transcendental equations, *Quarterly Journal of Mathematics*, 35, 261–282. IV. 1 (*a*)
8. N.I.C. XIII: On differentiation under the integral sign (continued), *Messenger of Mathematics*, 33, 62–67. V
9. The cardinal number of a closed set of points, *Messenger of Mathematics*, 33, 67–69. VII. 2
10. N.I.C. XIV: Integrals whose discontinuities are everywhere dense, *Messenger of Mathematics*, 33, 80–85. V
11. Note on divergent Fourier series, *Messenger of Mathematics*, 33, 137–144. III. 1 (*b*)
12. On the zeroes of two classes of Taylor series, *British Association Report*, 441–443. IV. 1 (*a*)
13. Question 15300, *Educational Times*, (2) 5, 61.
14. Additional note on Question 15282, *Educational Times*, (2) 5, 113–114.
15. Question 15361, *Educational Times*, (2) 5, 118.
16. Question 15125, *Educational Times*, (2) 6, 31.

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1905

1. On the roots of the equation $\frac{1}{\Gamma(x+1)} = c$, *Proceedings of the London Mathematical Society*, (2) 2, 1–7. IV. 1 (*a*)
2. (With T. J. I'A. Bromwich.) Some extensions to multiple series of Abel's theorem on the continuity of power series, *Proceedings of the London Mathematical Society*, (2) 2, 161–189. VI
3. Note in addition to a former paper on conditionally convergent multiple series. *Proceedings of the London Mathematical Society*, (2) 2, 190–191. VI
4. On the zeroes of certain classes of integral Taylor series. Part I: On the integral function $\sum_{n=0}^{\infty} \frac{x^{\phi(n)}}{\{\phi(n)\}!}$, *Proceedings of the London Mathematical Society*, (2) 2, 332–39. IV. 1 (*a*)
5. On the zeroes of certain classes of integral Taylor series. Part II: On the integral function $\sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}$ and other similar functions, *Proceedings of the London Mathematical Society*, (2) 2, 401–431. IV. 1 (*a*)
6. A method for determining the behaviour of certain classes of power series near a singular point on the circle of convergence, *Proceedings of the London Mathematical Society*, (2) 3, 381–389. IV. 1 (*b*)
7. On a class of analytic functions, *Proceedings of the London Mathematical Society*, (2) 3, 441–460. IV. 1 (*b*)

1905 (*cont.*)

8. On certain series of discontinuous functions connected with the modular functions, *Quarterly Journal of Mathematics*, 36, 93–123. IV. 1 (*d*)
9. Note on an integral function, *Messenger of Mathematics*, 34, 1–2. IV. 1 (*a*)
10. N.I.C. XV: On upper and lower integration, *Messenger of Mathematics*, 34, 3–6. V
11. N.I.C. XVI: A class of conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 34, 6–10. V
12. A generalization of Frullani's integral, *Messenger of Mathematics*, 34, 11–18, and note, p. 102. V
13. On the zeroes of a class of integral functions, *Messenger of Mathematics*. 34, 97–101. IV. 1 (*a*)
14. On certain conditionally convergent multiple series connected with the elliptic functions, *Messenger of Mathematics*, 34, 146–153. VI
15. Question 15686, *Educational Times*, (2) 8, 74. VII. 3
16. The expression of the double zeta function and double gamma function in terms of elliptic functions, *Transactions of the Cambridge Philosophical Society*, 20, 1–35. IV. 1 (*d*)

1906

1. On Kummer's series for log $\Gamma(a)$, *Quarterly Journal of Mathematics*, 37, 49–53. IV. 1 (*d*)
2. On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters, *Quarterly Journal of Mathematics*, 37, 53–79. IV. 1 (*d*)
3. On the function $P_\rho(x)$, *Quarterly Journal of Mathematics*, 37, 146–172 (correction at end of 1906, 5). IV. 1 (*a*)
4. On certain double integrals, *Quarterly Journal of Mathematics*, 37, 360–9. V
5. On the integral function $\Phi_{a,\alpha,\beta}(x) = \sum_0^\infty \frac{x^n}{(n+a)^{\alpha n+\beta}}$, *Quarterly Journal of Mathematics*, 37, 369–378. IV. 1 (*a*)
6. N.I.C. XVII: On the integration of series, *Messenger of Mathematics*, 35, 126–130. V
7. A formula for the prime factors of any number, *Messenger of Mathematics*, 35, 145–146. II. 1
8. N.I.C. XVIII: On some discontinuous integrals, *Messenger of Mathematics*, 35, 158–166. V
9. Some notes on certain theorems in higher trigonometry, *Mathematical Gazette*, 3, 284–288. V

1907

1. The continuum and the second number class, *Proceedings of the London Mathematical Society*, (2) 4, 10–17. VII. 2
2. Some theorems connected with Abel's theorem on the continuity of power series, *Proceedings of the London Mathematical Society*, (2) 4, 247–265. VI
3. On the singularities of functions defined by Taylor's series (Remarks in addition to a former paper), *Proceedings of the London Mathematical Society*, (2) 5, 197–205. IV. 1 (*b*)
4. The singular points of certain classes of functions of several variables, *Proceedings of the London Mathematical Society*, (2) 5, 342–360. IV. 1 (*b*)
5. On certain oscillating series, *Quarterly Journal of Mathematics*, 38, 269–288. VI
6. Some theorems concerning infinite series, *Mathematische Annalen*, 64, 77–94. VI
7. N.I.C. XIX: On Abel's lemma and the second theorem of the mean, *Messenger of Mathematics*, 36, 10–13. V
8. Higher trigonometry, *Mathematical Gazette*, 4, 13–14. VII. 2
9. A curious imaginary curve, *Mathematical Gazette*, 4, 14. VII. 2
10. The line at infinity, etc., *Mathematical Gazette*, 4, 14–15. VII. 2

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1908

1. Generalization of a theorem in the theory of divergent series, *Proceedings of the London Mathematical Society*, (2) 6, 255–264. VI
2. The multiplication of conditionally convergent series, *Proceedings of the London Mathematical Society*, (2) 6, 410–423. VI
3. Further researches in the theory of divergent series and integrals, *Transactions of the Cambridge Philosophical Society*, 21, 1–48. VI
4. (With T. J. I'A. Bromwich.) The definition of an infinite integral as the limit of a finite or infinite series, *Quarterly Journal of Mathematics*, 39, 222–240. V
5. Some multiple integrals, *Quarterly Journal of Mathematics*, 39, 357–375. V
6. N.I.C. XX: On double Frullanian integrals, *Messenger of Mathematics*, 37, 96–103. V
7. N.I.C. XXI: On a conditionally convergent multiple integral, *Messenger of Mathematics*, 37, 127–130. V
8. N.I.C. XXII: On double Frullanian integrals (cont.), *Messenger of Mathematics*, 37, 154–161. V
9. Question 16257, *Educational Times*, (2) 13, 79–80. VII. 3
10. Mendelian proportions in a mixed population, *Science* (American Association for the Advancement of Science), new series 28, 49–50. VII. 2

1909

1. A note on the continuity or discontinuity of a function defined by an infinite product, *Proceedings of the London Mathematical Society*, (2) 7, 40–48. VI
2. The theory of Cauchy's principal values (Fourth Paper: the integration of principal values—continued—with applications to the inversion of definite integrals), *Proceedings of the London Mathematical Society*, (2) 7, 181–208. V
3. On an integral equation, *Proceedings of the London Mathematical Society*, (2) 7, 445–472. VII. I
4. N.I.C. XXIII: On certain oscillating cases of Dirichlet's integral, *Messenger of Mathematics*, 38, 1–8. V
5. On certain definite integrals whose values can be expressed in terms of Bessel's functions, *Messenger of Mathematics*, 38, 129–132. V
6. N.I.C. XXIV: Oscillating cases of Dirichlet's integral (cont.), *Messenger of Mathematics*, 38, 176–185 (correction at end of 1911, 3). V
7. The integral $\int_0^\infty \frac{\sin x}{x}\,dx$, *Mathematical Gazette*, 5, 98–103. V

1910

1. The application to Dirichlet's series of Borel's exponential method of summation, *Proceedings of the London Mathematical Society*, (2) 8, 277–294. VI
2. The ordinal relations of the terms of a convergent sequence, *Proceedings of the London Mathematical Society*, (2) 8, 295–300. VII. 2
3. Theorems relating to the summability and convergence of slowly oscillating series, *Proceedings of the London Mathematical Society*, (2) 8, 301–320. VI
4. The maximum modulus of an integral function, *Quarterly Journal of Mathematics*, 41, 1–9. IV. 2
5. On certain definite integrals considered by Airy and by Stokes, *Quarterly Journal of Mathematics*, 41, 226–240. IV. 1 (*d*)
6. N.I.C. XXV: Absolutely convergent integrals of irregular types (cont.), *Messenger of Mathematics*, 39, 28–32. V
7. The zeroes of the integral function $\sum \frac{x^{n^2}}{n^2!}$, and of some similar functions, *Messenger of Mathematics*, 39, 88–96. IV. 1 (*a*)

1910 (*cont.*)

8. N.I.C. XXVI: On a case of term-by-term integration of an infinite series, *Messenger of Mathematics*, 39, 136–139. V
9. To find an approximation to the large positive root of the equation $e^{e^x} = 10^{10}x^{10}e^{10^{10}x^{10}}$, *Mathematical Gazette*, 5, 333–334. VII. 2

1911

1. Theorems connected with Maclaurin's test for the convergence of series, *Proceedings of the London Mathematicsl Society*, (2), 9, 126–144. VI
2. (With S. Chapman.) A general view of the theory of summable series, *Quarterly Journal of Mathematics*, 42, 181–215. VI
3. N.I.C. XXVII: Oscillating cases of Dirichlet's integral (cont.), *Messenger of Mathematics*, 40, 44–53. V
4. A class of definite integrals, *Messenger of Mathematics*, 40, 53–54. V
5. N.I.C. XXVIII: A conditionally convergent double integral, *Messenger of Mathematics*, 40, 62–69. V
6. N.I.C. XXIX: Two convergence theorems, *Messenger of Mathematics*, 40, 87–91. V
7. N.I.C. XXX: A theorem concerning summable integrals, *Messenger of Mathematics*, 40, 108–112. VI
8. N.I.C. XXXI: The uniform convergence of Borel's integral, *Messenger of Mathematics*, 40, 161–165. VI
9. Fourier's double integral and the theory of divergent integrals, *Transactions of the Cambridge Philosophical Society*, 21, 427–451. VII. 1

1912

1. Properties of logarithmico-exponential functions, *Proceedings of the London Mathematical Society*, (2) 10, 54–90. IV. 1 (*c*)
2. On the multiplication of Dirichlet's series, *Proceedings of the London Mathematical Society*, (2) 10, 396–405. VI
3. Some results concerning the behaviour at infinity of a real and continuous solution of an algebraic differential equation of the first order, *Proceedings of the London Mathematical Society*, (2) 10, 451–468. VII. 2
4. (With J. E. L.) Some problems of Diophantine approximation, *5th International Congress of Mathematicians, Cambridge*, 1, 223–229. I. 1
5. Generalizations of a limit theorem of Mr. Mercer, *Quarterly Journal of Mathematics*, 43, 143–150. VI
6. Note on Dr. Vacca's series for γ, *Quarterly Journal of Mathematics*, 43, 215–216. IV. 1 (*d*)
7. Note on a theorem of Cesàro, *Messenger of Mathematics*, 41, 17–22. VI
8. N.I.C. XXXII: On double series and double integrals, *Messenger of Mathematics*, 41, 44–48. V
9. N.I.C. XXXIII: Some cases of the inversion of the order of integration, *Messenger of Mathematics*, 41, 102–109. VII. 1

1913

1. (With J. E. L.) The relations between Borel's and Cesàro's methods of summation, *Proceedings of the London Mathematical Society*, (2) 11, 1–16. VI
2. (With J. E. L.) Contributions to the arithmetic theory of series, *Proceedings of the London Mathematical Society*, (2) 11, 411–478. VI
3. An extension of a theorem on oscillating series, *Proceedings of the London Mathematical Society*, (2) 12, 174–180. VI

1913 (*cont.*)

4. On the summability of Fourier's series, *Proceedings of the London Mathematical Society*, 12, 365–372. III. 1 (*b*)
5. Oscillating Dirichlet's integrals: an essay in the 'Infinitärcalcül' of Paul Du Bois-Reymond, *Quarterly Journal of Mathematics*, 44, 1–40. IV. 1 (*c*)
6. A theorem concerning Taylor's series, *Quarterly Journal of Mathematics*, 44, 147–60. IV. 2
7. Oscillating Dirichlet's integrals (Second Paper), *Quarterly Journal of Mathematics*, 44, 242–263. IV. 1 (*c*)
8. N.I.C. XXXIV: Absolutely convergent integrals of irregular types (cont.), *Messenger of Mathematics*, 42, 13–18. V
9. N.I.C. XXXV: On an integral equation, *Messenger of Mathematics*, 42, 89–93. VII. 1
10. (With J. E. L.) Tauberian theorems concerning series of positive terms, *Messenger of Mathematics*, 42, 191–192. VI
11. (With J. E. L.) Sur la série de Fourier d'une fonction à carré sommable, *Comptes Rendus*, 156, 1307–1309. III. 1 (*b*)

1914

1. Sur les zéros de la fonction $\zeta(s)$ de Riemann, *Comptes Rendus*, 158, 1012–1014. II. 1
2. (With J. E. L.) D.A. I: The fractional part of $n^k\theta$, *Acta Mathematica*, 37, 155–191. I. 1
3. (With J. E. L.) D.A. II: The trigonometrical series associated with the elliptic ϑ-functions, *Acta Mathematica*, 37, 193–239. I. 1
4. (With J. E. L.) Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, *Proceedings of the London Mathematical Society*, (2) 13, 174–191. VI
5. Note on Lambert's series, *Proceedings of the London Mathematical Society*, (2) 13, 192–198. VI
6. Note in addition to a paper on Taylor's series, *Quarterly Journal of Mathematics*, 45, 77–84. IV. 2
7. A function of two variables, *Quarterly Journal of Mathematics*, 45, 85–113. IV. 1 (*b*)
8. N.I.C. XXXVI: On the asymptotic values of certain integrals, *Messenger of Mathematics*, 43, 9–13. V
9. N.I.C. XXXVII: On the region of convergence of Borel's integral, *Messenger of Mathematics*, 43, 22–24. VI
10. N.I.C. XXXVIII: On the definition of an analytic function by means of a definite integral, *Messenger of Mathematics*, 43, 29–33. IV. 2
11. (With J. E. L.) Some theorems concerning Dirichlet's series, *Messenger of Mathematics*, 43, 134–147. VI

1915

1. (With J. E. L.) New proofs of the prime-number theorem and similar theorems, *Quarterly Journal of Mathematics*, 46, 215–219. II. 1
2. Correction of an error, *Quarterly Journal of Mathematics*, 46, 261–262. V
3. On the expression of a number as the sum of two squares, *Quarterly Journal of Mathematics*, 46, 263–283. II. 2
4. The mean value of the modulus of an analytic function, *Proceedings of the London Mathematical Society*, (2) 14, 269–277. III. 2
5. Proof of a formula of Mr. Ramanujan, *Messenger of Mathematics*, 44, 18–21. V
6. N.I.C. XXXIX: Further examples of conditionally convergent infinite double integrals, *Messenger of Mathematics*, 44, 57–63. V
7. N.I.C. XL: Some cases of term-by-term integration of an infinite series, *Messenger of Mathematics*, 44, 145–149. V

1915 (*cont.*)

8. N.I.C. XLI: On the convergence of certain integrals and series, *Messenger of Mathematics*, 44, 163–166. V
9. Sur le problème des diviseurs de Dirichlet, *Comptes Rendus*, 160, 617–619. II. 2
10. Prime numbers, *British Association Report*, 350–354. II. 1
11. Example to illustrate a point in the theory of Dirichlet's series, *The Tôhoku Mathematical Journal*, 8, 59–66. VI
12. The definition of a complex number, *Mathematical Gazette*, 8, 48–49. VII. 2

1916

1. The application of Abel's method of summation to Dirichlet's series, *Quarterly Journal of Mathematics*, 47, 176–192. VI
2. Weierstrass's non-differentiable function, *Transactions of the American Mathematical Society*, 17, 301–325. IV. 1 (*d*)
3. (With J. E. L.) D.A.: A remarkable trigonometrical series, *Proceedings of the National Academy of Sciences*, 2, 583–586. I. 1
4. On Dirichlet's divisor problem, *Proceedings of the London Mathematical Society*, (2) 15, 1–25. II. 2
5. The second theorem of consistency for summable series, *Proceedings of the London Mathematical Society*, (2) 15, 72–88. VI
6. The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, *Proceedings of the London Mathematical Society*, (2) 15, 192–213. II. 2
7. Sur la sommation des séries de Dirichlet, *Comptes Rendus*, 162, 463–465. VI
8. (With J. E. L.) Theorems concerning the summability of series by Borel's exponential method, *Rendiconti del Circolo matematico di Palermo*, 41, 36–53. VI
9. (With J. E. L.) D.A.: The series $\Sigma\, e(\lambda_n)$ and the distribution of the points $(\lambda_n \alpha)$, *Proceedings of the National Academy of Sciences*, 3, 84–88. I. 1
10. Asymptotic formulae in combinatory analysis, *Quatrième Congrès des Mathématiciens Scandinaves*, 45–53. I. 2 (*a*)
11. Further remarks on the integral $\int_0^\infty \frac{\sin x}{x}\,dx$, *Mathematical Gazette*, 8, 301–303. V

1917

1. (With S. Ramanujan) Une formule asymptotique pour le nombre des partitions de n, *Comptes Rendus*, 164, 35–38. I. 2 (*a*)
2. On a theorem of Mr G. Pólya, *Proceedings of the Cambridge Philosophical Society*, 19, 60–63. IV. 2
3. On the convergence of certain multiple series, *Proceedings of the Cambridge Philosophical Society*, 19, 86–95. VI
4. (With S. Ramanujan) Asymptotic formulae for the distribution of integers of various types, *Proceedings of the London Mathematical Society*, (2) 16, 112–132. I. 2 (*a*)
5. N.I.C. XLII: On Weierstrass's singular integral, and on a theorem of Lerch, *Messenger of Mathematics*, 46, 43–48. VII. 1
6. N.I.C. XLIII: On the asymptotic value of a definite integral, and the coefficient in a power series, *Messenger of Mathematics*, 46, 70–73. V
7. N.I.C. XLIV: On certain multiple integrals and series which occur in the analytic theory of numbers, *Messenger of Mathematics*, 46, 104–107. V
8. N.I.C. XLV: On a point in the theory of Fourier series, *Messenger of Mathematics*, 46, 146–149. III. 1 (*a*)
9. N.I.C. XLVI: On Stieltjes' 'problème des moments', *Messenger of Mathematics*, 46, 175–182. VII. 1

1917 (*cont.*)

10. (With J. E. L.) Sur la convergence des séries de Fourier et des séries de Taylor, *Comptes Rendus*, 165, 1047–1049. III. 1 (*a*)
11. Mr. S. Ramanujan's mathematical work in England, *Journal of the Indian Mathematical Society*, 9, 30–45. VII. 2

1918

1. (With J. E. L.) Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Mathematica*, 41, 119–196. II. 1
2. (With S. Ramanujan) On the coefficients in the expansions of certain modular functions, *Proceedings of the Royal Society*, (A) 95, 144–155. I. 2 (*a*)
3. Sir George Stokes and the concept of uniform convergence, *Proceedings of the Cambridge Philosophical Society*, 19, 148–156. VII. 2
4. (With J. E. L.) On the Fourier series of a bounded function, *Proceedings of the London Mathematical Society*, (2) 17, xiii–xv. III. 1 (*b*)
5. (With S. Ramanujan) Asymptotic formulae in combinatory analysis, *Proceedings of the London Mathematical Society*, (2) 17, 75–115. I. 2 (*a*)
6. N.I.C. XLVII: On Stieltjes' 'problème des moments' (cont.), *Messenger of Mathematics*, 47, 81–88. VII. 1
7. N.I.C. XLVIII: On some properties of integrals of fractional order, *Messenger of Mathematics*, 47, 145–150. V
8. N.I.C. XLIX: On Mellin's inversion formula, *Messenger of Mathematics*, 47, 178–184. VII. 1
9. Note on an expression of Lambert's series as a definite integral, *Messenger of Mathematics*, 47, 190–192. IV. 1 (*d*)
10. On the representation of a number as the sum of any number of squares, and in particular of five or seven, *Proceedings of the National Academy of Sciences*, 4, 189–193. I. 2 (*a*)

1919

1. (With J. E. L.) Note on Messrs. Shah and Wilson's paper entitled: 'On an empirical formula connected with Goldbach's theorem', *Proceedings of the Cambridge Philosophical Society*, 19, 245–254. I. 2 (*c*)
2. N.I.C. L. On the integral of Stieltjes and the formula for integration by parts, *Messenger of Mathematics*, 48, 90–100. V
3. N.I.C. LI: On Hilbert's double-series theorem, and some connected theorems concerning the convergence of infinite series and integrals, *Messenger of Mathematics*, 48, 107–112. II. 3
4. A problem of Diophantine approximation, *Journal of the Indian Mathematical Society*, 11, 162–166. I. 1

1920

1. (With S. Ramanujan) The normal number of prime factors of a number n, *Quarterly Journal of Mathematics*, 48, 76–92. II. 1
2. (With J. E. L.) A new solution of Waring's problem, *Quarterly Journal of Mathematics*, 48, 272–293. I. 2 (*b*)
3. Note on a theorem of Hilbert, *Mathematische Zeitschrift*, 6, 314–317. II. 3
4. On two theorems of F. Carlson and S. Wigert, *Acta Mathematica*, 42, 327–339. IV. 2
5. (With J. E. L.) P.N. I: A new solution of Waring's problem, *Göttinger Nachrichten* (1920), 33–54. I. 2 (*b*)
6. Additional note on two problems in the analytic theory of numbers, *Proceedings of the London Mathematical Society*, (2) 18, 201–204. II. 2
7. (With J. E. L.) Abel's theorem and its converse, *Proceedings of the London Mathematical Society*, (2) 18, 205–235. VI

1920 (*cont.*)

8. N.I.C. LII: On some definite integrals considered by Mellin, *Messenger of Mathematics*, 49, 85–91. VII. 1

9. N.I.C. LIII: On certain criteria for the convergence of the Fourier series of a continuous function, *Messenger of Mathematics*, 49, 149–155. III. 1 (*a*)

10. On the representation of a number as the sum of any number of squares, and in particular of five, *Transactions of the American Mathematical Society*, 21, 255–284. I. 2 (*a*)

11. *Some famous problems of the theory of numbers and in particular Waring's problem*, Inaugural lecture, Oxford, 1920. I. 2 (*d*)

1921

1. (With J. E. L.) P.N. II: Proof that every large number is the sum of at most 21 biquadrates, *Mathematische Zeitschrift*, 9, 14–27. I. 2 (*b*)

2. (With J. E. L.) The zeros of Riemann's zeta-function on the critical line, *Mathematische Zeitschrift*, 10, 283–317. II. 1

3. Note on Ramanujan's trigonometrical function $c_q(n)$, and certain series of arithmetical functions, *Proceedings of the Cambridge Philosophical Society*, 20, 263–271. II. 2

4. A theorem concerning summable series, *Proceedings of the Cambridge Philosophical Society*, 20, 304–307. VI

5. A convergence theorem, *Proceedings of the London Mathematical Society*, (2) 19, vi–vii. II. 3

6. (With J. E. L.) On a Tauberian theorem for Lambert's series, and some fundamental theorems in the analytic theory of numbers, *Proceedings of the London Mathematical Society*, (2) 19, 21–29. II. 1

7. N.I.C. LIV: Further notes on Mellin's inversion formulae, *Messenger of Mathematics*, 50, 165–171. VII. 1

1922

1. Goldbach's Theorem, *Matematisk Tidsskrift B*, 1–16. I. 2 (*c*)

2. A new proof of the functional equation for the zeta-function, *Matematisk Tidsskrift B*, 71–73. II. 1

3. (With J. E. L.) P.N. III: On the expression of a number as a sum of primes, *Acta Mathematica*, 44, 1–70. I. 2 (*c*)

4. (With J. E. L.) P.N. IV: The singular series in Waring's problem and the value of the number $G(k)$, *Mathematische Zeitschrift*, 12, 161–188. I. 2 (*b*)

5. (With J. E. L.) D.A.: A further note on the trigonometrical series associated with the elliptic theta-functions, *Proceedings of the Cambridge Philosophical Society*, 21, 1–5. I. 1

6. (With J. E. L.) D.A.: The lattice-points of a right-angled triangle, *Proceedings of the London Mathematical Society*, (2) 20, 15–36. I. 1

7. (With T. Carleman) Fourier's series and analytic functions, *Proceedings of the Royal Society*, (A), 101, 124–133. IV. 2

8. (With J. E. L.) Summation of a certain multiple series, *Proceedings of the London Mathematical Society*, (2) 20, xxx. I. 2 (*c*)

9. (With J. E. L.) D.A.: The lattice-points of a right-angled triangle, *Hamburg Abhandlungen*, 1, 212–249. I. 1

10. N.I.C. LV: On the integration of Fourier series, *Messenger of Mathematics*, 51, 186–192. III. 1 (*e*)

11. The theory of numbers, *British Association Report*, 90, 16–24. VII. 2

1923

1. (With J. E. L.) On Lindelöf's hypothesis concerning the Riemann zeta-function, *Proceedings of the Royal Society*, (A) 103, 403–412. II. 1

1923 *(cont.)*

2. A chapter from Ramanujan's notebook, *Proceedings of the Cambridge Philosophical Society*, 21, 492–503. IV. 1 (*d*)
3. (With J. E. L.) D.A.: The analytic character of the sum of a Dirichlet's series considered by Hecke, *Hamburg Abhandlungen* 3, 57–68. I. 1
4. (With J. E. L.) D.A.: The analytic properties of certain Dirichlet's series associated with the distribution of numbers to modulus unity, *Transactions of the Cambridge Philosophical Society*, 22, 519–533. I. 1
5. (With J. E. L.) The approximate functional equation in the theory of the zeta-function with applications to the divisor-problems of Dirichlet and Piltz, *Proceedings of the London Mathematical Society*, (2) 21, 39–74. II. 1
6. N.I.C. LVI: On Fourier's series and Fourier's integral, *Messenger of Mathematics*, 52, 49–53. III. 1 (*e*)

1924

1. (With J. E. L.) Solution of the Cesàro summability problem for power-series and Fourier series, *Mathematische Zeitschrift*, 19, 67–96. III. 1 (*b*)
2. Some formulae of Ramanujan, *Proceedings of the London Mathematical Society*, (2) 22, xii–xiii. IV. 1 (*d*)
3. (With J. E. L.) Note on a theorem concerning Fourier series, *Proceedings of the London Mathematical Society*, (2) 22, xviii–xix. III. 1 (*b*)
4. (With J. E. L.) The equivalence of certain integral means, *Proceedings of the London Mathematical Society*, (2) 22, xl–xliii. VI
5. (With J. E. L.) The allied series of a Fourier series, *Proceedings of the London Mathematical Society*, (2) 22, xliii–xlv. III. 1 (*b*)
6. (With J. E. L.) P.N. V: A further contribution to the study of Goldbach's problem, *Proceedings of the London Mathematical Society* (2) 22, 46–56. I. 2 (*c*)
7. (With J. E. L.) Abel's theorem and its converse II, *Proceedings of the London Mathematical Society*, (2) 22, 254–269. VI
8. N.I.C. LVII: On Fourier transforms, *Messenger of Mathematics*, 53, 135–142. VII. 1
9. (With E. Landau) The lattice points of a circle, *Proceedings of the Royal Society* (A), 105, 244–258. II. 2

1925

1. (With J. E. L.) P.N. VI: Further researches in Waring's problem, *Mathematische Zeitschrift*, 23, 1–37. I. 2 (*b*)
2. The lattice points of a circle, *Proceedings of the Royal Society* (A), 107, 623–635. II. 2
3. What is geometry? *Mathematical Gazette*, 12, 309–316. VII. 2
4. (With J. E. L.) D.A.: An additional note on the trigonometrical series associated with the elliptic theta-functions, *Acta Mathematica*, 47, 189–198. I. 1
5. (With J. E. L.) A theorem concerning series of positive terms, with applications to the theory of functions, *Meddelelser København*, 7, Nr. 4. IV. 2
6. Note on a theorem of Hilbert concerning series of positive terms, *Proceedings of the London Mathematical Society*, (2) 23, xlv–xlvi. II. 3
7. Some formulae in the theory of Bessel functions, *Proceedings of the London Mathematical Society*, (2) 23, lxi–lxiii. IV. 1 (*d*)
8. (With E. C. Titchmarsh) Solutions of some integral equations considered by Bateman, Kapteyn, Littlewood and Milne, *Proceedings of the London Mathematical Society*, (2) 23, 1–26, and Correction ibid. 24, xxxi–xxxiii. VII. 1
9. N.I.C. LVIII: On Hilbert transforms, *Messenger of Mathematics*, 54, 20–27. VII. 1
10. N.I.C. LIX: On Hilbert transforms (cont.), *Messenger of Mathematics*, 54, 81–88. VII. 1
11. N.I.C. LX: An inequality between integrals, *Messenger of Mathematics*, 54, 150–156. II. 3

1926

1. A definite integral which occurs in physical optics, *Proceedings of the London Mathematical Society*, (2) 24, xxx–xxxi. IV. 1 (*d*)
2. (With J. E. L.) Some properties of fractional integrals, *Proceedings of the London Mathematical Society*, (2) 24, xxxvii–xli. III. 2
3. Note on the inversion of a repeated integral, *Proceedings of the London Mathematical Society*, (2) 24, l–li. V
4. (With J. E. L.) The allied series of a Fourier series, *Proceedings of the London Mathematical Society*, (2) 24, 211–246. III. 1 (*b*)
5. (With J. E. L.) A further note on the converse of Abel's theorem, *Proceedings of the London Mathematical Society*, (2) 25, 219–236. VI
6. (With J. E. L. and G. Pólya) The maximum of a certain bilinear form, *Proceedings of the London Mathematical Society*, (2) 25, 265–282. II. 3
7. (With J. E. L.) Some new properties of Fourier constants, *Mathematische Annalen*, 97, 159–209. III. 1 (*c*)
8. (With J. E. L.) N.S. I: Two theorems concerning Fourier series, *Journal of the London Mathematical Society*, 1, 19–25. III. 1 (*a*)
9. A theorem concerning harmonic functions, *Journal of the London Mathematical Society*, 1, 130–131. IV. 2
10. (With J. E. L.) N.S. II: The Fourier series of a positive function, *Journal of the London Mathematical Society*, 1, 134–138. III. 1 (*b*)
11. (With S. Bochner) Notes on two theorems of Norbert Wiener, *Journal of the London Mathematical Society*, 1, 240–244. VII. 1
12. N.I.C. LXI: On the term by term integration of a series of Bessel functions, *Messenger of Mathematics*, 55, 140–144. IV. 1 (*d*)
13. The case against the Mathematical Tripos, *Mathematical Gazette*, 13, 61–71. VII. 2

1927

1. Note on Ramanujan's arithmetical function $\tau(n)$, *Proceedings of the Cambridge Philosophical Society*, 23, 675–680. II. 2
2. (With J. E. L.) N.S. III: On the summability of the Fourier series of a nearly continuous function, *Proceedings of the Cambridge Philosophical Society*, 23, 681–684. III. 1 (*b*)
3. (With J. E. L.) N.S. IV: On the strong summability of Fourier series, *Proceedings of the London Mathematical Society*, (2) 26, 273–286. III. 1 (*b*)
4. (With J. E. L.) N.S. V: On Parseval's theorem, *Proceedings of the London Mathematical Society*, (2) 26, 287–294. III. 1 (*e*)
5. (With A. E. Ingham and G. Pólya) Theorems concerning mean values of analytic functions, *Proceedings of the Royal Society* (A), 113, 542–569. IV. 2
6. (With J. E. L.) Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, *Journal für Mathematik* 157, 141–158. II. 3
7. N.I.C. LXII: A singular integral, *Messenger of Mathematics*, 56, 10–16. VII. 1
8. N.I.C. LXIII: Some further applications of Mellin's inversion formula, *Messenger of Mathematics*, 56, 186–192. VII. 1
9. Note on a theorem of Mertens, *Journal of the London Mathematical Society*, 2, 70–72. II. 1
10. Note on the multiplication of series, *Journal of the London Mathematical Society*, 2, 169–171. VI
11. (With J. E. L.) N.S. VI: Two inequalities, *Journal of the London Mathematical Society*, 2, 196–201. II. 3

1928

1. (With J. E. L.) A theorem in the theory of summable divergent series, *Proceedings of the London Mathematical Society*, (2) 27, 327–348. VI
2. (With A. E. Ingham and G. Pólya) Notes on moduli and mean values, *Proceedings of the London Mathematical Society*, (2) 27, 401–409. IV. 2
3. (With J. E. L.) N.S. VII: On Young's convergence criterion for Fourier series, *Proceedings of the London Mathematical Society*, (2) 28, 301–311. III. 1 (*b*)
4. (With J. E. L.) P.N. VIII: The number $\Gamma(k)$ in Waring's problem, *Proceedings of the London Mathematical Society*, (2) 28, 518–542. I. 2 (*b*)
5. (With J. E. L.) Some properties of fractional integrals I, *Mathematische Zeitschrift*, 27, 565–606. III. 2
6. (With J. E. L.) A convergence criterion for Fourier series, *Mathematische Zeitschrift*, 28, 612–634. III. 1 (*a*)
7. N.I.C. LXIV: Further inequalities between integrals, *Messenger of Mathematics*, 57, 12–16. II. 3
8. N.I.C. LXV: A discontinuous integral, *Messenger of Mathematics*, 57, 113–120. IV. 1 (*d*)
9. A theorem concerning trigonometrical series, *Journal of the London Mathematical Society*, 3, 12–13. III. 1 (*d*)
10. (With J. E. L.) N.S. VIII: An inequality, *Journal of the London Mathematical Society*, 3, 105–110. II. 3
11. Remarks on three recent notes in the *Journal*, *Journal of the London Mathematical Society*, 3, 166–169. II. 3
12. A formula of Ramanujan, *Journal of the London Mathematical Society*, 3, 238–240. IV. 1 (*d*)
13. (With J. E. L.) N.S. IX: On the absolute convergence of Fourier series, *Journal of the London Mathematical Society*, 3, 250–253. III. 1 (*a*)
14. (With J. E. L.) N.S. X: Some more inequalities, *Journal of the London Mathematical Society*, 3, 294–299. II. 3

1929

1. (With J. E. L.) The approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$, *Proceedings of the London Mathematical Society*, (2) 29, 81–97. II. 1
2. Prolegomena to a chapter on inequalities (Presidential Address), *Journal of the London Mathematical Society*, 4, 61–78, and addenda, ibid. 5, 80. II. 3
3. Remarks in addition to Dr. Widder's note on inequalities, *Journal of the London Mathematical Society*, 4, 199–202. II. 3
4. (With J. E. L.) A point in the theory of conjugate functions, *Journal of the London Mathematical Society*, 4, 242–245. III. 1 (*e*)
5. (With E. C. Titchmarsh) Solution of an integral equation, *Journal of the London Mathematical Society*, 4, 300–304. VII. 1
6. An introduction to the theory of numbers, *Bulletin of the American Mathematical Society*, 35, 778–818. VII. 2
7. N.I.C. LXVI: The arithmetic mean of a Fourier constant, *Messenger of Mathematics*, 58, 50–52. III. 1 (*e*)
8. N.I.C. LXVII: On the repeated integral which occurs in the theory of conjugate functions, *Messenger of Mathematics*, 58, 53–58. VII. 1
9. N.I.C. LXVIII: The limit of an integral mean value, *Messenger of Mathematics*, 58, 115–120. II. 3
10. N.I.C. LXIX: On asymptotic values of Fourier constants, *Messenger of Mathematics*, 58, 130–135. III. 1 (*a*)
11. (With J. E. L. and G. Pólya) Some simple inequalities satisfied by convex functions, *Messenger of Mathematics*, 58, 145–152. II. 3
12. Mathematical proof, *Mind*, 38, 1–25. VII. 2

1930

1. (With J. E. L.) A maximal theorem with function-theoretic applications, *Acta Mathematica*, 54, 81–116. II. 3
2. (With E. C. Titchmarsh) Self-reciprocal functions, *Quarterly Journal of Mathematics*, 1, 196–231. VII. 1
3. (With J. E. L.) D.A.: A series of cosecants, *Bulletin of the Calcutta Mathematical Society*, 20, 251–266. I. 1
4. (With J. E. L.) N.S. XI: On Tauberian theorems, *Proceedings of the London Mathematical Society*, (2) 30, 23–37. VI
5. (With E. C. Titchmarsh) Additional note on certain integral equations, *Proceedings of the London Mathematical Society*, (2) 30, 95–106. VII. 1
6. (With J. E. L.) N.S. XII: On certain inequalities connected with the calculus of variations, *Journal of the London Mathematical Society*, 5, 34–39. II. 3

1931

1. Some theorems concerning trigonometrical series of a special type, *Proceedings of the London Mathematical Society*, (2) 32, 441–448. III. 1 (*d*)
2. (With J. E. L.) Some properties of conjugate functions, *Journal für Mathematik*, 167, 405–423. III. 2
3. The summability of a Fourier series by logarithmic means, *Quarterly Journal of Mathematics*, 2, 107–112. III. 1 (*b*)
4. (With J. E. L.) N.S. XIII: Some new properties of Fourier constants, *Journal of the London Mathematical Society*, 6, 3–9. III. 1 (*c*)
5. (With J. E. L.) N.S. XIV: An additional note on the summability of Fourier series, *Journal of the London Mathematical Society*, 6, 9–12. III. 1 (*b*)
6. (With E. C. Titchmarsh) A note on Parseval's theorem for Fourier transforms, *Journal of the London Mathematical Society*, 6, 44–48. VII. 1
7. (With J. E. L.) N.S. XV: On the series conjugate to the Fourier series of a bounded function, *Journal of the London Mathematical Society*, 6, 278–281. III. 1 (*b*)
8. (With J. E. L.) N.S. XVI: Two Tauberian theorems, *Journal of the London Mathematical Society*, 6, 281–286. VI

1932

1. (With E. C. Titchmarsh) Formulae connecting different classes of self-reciprocal functions, *Proceedings of the London Mathematical Society*, (2) 33, 225–232. VII. 1
2. On Hilbert transforms, *Quarterly Journal of Mathematics*, 3, 102–112. VII. 1
3. (With J. E. L.) Some integral inequalities connected with the calculus of variations, *Quarterly Journal of Mathematics*, 3, 241–252. II. 3
4. (With J. E. L.) Some properties of fractional integrals II, *Mathematische Zeitschrift*, 34, 403–439. III. 2
5. (With J. E. L.) Some new cases of Parseval's theorem, *Mathematische Zeitschrift*, 34, 620–633. III. 1 (*c*)
6. (With J. E. L.) An additional note on Parseval's theorem, *Mathematische Zeitschrift*, 34, 634–636. III. 1 (*c*)
7. (With E. C. Titchmarsh) An integral equation, *Proceedings of the Cambridge Philosophical Society*, 28, 165–173. VII. 1
8. Summation of a series of polynomials of Laguerre, *Journal of the London Mathematical Society*, 7, 138–139, and addendum, ibid. 192. III. 1 (*e*)
9. (With J. E. L.) N.S. XVII: Some new convergence criteria for Fourier series, *Journal of the London Mathematical Society*, 7, 252–256. III. 1 (*a*)

1933

1. (With E. C. Titchmarsh) A class of Fourier kernels, *Proceedings of the London Mathematical Society*, (2) 35, 116–155. VII. 1
2. (With J. E. L.) Some more integral inequalities, *The Tôhoku Mathematical Journal*, 37, 151–159. II. 3
3. The constants of certain inequalities, *Journal of the London Mathematical Society*, 8, 114–119. II. 3
4. A theorem concerning Fourier transforms, *Journal of the London Mathematical Society*, 8, 227–231. VII. 1

1934

1. (With J. E. L.) Theorems concerning Cesàro means of power series, *Proceedings of the London Mathematical Society*, (2) 36, 516–531. III. 2
2. (With J. E. L.) Bilinear forms bounded in space [*p*, *q*], *Quarterly Journal of Mathematics*, 5, 241–254. II. 3
3. (With J. E. L.) Some new convergence criteria for Fourier series, *Annali Pisa*, (2) 3, 43–62. III. 1 (*a*)
4. (With E. M. Wright) Leudesdorf's extension of Wolstenholme's theorem, *Journal of the London Mathematical Society*, 9, 38–41, and corrigendum, ibid. 240. II. 2
5. On the summability of series by Borel's and Mittag-Leffler's methods, *Journal of the London Mathematical Society*, 9, 153–157. VI

1935

1. Remarks on some points in the theory of divergent series, *Annals of Mathematics*, (2) 36, 167–181. VI
2. The resultant of two Fourier kernels, *Proceedings of the Cambridge Philosophical Society*, 31, 1–6. VII. 1
3. (With E. Landau and J. E. L.) Some inequalities satisfied by the integrals or derivatives of real or analytic functions, *Mathematische Zeitschrift*, 39, 677–695. II. 3
4. (With J. E. L.) An inequality, *Mathematische Zeitschrift*, 40, 1–40. II. 3
5. (With J. E. L.) The strong summability of Fourier series, *Fundamenta Mathematicae*, 25, 162–189. III. 1 (*b*)
6. (With J. E. L.) N.S. XVIII: On the convergence of Fourier series, *Proceedings of the Cambridge Philosophical Society*, 31, 317–323. III. 1 (*c*)
7. (With J. E. L.) N.S. XIX: A problem concerning majorants of Fourier series, *Quarterly Journal of Mathematics*, 6, 304–315. III. 1 (*c*)
8. Second note on a theorem of Mertens, *Journal of the London Mathematical Society*, 10, 91–94. II. 1
9. Some identities satisfied by infinite series, *Journal of the London Mathematical Society*, 10, 217–220. IV. 1 (*d*)

1936

1. (With J. E. L.) N.S. XX: On Lambert series, *Proceedings of the London Mathematical Society*, (2) 41, 257–270. VI
2. (With J. E. L.) Some more theorems concerning Fourier series and Fourier power series, *Duke Mathematical Journal*, 2, 354–382. III 1 (*b*)
3. (With E. C. Titchmarsh) New solution of an integral equation, *Proceedings of the London Mathematical Society*, (2) 41, 1–15. VII. 1
4. A note on two inequalities, *Journal of the London Mathematical Society*, 11, 167–170. II. 3

1937

1. Ramanujan and the theory of Fourier transforms, *Quarterly Journal of Mathematics*, 8, 245–254. VII. 1

2. The Indian mathematician Ramanujan, *American Mathematical Monthly*, 44, 137–155. VII. 2

3. (With J. E. L.) N.S. XXI: Generalizations of a theorem of Paley, *Quarterly Journal of Mathematics*, 8, 161–171. III. 2

4. On a theorem of Paley and Wiener, *Proceedings of the Cambridge Philosophical Society*, 33, 1–5. VII. 1

5. (With N. Levinson) Inequalities satisfied by a certain definite integral, *Bulletin of the American Mathematical Society*, 43, 709–716. II. 3

6. A formula of Ramanujan in the theory of primes, *Journal of the London Mathematical Society*, 12, 94–98. II. 1

7. Another formula of Ramanujan, *Journal of the London Mathematical Society*, 12, 314–318. IV. 1 (*d*)

1938

1. A further note on Ramanujan's arithmetical function $\tau(n)$, *Proceedings of the Cambridge Philosophical Society*, 34, 309–315. II. 2

1939

1. A note on a differential equation, *Proceedings of the Cambridge Philosophical Society*, 35, 652–653. VII. 2

2. Notes on special systems of orthogonal functions (I): The boundedness of the generalized Laguerre system, *Journal of the London Mathematical Society*, 14, 34–36. IV. 1 (*d*)

3. Notes on special systems of orthogonal functions (II): On functions orthogonal with respect to their own zeros, *Journal of the London Mathematical Society*, 14, 37–44. III. 1 (*e*)

1940

1. Notes on special systems of orthogonal functions (III): A system of orthogonal polynomials, *Proceedings of the Cambridge Philosophical Society*, 36, 1–8. IV. 1 (*d*)

1941

1. (With J. E. L.) Theorems concerning mean values of analytic or harmonic functions, *Quarterly Journal of Mathematics*, 12, 221–256. III. 2

2. Note on a divergent series, *Proceedings of the Cambridge Philosophical Society*, 37, 1–8. VI

3. Notes on special systems of orthogonal functions (IV): The orthogonal functions of Whittaker's cardinal series, *Proceedings of the Cambridge Philosophical Society*, 37, 331–348. III. 1 (*d*)

4. A double integral, *Journal of the London Mathematical Society*, 16, 89–94. VII. 1

1942

1. Note on Lebesgue's constants in the theory of Fourier series, *Journal of the London Mathematical Society*, 17, 4–13. III. 1 (*a*)

1943

1. An inequality for Hausdorff means, *Journal of the London Mathematical Society*, 18, 46–50. II. 3

1943 (*cont.*)

2. (With W. W. Rogosinski) Notes on Fourier series (I): On sine series with positive coefficients, *Journal of the London Mathematical Society*, 18, 50–57. III. 1 (*d*)

3. (With W. W. Rogosinski) Notes on Fourier series (II): On the Gibbs phenomenon, *Journal of the London Mathematical Society*, 18, 83–87. III. 1 (*a*)

4. (With J. E. L.) N.S. XXII: On the Tauberian theorem for Borel summability, *Journal of the London Mathematical Society*, 18, 194–200. VI

1944

1. (With J. E. L.) N.S. XXIII: On the partial sums of Fourier series, *Proceedings of the Cambridge Philosophical Society*, 40, 103–107. III. 1 (*c*)

2. Note on the multiplication of series by Cauchy's rule, *Proceedings of the Cambridge Philosophical Society*, 40, 251–252. VI

1945

1. (With W. W. Rogosinski) Notes on Fourier series (III): Asymptotic formulae for the sums of certain trigonometrical series, *Quarterly Journal of Mathematics*, 16, 49–58. III. 1 (*d*)

2. (With N. Aronszajn) Properties of a class of double integrals, *Annals of Mathematics*, (2) 46, 220–241, and corrigendum in 47, 166. V

3. Riemann's form of Taylor's series, *Journal of the London Mathematical Society*, 20, 48–57. VI

4. A mathematical theorem about golf, *Mathematical Gazette*, 29, 226–227. VII. 2

1946

1. (With J. E. L.) N.S. XXIV: A curious power-series, *Proceedings of the Cambridge Philosophical Society*, 42, 85–90. I. 1

2. (With W. W. Rogosinski) Theorems concerning functions subharmonic in a strip, *Proceedings of the Royal Society* (A), 185, 1–14. IV. 2

1947

1. (With W. W. Rogosinski) Notes on Fourier series (IV): Summability (R_2), *Proceedings of the Cambridge Philosophical Society*, 43, 10–25. III. 1 (*b*)

2. A double integral, *Journal of the London Mathematical Society*, 22, 242–247. VII. 1

1948

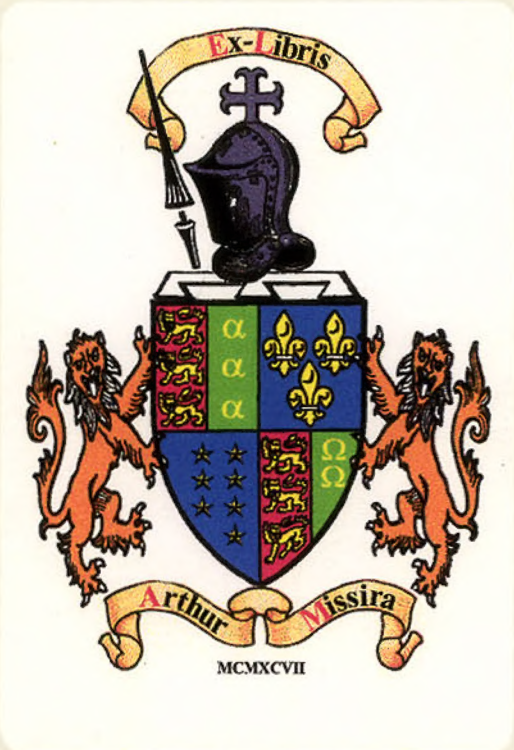
1. (With J. E. L.) A new proof of a theorem on rearrangements, *Journal of the London Mathematical Society*, 23, 163–168. II. 3

1949

1. (With W. W. Rogosinski) Notes on Fourier series (V): Summability (R_1), *Proceedings of the Cambridge Philosophical Society*, 45, 173–185. III. 1 (*b*)

OBITUARY NOTICES BY G. H. HARDY

S. Ramanujan, *Proceedings of the London Mathematical Society*, (2) 19 (1921), xl–lviii.
C. Jordan, *Proceedings of the Royal Society* A, 104 (1922), xxiii–xxvi.
G. Mittag-Leffler, *Journal of the London Mathematical Society*, 3 (1928), 156–60.
J. W. L. Glaisher, *Messenger of Mathematics*, 58 (1929), 159–160.
T. J. I'A. Bromwich, *Journal of the London Mathematical Society*, 5 (1930), 209–220.
R. E. A. C. Paley, *Journal of the London Mathematical Society*, 9 (1934), 76–80.
E. W. Hobson, *Journal of the London Mathematical Society*, 9 (1934), 225–237.
E. Landau (with H. Heilbronn), *Journal of the London Mathematical Society*, 13 (1938), 302–310.
W. H. Young, *Journal of the London Mathematical Society*, 17 (1942), 218–237.
J. R. Wilton (with H. S. Carslaw), *Journal of the London Mathematical Society*, 20 (1945), 58–64.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